NZQA Scholarship Calculus 2024

Question One — Full Worked Solutions

1(a)

Consider the curve

$$y = \frac{2x^2 + 1}{3x^2 - 4x - 2}.$$

(i) Find the coordinates of any stationary points on the curve, and determine their nature.

A stationary point occurs where

$$\frac{dy}{dx} = 0,$$

provided the function is defined there.

Let

$$u = 2x^2 + 1, v = 3x^2 - 4x - 2.$$

Then

$$y = \frac{u}{v},$$

so by the quotient rule,

$$\frac{dy}{dx} = \frac{u'v - uv'}{v^2}.$$

Now

$$u' = 4x, v' = 6x - 4.$$

So

$$dy/dx = (4x(3x^2 - 4x - 2) - (2x^2 + 1)(6x - 4))/(3x^2 - 4x - 2)^2 .$$

Expand the top:

$$4x(3x^2 - 4x - 2) = 12x^3 - 16x^2 - 8x,$$

and

$$(2x^2 + 1)(6x - 4) = 12x^3 - 8x^2 + 6x - 4.$$

Therefore

$$dy/dx = ((12x^3 - 16x^2 - 8x) - (12x^3 - 8x^2 + 6x - 4))/(3x^2 - 4x - 2)^2 .$$

Simplify:

$$dy/dx = (-8x^2 - 14x + 4)/(3x^2 - 4x - 2)^2 .$$

Factor the numerator:

$$-8x^2 - 14x + 4 = -2(4x^2 + 7x - 2).$$

So stationary points come from

$$4x^2 + 7x - 2 = 0.$$

Solve:

$$x = \frac{-7 \pm \sqrt{49 + 32}}{8} = \frac{-7 \pm 9}{8}.$$

Thus

$$x = \frac{1}{4} \text{ or } x = -2.$$

Now find the corresponding y-values.

At $x = \frac{1}{4}$:

$$y = \frac{2\left(\frac{1}{4}\right)^2 + 1}{3\left(\frac{1}{4}\right)^2 - 4\left(\frac{1}{4}\right) - 2} = \frac{2 \cdot \frac{1}{16} + 1}{3 \cdot \frac{1}{16} - 1 - 2} = \frac{\frac{1}{8} + 1}{\frac{3}{16} - 3} = \frac{\frac{9}{8}}{-\frac{45}{16}} = \frac{9}{8} \cdot \frac{16}{-45} = -\frac{2}{5}.$$

So one stationary point is

$$\left(\frac{1}{4}, -\frac{2}{5}\right).$$

At $x = -2$:

$$y = \frac{2(-2)^2 + 1}{3(-2)^2 - 4(-2) - 2} = \frac{8 + 1}{12 + 8 - 2} = \frac{9}{18} = \frac{1}{2}.$$

So the other stationary point is

$$\left(-2, \frac{1}{2}\right).$$

Determining their nature

Since

$$dy/dx = (-8x^\wedge 2 - 14x + 4)/(3x^\wedge 2\text{ⓜ} - 4x\text{ⓜ} - 2)^\wedge 2 \dashv ,$$

the denominator is always positive where defined, so the sign of $\frac{dy}{dx}$ is controlled by the numerator

$$-8x^2 - 14x + 4.$$

Test intervals around the roots $x = -2$and $x = \frac{1}{4}$:

- For $x = -3$: numerator $= -72 + 42 + 4 = -26 < 0$, so decreasing.
- For $x = -1$: numerator $= -8 + 14 + 4 = 10 > 0$, so increasing.
- For $x = 1$: numerator $= -8 - 14 + 4 = -18 < 0$, so decreasing.

So:

- at $x = -2$, the gradient changes from negative to positive: **local minimum**;
- at $x = \frac{1}{4}$, the gradient changes from positive to negative: **local maximum**.

