1) a) $\sqrt[3]{125^{x^3-5x^2+11x-3}} = \sqrt[4]{5^{4x^2+12}}$

Rewrite purely with indices, noting $125^{\frac{1}{3}} = 5$:

$$5^{x^3-5x^2+11x-3} = 5^{\frac{1}{4}(4x^2+12)} = 5^{x^2+3}$$

Since exponentiation is a 1-1 operation we can say:

$$x^3 - 5x^2 + 11x - 3 = x^2 + 3$$

$$x^3 - 6x^2 + 11x - 6 = 0$$

$x = 1$ is a root:

$$(x-1)(x^2 - 5x + 6) = 0$$

$$(x-1)(x-2)(x-3) = 0$$

So $x = 1, 2$ or 3. ✓

b) $z = 2 - \dfrac{2+i}{iz} = \dfrac{2iz - 2 - i}{iz} = \dfrac{-2+i(2z-1)}{iz}$

So $iz^2 = 2iz - (2+i)$

$$z^2 - 2z + \frac{(2+i)}{i} = 0$$

$$z^2 - 2z + (1-2i) = 0$$

$$z = \frac{2 \pm \sqrt{4 - 4(1-2i)}}{2}$$

$$= 1 \pm \sqrt{1-(1-2i)}$$

$$= 1 \pm \sqrt{2i}$$

$$= 1 \pm \sqrt{2}\sqrt{i}$$

and $\sqrt{i} = \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}i = \frac{\sqrt{2}+\sqrt{2}i}{2}$

So the solutions are

$$1 \pm \frac{\sqrt{2}(\sqrt{2}+\sqrt{2}i)}{2}$$

$$= 1 \pm (1+i)$$

$$= \underline{\underline{2-i \text{ or } -i}} \checkmark$$

c) $z^{2025} = \bar{z}$

Let $z = r\operatorname{cis}\theta$: then $r^{2025} = r$, so $r = 1$.

So $z\bar{z} = 1$.

Mult. through by z:

$$z^{2026} = 1.$$

The complex roots of unity are therefore

$$z = \operatorname{cis}\left(\frac{2\pi n}{2026}\right) \text{ for } n = 0, \dots 2025$$

i.e. there are 2026 separate roots, including 1 and -1 but no purely imaginary numbers.

But there's also a solution $z = 0$; which essentially comes out as a factor.

So there are 2027 ('complex') roots. ✓

d) I've done this part and got the right answer ✓ $\left(\frac{7}{4} + \sqrt{3}\right)$ but I've lost it. I hate this whole paper so much that I'm not going to bother writing it out again... see the published solution or the Chat GPT one.

2) a) $x = 2t + t^2$

$y = p(t)$

$p(t) = at^3 + bt^2 + ct + d$

$p(1) = 2.5$

$p'(1) = 6$

we assume (second time round) that p' indicates $\frac{d}{dt}$, not $\frac{d}{dx}$.

We know $p(1) = a + b + c + d = 2.5$ ②

$p'(t) = 3at^2 + 2bt + c$

$p'(1) = 3a + 2b + c = 6$ ①

Now we have to find $\frac{d^2y}{dx^2}$.

$$\frac{dy}{dx} = \frac{dy}{dt} \Big/ \frac{dx}{dt}$$

and $\frac{dx}{dt} = 2 + 2t = 2(1+t)$

So $\frac{dy}{dx} = \frac{3at^2 + 2bt + c}{2(1+t)}$

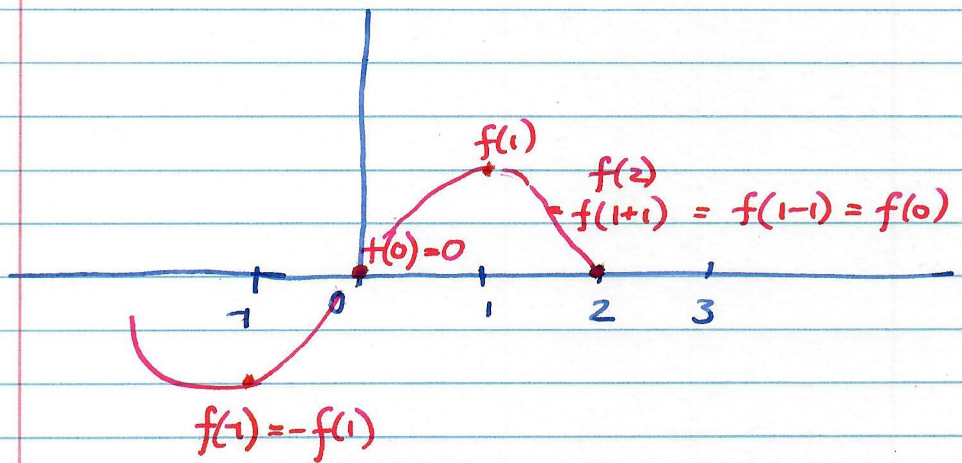
And $\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dt}\left(\frac{dy}{dx}\right) \Big/ \frac{dx}{dt}$