Answer to 1(a)(i):

$$\boxed{\left(-2, \frac{1}{2}\right) \text{ is a local minimum}}$$
$$\boxed{\left(\frac{1}{4}, -\frac{2}{5}\right) \text{ is a local maximum}}$$

(ii) Find the value of $\lim_{x \to \infty} \frac{2x^2+1}{3x^2-4x-2}$. Hence, find the coordinate(s) where the curve intersects its own asymptote(s).

For large x, the highest power terms dominate, so

$$\lim_{x \to \infty} \frac{2x^2 + 1}{3x^2 - 4x - 2} = \frac{2}{3}.$$

Thus the horizontal asymptote is

$$y = \frac{2}{3}.$$

To find where the curve intersects its asymptote, solve

$$\frac{2x^2 + 1}{3x^2 - 4x - 2} = \frac{2}{3}.$$

Cross-multiply:

$$3(2x^2 + 1) = 2(3x^2 - 4x - 2).$$

So

$$6x^2 + 3 = 6x^2 - 8x - 4.$$

Hence

$$3 = -8x - 4$$
$$7 = -8x$$

$$x = -\frac{7}{8}.$$

Then

$$y = \frac{2}{3}.$$

So the curve intersects its horizontal asymptote at

$$\boxed{\left(-\frac{7}{\frac{8,2}{3}}\right)}.$$

1(b)

For what real value(s) of α does the system

$$x^3 + y^3 = 2, x + y = \alpha$$

have no real solutions?

We use the identity

$$x^3 + y^3 = (x + y)^3 - 3xy(x + y).$$

Since $x + y = \alpha$, we get

$$2 = \alpha^3 - 3\alpha xy.$$

So

$$xy = \frac{\alpha^3 - 2}{3\alpha},$$

provided $\alpha \neq 0$.

Now $x$and $y$are the roots of

$$t^2 - \alpha t + xy = 0.$$

For real solutions, this quadratic must have real roots, so its discriminant must satisfy

$$\alpha^2 - 4xy \geq 0.$$

Substitute the expression for xy:

$$\alpha^2 - 4 \cdot \frac{\alpha^3 - 2}{3\alpha} \geq 0.$$

Multiply through by 3α, but it is cleaner first to simplify:

$$\alpha^2 - \frac{4(\alpha^3 - 2)}{3\alpha} \geq 0.$$

Put over a common denominator:

$$\frac{3\alpha^3 - 4\alpha^3 + 8}{3\alpha} \geq 0$$

$$\frac{8 - \alpha^3}{3\alpha} \geq 0.$$

So we need

$$\frac{8-\alpha^3}{\alpha} \geq 0.$$

Now analyse the sign.

Critical values are $\alpha = 0$ and $\alpha = 2$.

- If $\alpha < 0$, then $8-\alpha^3 > 0$and $\alpha < 0$, so the fraction is negative.
- If $0 < \alpha < 2$, then $8-\alpha^3 > 0$and $\alpha > 0$, so the fraction is positive.
- If $\alpha > 2$, then $8-\alpha^3 < 0$and $\alpha > 0$, so the fraction is negative.
- At $\alpha = 2$, the fraction is 0, so real solutions exist.
- At $\alpha = 0$, the original system gives $y = -x$, so
-

$$x^3 + (-x)^3 = 0 \neq 2,$$

hence there are **no** solutions.

Therefore the system has real solutions only for

$$0 < \alpha \leq 2.$$

So it has **no** real solutions for

$$\boxed{\alpha \leq 0 \text{ or } \alpha > 2}.$$

1(c)

A landscaper models one side of a 48 m high hill by

$$y = kx^2(126 - x),$$

where $x$and $y$are in metres. Each step has depth $280\text{mm} = 0.28\text{m}$, and maximum height $190\text{mm} = 0.19\text{m}$. If every step has depth 0.28m, would the steps all satisfy the maximum-height regulation?

Use calculus.

We first determine k.

The hill is 48m high. The model

$$y = kx^2(126 - x)$$

has its maximum where

$$\frac{dy}{dx} = 0.$$

Differentiate:

$$y = k(126x^2 - x^3),$$

so

$$\frac{dy}{dx} = k(252x - 3x^2) = 3kx(84 - x).$$

Thus stationary points occur at

$$x = 0 \text{or} x = 84.$$

The maximum is at $x = 84$. So

$$48 = y(84) = k \cdot 84^2 \cdot (126 - 84) = k \cdot 84^2 \cdot 42.$$

Now

$$84^2 = 7056, 7056 \cdot 42 = 296352.$$

Hence

$$k = \frac{48}{296352} = \frac{1}{6174}.$$

So

$$y = \frac{1}{6174} x^2 (126 - x).$$

Step height and derivative

If a step has horizontal depth 0.28m, then the vertical rise over one step is approximately

$$\Delta y \approx \frac{dy}{dx} \cdot 0.28.$$

To guarantee all steps satisfy the maximum height regulation, we need the **largest** slope on the hill.

We already have

$$\frac{dy}{dx} = 3kx(84 - x).$$

This is a quadratic opening downwards, so its maximum occurs halfway between $x = 0$ and $x = 84$, that is at

$$x = 42.$$

Then

$$\left(\frac{dy}{dx}\right)_{\max} = 3k(42)(42) = 3k(1764) = 5292k.$$

Using $k = \frac{1}{6174}$,

$$\left(\frac{dy}{dx}\right)_{\max} = \frac{5292}{6174} = \frac{6}{7}.$$

So the largest possible rise over one 0.28m step is approximately

$$\Delta y_{\max} \approx \frac{6}{7} \cdot 0.28 = 0.24 \text{ m}.$$

That is

$$0.24 \text{ m} = 240 \text{ mm},$$

which is greater than the allowed maximum of 190mm.

So the steps would **not** all satisfy the regulation.

A slightly stronger argument

Since the derivative decreases after $x = 42$, the steepest part is around $x = 42$, and there the actual rise over a 0.28m horizontal interval will be very close to this maximum estimate. In fact, since $240" mm" > 190" mm"$, the regulation is definitely breached.

Answer to 1(c):

$$\boxed{\text{No, the steps would not all satisfy the maximum-height regulation.}}$$

The steepest gradient is

$$\boxed{\frac{6}{7}},$$

giving a step rise of about

$$\boxed{0.24 \text{ m} = 240 \text{ mm}},$$

which exceeds 190mm.

Final answers for Question One

$$\boxed{\left(-2, \frac{1}{2}\right) \text{ local minimum}, \left(\frac{1}{4}, -\frac{2}{5}\right) \text{ local maximum}}$$

$$\boxed{\lim_{x \to \infty} \frac{2x^2 + 1}{3x^2 - 4x - 2} = \frac{2}{3}}$$

$$\boxed{\text{Curve meets its asymptote at } \left(-\frac{7}{\frac{8,2}{3}}\right)}$$

$\boxed{\text{No real solutions when } \alpha \leq 0 \text{ or } \alpha > 2}$

$\boxed{\text{In part (c): No, the 280 mm steps would be too high at the steepest point}}$

Question Two — Full Worked Solutions

2(a)

For $0 \leq x \leq 3$, consider

$$h(x) = \max\left(2\sqrt{x}, 2x, x^2\right).$$

Evaluate

$$\int_0^3 h(x)\, dx.$$

The key idea is that $h(x)$is whichever of the three functions is largest at each value of x. So we first find where the graphs cross.

We compare the three functions:

$$f_1(x) = 2\sqrt{x}, f_2(x) = 2x, f_3(x) = x^2.$$

Step 1: Find intersections

Compare $2\sqrt{x}$and $2x$

Set them equal:

$$2\sqrt{x} = 2x.$$

So

$$\sqrt{x} = x.$$

Squaring gives

$$x = x^2$$
$$x(x-1) = 0.$$

Thus

$$x = 0 \text{ or } x = 1.$$

Compare $2x$and x^2

Set them equal:

$$2x = x^2$$
$$x(x-2) = 0.$$

Thus

$$x = 0 \text{ or } x = 2.$$

Compare $2\sqrt{x}$ and x^2

Set them equal:

$$2\sqrt{x} = x^2.$$

For $x \geq 0$, square both sides:

$$4x = x^4$$
$$x(x^3 - 4) = 0.$$

So

$$x = 0 \text{ or } x = \sqrt[3]{4}.$$

That last crossing is useful to know, but for the **maximum** function, the real issue is which graph is on top.