$$\frac{d}{dt}\left(\frac{dy}{dx}\right) = \frac{2(1+t)(6at+2b) - (3at^2+2bt+c)(2)}{4(1+t)^2}$$

2) b) $f(-x) = -f(x)$

$f(1-x) = f(1+x)$

This means the function is <u>rotationally</u> symmetric about $x=0$ (the origin) and <u>reflectionally</u> symmetric about $x=1$ - for example:



Also, because $f(x)$ is odd $f(0)$ must be 0 (assuming f is continuous at 0).

We can then calculate $f(x)$ $\forall$ $x \in \mathbb{N}$ by using symmetries.

$f(0) = 0$
$f(1) = 2025$
$f(2) = f(1+1) = f(1-1) = f(0) = 0$
$f(3) = f(1+2) = f(1-2) = f(-1) = -f(1) = -2025$
$f(4) = f(1+3) = f(1-3) = f(-2) = -f(2) = 0$
$f(5) = f(1+4) = f(1-4) = f(-3) = -f(3) = 2025$

Proceding in this manner we can see in general

$$\left.\begin{aligned} f(4p) &= f(4p+2) = 0 \\ f(4p+1) &= 2025 \\ f(4p+3) &= -2025 \end{aligned}\right\} \forall\, p \in \mathbb{N}$$

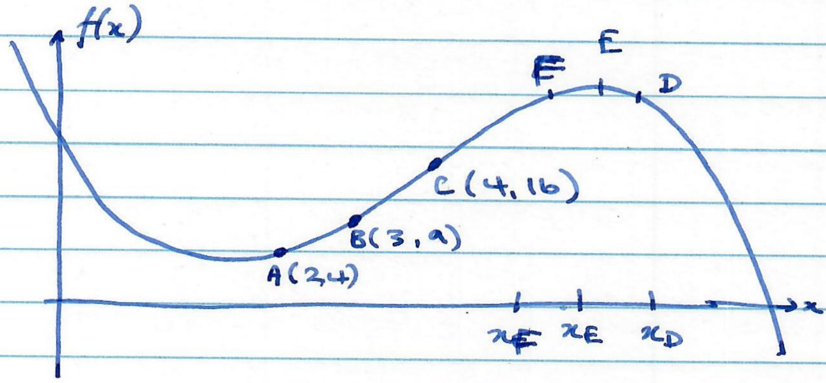
So each block of 4 ($n = 1, 2, 3, 4$ etc) sums to 0, and the only term left is the 2025^{th}:

$$\underbrace{f(1)+f(2)+f(3)+f(4)}_{=0} + \text{———} + \underbrace{f(2021)+f(2022)+f(2023)+f(2024)}_{=0}$$

$$\underbrace{+f(2025)}_{=2025}$$

and the sum required is simply $\underline{\underline{2025}}$. ✓

2) c)



$f(x) = ax^3 + bx^2 + cx + d$

A : (2,4)
B : (3,9)
C : (4,16)

Note these lie on $y = x^2$; so the suggested function $g(x) = f(x) - x^2$ has these as its roots.

This means
$$\begin{aligned} g(x) &= ax^3 + (b-1)x^2 + cx + d \\ &= f(x) - x^2 \\ &= a(x-2)(x-3)(x-4) \\ &= ax^3 + a(-2-3-4)x^2 \\ &\quad + a(6+12+8)x - 24a \\ &= ax^3 - 9ax^2 + 26ax - 24a \end{aligned}$$

So (if this is useful)
$$-9a = b-1 \quad (1)$$
$$26a = c \quad (2)$$
$$-24a = d. \quad (3)$$

And also $f(0) = g(0) = d$ (which is what we're asked for).