Step 2: Determine which function is largest on each interval

Test values:

- At $x = \frac{1}{4}$:

$$2\sqrt{x} = 1, 2x = \frac{1}{2}, x^2 = \frac{1}{16}.$$

Largest is $2\sqrt{x}$.

- At $x = \frac{3}{2}$:

$$2\sqrt{x} \approx 2.45, 2x = 3, x^2 = 2.25.$$

Largest is $2x$.

- At $x = \frac{5}{2}$:

$$2\sqrt{x} \approx 3.16, 2x = 5, x^2 = 6.25.$$

Largest is x^2.

So:

- on $0 \leq x \leq 1$, $h(x) = 2\sqrt{x}$,
- on $1 \leq x \leq 2$, $h(x) = 2x$,
- on $2 \leq x \leq 3$, $h(x) = x^2$.

Thus

$$\int_0^3 h(x)\,dx = \int_0^1 2\sqrt{x}\,dx + \int_1^2 2x\,dx + \int_2^3 x^2\,dx.$$

Step 3: Evaluate each integral

First integral:

$$\int_0^1 2\sqrt{x}\,dx = 2\int_0^1 x^{\frac{1}{2}}\,dx = 2\left[\frac{2}{3}x^{\frac{3}{2}}\right]_0^1 = \frac{4}{3}.$$

Second integral:

$$\int_1^2 2x\,dx = [x^2]_1^2 = 4 - 1 = 3.$$

Third integral:

$$\int_2^3 x^2\,dx = \left[\frac{x^3}{3}\right]_2^3 = \frac{27-8}{3} = \frac{19}{3}.$$

So

$$\int_0^3 h(x)\,dx = \frac{4}{3} + 3 + \frac{19}{3} = \frac{4}{3} + \frac{9}{3} + \frac{19}{3} = \frac{32}{3}.$$

Answer to 2(a):

$$\boxed{\int_0^3 h(x)\,dx = \frac{32}{3}}$$

2(b)
Given that

$$\frac{dy}{dx} = 1 + y^2,$$

find the value of M for which

$$\frac{d^3y}{dx^3} = M(1+y^2)(1+3y^2).$$

We already know the answer from the earlier question, but here is the full worked version in the same style.

Let

$$y' = 1 + y^2.$$

Differentiate once:

$$y'' = \frac{d}{dx(1+y^2)} = 2y\,y'.$$

Substitute $y' = 1 + y^2$:

$$y'' = 2y(1+y^2).$$

Differentiate again:

$$y''' = \frac{d}{dx\left(2y(1+y^2)\right)}.$$

Use the product rule:

$$y''' = 2\left[y'(1+y^2) + y\frac{d}{dx(1+y^2)}\right].$$

Now

$$\frac{d}{dx(1+y^2)} = 2y\,y'.$$

So

$$y''' = 2[y'(1+y^2) + 2y^2y'].$$

Factor out y':

$$y''' = 2y'(1+y^2+2y^2) = 2y'(1+3y^2).$$

Now substitute $y' = 1+y^2$:

$$y''' = 2(1+y^2)(1+3y^2).$$

Comparing with

$$y''' = M(1+y^2)(1+3y^2),$$

we get

$$\boxed{M=2}.$$

Answer to 2(b):

$$\boxed{M=2}$$

2(c)
For $n > 2$, define

$$s_n(x) = |\, x-1 \,| + |\, x-2 \,| + \cdots + |\, x-n \,|.$$

(i) Find s_3' " " $\left(\frac{5}{2}\right)$.

First write

$$s_3(x) = |\, x-1 \,| + |\, x-2 \,| + |\, x-3 \,|.$$

At $x = \frac{5}{2}$:

- $x-1 > 0$, so $|\, x-1 \,| = x-1$,
- $x-2 > 0$, so $|\, x-2 \,| = x-2$,
- $x-3 < 0$, so $|\, x-3 \,| = 3-x$.

So near $x = \frac{5}{2}$,

$$s_3(x) = (x-1) + (x-2) + (3-x).$$

Simplify:

$$s_3(x) = x.$$

Therefore

$$s_3'(x) = 1$$

near $x = \frac{5}{2}$, so

$$\boxed{s_3'\left(\frac{5}{2}\right) = 1}.$$

Answer to 2(c)(i):

$$\boxed{1}$$

(ii) Find the minimum value of $s_{2024}(x)$. Justify your answer.

We need the minimum of

$$s_{2024}(x) = |x-1| + |x-2| + \cdots + |x-2024|.$$

This is the sum of the distances from x to each of the integers $1, 2, \ldots, 2024$.

The general principle is:

The sum of distances to a set of points is minimised at the **median**.

Since there are 2024 points, an even number, the median is any point between the two middle values:

$$1012 \text{ and } 1013.$$

So $s_{2024}(x)$ is minimised for every

$$x \in [1012, 1013].$$

Now find that minimum value by choosing a convenient point, say $x = 1012$.

Then

$$s_{2024}(1012) = \sum_{k=1}^{2024} |1012 - k|.$$

Split this into the points on the left and right.

- For $k = 1, \ldots, 1012$:

$$|1012 - k| = 1012 - k,$$

giving

$$1011 + 1010 + \cdots + 1 + 0.$$

- For $k = 1013, \ldots, 2024$:

$$|\,1012 - k\,| = k - 1012,$$

giving

$$1 + 2 + \cdots + 1012.$$

So

$$s_{2024}(1012) = (1 + 2 + \cdots + 1011) + (1 + 2 + \cdots + 1012).$$

Use

$$1 + 2 + \cdots + m = \frac{m(m+1)}{2}.$$

Then

$$1 + 2 + \cdots + 1011 = \frac{1011 \cdot 1012}{2},$$

and

$$1 + 2 + \cdots + 1012 = \frac{1012 \cdot 1013}{2}.$$

Therefore

$$s_{2024}(1012) = \frac{1011 \cdot 1012}{2} + \frac{1012 \cdot 1013}{2}.$$

Factor out $\frac{1012}{2}$:

$$s_{2024}(1012) = \frac{1012}{2(1011 + 1013)} = \frac{1012}{2(2024)}.$$

So

$$s_{2024}(1012) = 1012^2.$$

Now

$$1012^2 = (1000 + 12)^2 = 1000000 + 24000 + 144 = 1024144.$$

Thus the minimum value is

$$\boxed{1024144}.$$

Why this is the minimum

A clean derivative-style justification is as follows.

For values of $x$not equal to one of the integers $1, 2, \ldots, 2024$, each term contributes:

- $+1$if $x > k$,
- -1if $x < k$.

So

$$s\_2024^\wedge{}' (x) = \#(\text{"}points\ to\ the\ left\ of\text{ "}\ x) - \#(\text{"}points\ to\ the\ right\ of\text{ "}\ x).$$

- If $x < 1012$, there are more points to the right than to the left, so the derivative is negative.
- If $x > 1013$, there are more points to the left, so the derivative is positive.
- If $1012 < x < 1013$, there are exactly 1012points on each side, so the derivative is 0.

Hence $s_{2024}(x)$decreases, then is flat, then increases — so the minimum occurs on the interval $[1012, 1013]$.