Now consider the lines AB etc.
These intercept at D, E, F so "f(x) - each line" has roots defined by $x = 2, 3, x_D$ etc.

Eq^n of AB: $y = \frac{9-4}{1}x + c = 5x + c = \underline{5x - 6.}$

AC: $y = \frac{16-4}{2}x + c = 6x + c = \underline{6x - 8}$

BC: $y = \frac{16-9}{}x + c = 7x + c = \underline{7x - 12.}$

We can subs. these into $f(x)$ to find x_D, x_E, x_F

and we know $x_D + x_E + x_F = 25$.

(*) An important step is now to say:

$$f(x) = g(x) + x^2 = a(x-2)(x-3)(x-4) + x^2.$$

Equate this with the line $y = 5x - 6$:

$$a(x_D-2)(x_D-3)(x_D-4) + x_D^2 = 5x_D - 6$$

$$a(x_D-2)(x_D-3)(x_D-4) + (x_D^2 - 5x_D + 6) = 0$$

$$a\cancel{(x_D-2)}\cancel{(x_D-3)}(x_D-4) + \cancel{(x_D-2)}\cancel{(x_D-3)} = 0$$

So $\quad a(x_D - 4) + 1 = 0$

Similarly $\quad a(x_E - 3) + 1 = 0$

$\quad a(x_F - 2) + 1 = 0$

So $\quad 25a - 9a + 3 = 0$

$$a = \frac{-3}{16}.$$

So from ③ $\quad f(0) = d = \frac{-3}{16} \times -24$

$$= \frac{72}{16} = \underline{\underline{4.5}}.$$

The key to this is the step (*), which leads to the nice cancellations later.

3) a)

$$\frac{1}{1+\sin x} = \frac{1}{1+\sin x} \times \frac{1-\sin x}{1-\sin x}$$

$$= \frac{1-\sin x}{1-\sin^2 x}$$

$$= \frac{1-\sin x}{\cos^2 x}$$

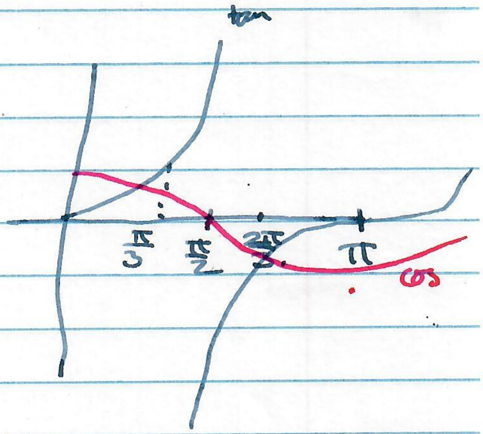
$$= \sec^2 x - \sec x \tan x$$

$$= \sec x(\sec x - \tan x) \text{ as required.}$$

$$\text{So } \int_0^{\frac{2\pi}{3}} \frac{1}{1+\sin x}\,dx = \int_0^{\frac{2\pi}{3}} \sec^2 x - \sec x \tan x \, dx$$

(using standard formulae)

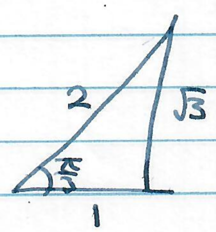
$$= \left[\tan x - \sec x\right]_0^{\frac{2\pi}{3}}$$



$$\tan\frac{\pi}{3} = \sqrt{3} \quad \text{so} \quad \tan\frac{2\pi}{3} = -\sqrt{3}$$

$$\cos\frac{\pi}{3} = \frac{1}{2} \quad \text{so} \quad \cos\frac{2\pi}{3} = -\frac{1}{2}$$

$$\sec\frac{2\pi}{3} = -2$$



So the expression is

$$\tan\frac{2\pi}{3} - \sec\frac{2\pi}{3} - \tan 0 + \sec 0$$

$$= -\sqrt{3} + 2 - 0 + 1 = 3 - \sqrt{3}$$

3) b) i) $\tan\left(\frac{\pi}{8}\right) = \sqrt{2}-1$

and $\tan\left(\frac{\pi}{4}\right) = 1$

So $\tan\left(\frac{\pi}{4}+\frac{\pi}{8}\right) = \dfrac{\tan\left(\frac{\pi}{4}\right)+\tan\left(\frac{\pi}{8}\right)}{1-\tan\left(\frac{\pi}{4}\right)\tan\left(\frac{\pi}{8}\right)}$

$$= \frac{1+\sqrt{2}-1}{1-1(\sqrt{2}-1)} = \frac{\sqrt{2}}{2-\sqrt{2}} = \cancel{\frac{1}{\sqrt{2}}\frac{\sqrt{2}}{(1-\sqrt{2})}}$$

$$\cancel{= \frac{1}{2}\frac{\sqrt{2}(1+\sqrt{2})}{(1-\sqrt{2})(1+\sqrt{2})} = \frac{\sqrt{2}(1+\sqrt{2})}{2(1-2)^2}}$$

$$= \frac{1}{\sqrt{2}-1}\frac{(\sqrt{2}+1)}{(\sqrt{2}+1)}$$

$$= \frac{\sqrt{2}+1}{(2-1)} = \underline{\underline{\sqrt{2}+1.}}$$

or quicker: $\tan\left(\frac{3\pi}{8}\right) = \cos\left(\frac{\pi}{8}\right) = \frac{1}{\sqrt{2}-1}$ ok

ii) $\tan\left(\frac{x}{2}\right)\tan y = \sqrt{2}-1$

$\tan\left(\frac{x}{2}+y\right) = \sqrt{2}+1.$

Clearly $y = x = \frac{\pi}{4}$ is a solution.

But is it the only one? Clearly we could add $n\pi$ onto y. And pres. $2n\pi$ onto x.

$\tan\left(\frac{\pi}{8}\right) = \sqrt{2}-1$

$\tan\left(\frac{\pi}{4}\right) = 1$

$\tan\left(\frac{3\pi}{8}\right) = \sqrt{2}+1.$

Given $\tan\left(\frac{x}{2}\right)\tan y = \sqrt{2}-1 = \tan\left(\frac{\pi}{8}+n\pi\right)$

$\tan\left(\frac{x}{2}+y\right) = \sqrt{2}+1 = \tan\left(\frac{3\pi}{8}+n\pi\right)$

So $\frac{x}{2}+y = \frac{3\pi}{8}+n\pi$

$y = \frac{3\pi}{8}+n\pi-\frac{x}{2} = \frac{3\pi}{8}-\frac{x}{2}$ (ignore $n\pi$ for the moment)

So $\tan\left(\frac{x}{2}\right)\tan\left(\frac{3\pi}{8}-\frac{x}{2}\right) = \sqrt{2}-1$

Let $\tan\left(\frac{x}{2}\right) = u$:

$$u\left(\frac{\tan\frac{3\pi}{8}-u}{1+u\tan\frac{3\pi}{8}}\right) = \sqrt{2}-1$$

$$-u^2+u\tan\frac{3\pi}{8} = (\sqrt{2}-1)\left(1+u\tan\frac{3\pi}{8}\right)$$

$$-u^2+u(\sqrt{2}+1) = (\sqrt{2}-1)(1+u(\sqrt{2}+1))$$

$$-u^2+u(\sqrt{2}+1) = (\sqrt{2}-1)+u(\sqrt{2}-1)(\sqrt{2}+1)$$

$$-u^2+u(\sqrt{2}+1) = (\sqrt{2}-1)+(2-1)u$$

$$-u^2+\sqrt{2}u-(\sqrt{2}-1) = 0$$

$$u^2-\sqrt{2}u+(\sqrt{2}-1) = 0.$$

~~$(1-\sqrt{2})^2 = 1-2\sqrt{2}+2$~~

Solve this:

$$u = \frac{\sqrt{2} \pm \sqrt{2 - 4(\sqrt{2}-1)}}{2}$$

$$= \frac{\sqrt{2} \pm \sqrt{6 - 4\sqrt{2}}}{2}$$

$$= \frac{1 \pm \sqrt{3 - 2\sqrt{2}}}{\sqrt{2}}$$

We note that $3 - 2\sqrt{2} = (1 - \sqrt{2})^2$

So $u = \dfrac{1 \pm (1 - \sqrt{2})}{\sqrt{2}}$

$$= \frac{2-\sqrt{2}}{\sqrt{2}} \text{ or } \frac{\sqrt{2}}{\sqrt{2}}$$

i.e. $\sqrt{2} - 1$ or 1

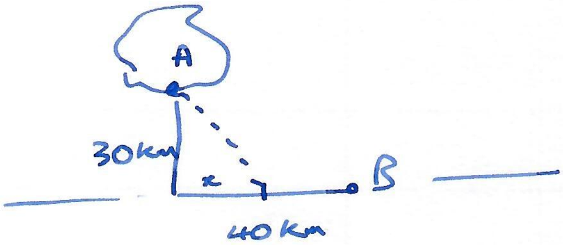
So $\dfrac{x}{2} = \dfrac{\pi}{8}$ or $\dfrac{\pi}{4}$ (+ $n\pi$ in each case)

So $\underline{\underline{x = \dfrac{\pi}{4} \text{ or } \dfrac{\pi}{2}}}$ ✓ (+ $2n\pi$ in each case)

and $y = \dfrac{3\pi}{8} - \dfrac{\pi}{8}$ or $\dfrac{3\pi}{8} - \dfrac{\pi}{4}$ (+ $n\pi$ in each case)

$= \underline{\underline{\dfrac{\pi}{4} \text{ or } \dfrac{\pi}{8}}}$ ✓ (+ $2n\pi$ in each case)

3) c)



C = Cost of the cable is $w\left(\sqrt{30^2+x^2}\right) + L(40-x)$

$$C = w(900+x^2)^{\frac{1}{2}} + L(40-x)$$

$$\frac{dC}{dx} = \frac{1}{2}w(900+x^2)^{-\frac{1}{2}}(2x) - L$$

$$= \frac{1}{2}w\frac{2x}{(900+x^2)^{\frac{1}{2}}} - L$$

This $=0$ for a minimum:

$$\frac{1}{\cancel{2}}w\frac{\cancel{2}x}{(900+x^2)^{\frac{1}{2}}} = L$$

$$wx = L(900+x^2)^{\frac{1}{2}}$$

$$w^2x^2 = L^2(900+x^2)$$

$$x^2(w^2-L^2) = 900L^2$$

$$x^2 = \frac{900L^2}{w^2-L^2}$$

when $w = 1.5L$: $x^2 = \frac{900}{1.25} = 720$ $x = 26.8$ km

i.e. run the cable to a point 13.2km from B.

$w = 1.2L$: $x^2 = \frac{900}{0.44} = 2045$ $x = 45.2$ km

which suggests run it directly to B.

4) a) i) I've no idea how to integrate $\sin^{-1}x$, but

we can clearly find $I = \int_0^{\frac{\pi}{2}} \sin y \, dy$ and

subtract this on the rectangle $[0, 1]$ by $[0, \frac{\pi}{2}]$

$$I = \int_0^{\frac{\pi}{2}} \sin y \, dy = [-\cos y]_0^{\frac{\pi}{2}} = 0 + 1 = 1.$$

So the integral required $= 1 \times \frac{\pi}{2} - 1$

i.e. $\frac{\pi}{2} - 1$.

ii) by similar reasoning,

$$\frac{dx}{dy} = \cos y$$

$$\text{So } \frac{dy}{dx} = \frac{1}{\cos y} = \frac{1}{\sqrt{1-\sin^2 y}} = \frac{1}{\sqrt{1-x^2}}$$

Using the chain rule,

$$\frac{d}{dx} \sin^{-1}(2x) = 2 \frac{1}{\sqrt{1-(2x)^2}}$$

$$\frac{d}{dx} \sin^{-1}\left(\frac{\pi}{4} - 2x\right) = -2 \frac{1}{\sqrt{1-\left(\frac{\pi}{4}-2x\right)^2}}$$

$$\frac{dg}{dx} = 2\left(\frac{1}{\sqrt{1-(2x)^2}} - \frac{1}{\sqrt{1-\left(\frac{\pi}{4}-2x\right)^2}} \right)$$

So for this to be 0 (assumed to be minimum):

$$\frac{1}{\sqrt{1-4x^2}} = \frac{1}{\sqrt{1-(\frac{\pi}{4}-2x)^2}}$$

$$\cancel{1-}\left(\frac{\pi}{4}-2x\right)^2 = \cancel{1-}4x^2$$

$$\frac{\pi^2}{16} - \pi x + \cancel{4x^2} = \cancel{4x^2}$$

$$\frac{\pi^2}{16} = \pi x$$

$$x = \frac{\pi}{16}.$$ ✓

So the minimum value is $g(\frac{\pi}{16}) = 2\sin^{-1}(\frac{\pi}{8})$ ✓

(note this extra part is required)

~~b) $f(x) = \left(\frac{1}{r^r}\right)^{\frac{1}{r+1}} \cdot x^r$~~

~~$\frac{df}{dx} = r\left(\frac{1}{r^r}\right)^{\frac{1}{r+1}} \cdot x^{r-1}$~~

~~$= \left(r^{\frac{r+1}{r+1}}\right)\left(\frac{1}{r^r}\right)^{\frac{1}{r+1}} = x^{r-1}$~~

~~$= r^{r+1}\left(\frac{r}{r^r}\right)^{\frac{1}{r+1}} x^{r-1}$~~

~~$= r^{r+1}\, r^{\frac{-(r-1)}{r+1}} x^{r-1} = r^{1-r} x^{r-1}$~~

~~And if $y = f(x) = \left(\frac{1}{r^r}\right)^{\frac{1}{r+1}} x^r = r^{\frac{-r}{r+1}} x^r$~~

~~then $x = y\, r^{\frac{-(1+r)}{r}}$~~

4) b) $f(x) = \left(\frac{1}{r^r}\right)^{\frac{1}{r+1}} x^r$

$= (r^{-r})^{\frac{1}{r+1}} x^r$

$= r^{\frac{-r}{r+1}} x^r$

So $\frac{df}{dx}(x) = r \cdot r^{\frac{-r}{r+1}} x^{r-1}$

$= r^{\frac{-r}{r+1}+1} x^{r-1}$

$= r^{\frac{-r+r+1}{r+1}} x^{r-1}$

$= r^{\frac{1}{r+1}} x^{r-1}$

$= f^{-1}(x)$ (given)

So using the result given,

$f(f^{-1}(x)) = x$

$= f\left(r^{\frac{1}{r+1}} x^{r-1}\right)$

$= r^{\frac{-r}{r+1}} \left(r^{\frac{1}{r+1}} x^{r-1}\right)^r$

$= r^{\frac{-r}{r+1}} \cdot r^{\frac{r}{r+1}} \cdot x^{r(r-1)} = r^0 x^{r(r-1)} = x^{r(r-1)}$

So $1 = r(r-1) = r^2 - r$: $r^2 - r - 1 = 0$

$r = \frac{1 \pm \sqrt{1+4}}{2} = \frac{1 \pm \sqrt{5}}{2}$ (ignore the -ve root since $r > 0$) ✓

4) c) i) $\lim_{x \to -\infty} h(x) = 0$ since $|h(x)|$ is $\sin(e^x)$

and the e^x term $\to 0$:

so the value is bounded by a value

which $\to \sin(0) = 0$.

$\lim_{x \to \infty} h(x)$ does not exist since the value

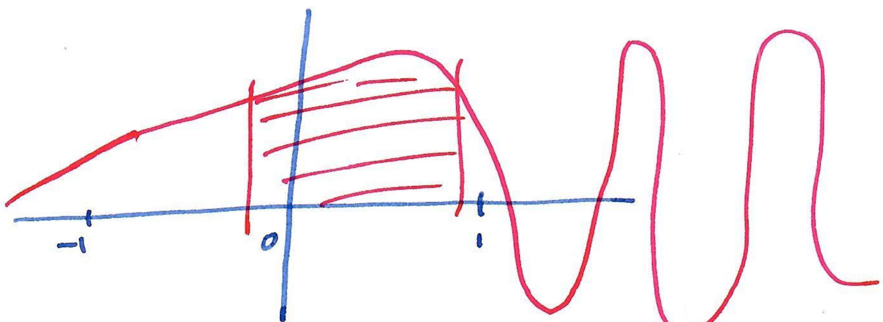
oscillates, with increasing frequency,

between values of ± 1. (it never 'settles down').

(thankfully we're not asked for deep mathematical reasoning.)

ii) $$I = \int_a^{a+1} \sin(e^x)\, dx.$$

The question asks for the maximum value of the definite integral... which is a very strange idea for a question. Looking at the diagram it appears to be $a \approx -0.2$, giving an area:



Chat GPT finds $a = \ln\left(\frac{\pi}{e+1}\right)$ quite easily.

The published answer goes into a more kind of iterative approach.. but it's not really any more convincing.