Answer to 2(c)(ii):

$$\boxed{\min s_{2024}(x) = 1012^2 = 1024144}$$

attained for every

$$\boxed{x \in [1012,1013]}.$$

Final answers for Question Two

$$\boxed{\int_0^3 h(x)\,dx = \frac{32}{3}}$$

$$\boxed{M = 2}$$

$$\boxed{s_3' \square \left(\frac{5}{2}\right) = 1}$$

$$\boxed{\min s_{2024}(x) = 1024144}$$

with the minimum attained for

$$\boxed{1012 \le x \le 1013}$$

Question Three — Full Worked Solutions

3(a)

Find the exact value of

$$\int_0^{\frac{\pi}{4}} \frac{\cos 2x}{\cos^2 x}\, dx.$$

We use the identity

$$\cos 2x = 2\cos^2 x - 1.$$

So

$$\frac{\cos 2x}{\cos^2 x} = \frac{2\cos^2 x - 1}{\cos^2 x} = 2 - \sec^2 x.$$

Therefore

$$\int_0^{\frac{\pi}{4}} \frac{\cos 2x}{\cos^2 x}\, dx = \int_0^{\frac{\pi}{4}} \llbracket(2 - \rrbracket \sec^2 x)\, dx\,.$$

Integrate:

$$\int (2 - \sec^2 x)\, dx = 2x - \tan x\,.$$

Hence

$$\int_0^{\frac{\pi}{4}} \llbracket(2 - \rrbracket \sec^2 x)\, dx = [2x - \tan x]_0^{\frac{\pi}{4}}.$$

Now

$$2 \cdot \frac{\pi}{4} - \frac{\tan \pi}{4} = \frac{\pi}{2} - 1,$$

and at $x = 0$,

$$2(0) - \tan 0 = 0.$$

So the exact value is

$$\boxed{\frac{\pi}{2} - 1}.$$

3(b)

Alice's motion is given parametrically by

$$x = 4\cos\theta + \cos 4\theta\,, y = 4\sin\theta + \sin 4\theta\,.$$

(i) Find the coordinates in the first quadrant where Alice is closest to the teapot. Justify your

answer.
The teapot is at the origin, so we minimise the distance from (x, y) to $(0, 0)$. It is easier to minimise the square of the distance:

$$r^2 = x^2 + y^2.$$

Substitute the parametric expressions:

$$r^2 = (4\cos\theta + \cos 4\theta)^2 + (4\sin\theta + \sin 4\theta)^2.$$

Expand:

$$r^2 = 16\cos^2\theta + 8\cos\theta\cos 4\theta + \cos^2 4\theta + 16\sin^2\theta + 8\sin\theta\sin 4\theta + \sin^2 4\theta.$$

Group terms:

$$r^2 = 16(\cos^2\theta + \sin^2\theta) + (\cos^2 4\theta + \sin^2 4\theta) + 8(\cos\theta\cos 4\theta + \sin\theta\sin 4\theta).$$

Using

$$\cos^2 t + \sin^2 t = 1$$

and

$$\cos A\cos B + \sin A\sin B = \cos(A - B),$$

we get

$$r^2 = 16 + 1 + 8\cos(\theta - 4\theta) = 17 + 8\cos 3\theta.$$

So we must minimise

$$r^2 = 17 + 8\cos 3\theta.$$

This is smallest when

$$\cos 3\theta = -1.$$

Hence

$$3\theta = \pi \Rightarrow \theta = \frac{\pi}{3}.$$

This does give a point in the first quadrant, because

$$x = \frac{4\cos\pi}{3} + \frac{\cos 4\pi}{3} = 4\cdot\frac{1}{2} + \left(-\frac{1}{2}\right) = 2 - \frac{1}{2} = \frac{3}{2},$$

and

$$y = \frac{4\sin\pi}{3} + \frac{\sin 4\pi}{3} = 4\cdot\frac{\sqrt{3}}{2} + \left(-\frac{\sqrt{3}}{2}\right) = 2\sqrt{3} - \frac{\sqrt{3}}{2} = \frac{3\sqrt{3}}{2}.$$

So the coordinates are

$$\boxed{\left(\frac{3}{2}, \frac{3\sqrt{3}}{2}\right)}.$$

(ii) Find the distance Alice travels in one complete rotation around the teapot.

A complete rotation corresponds to

$$0 \leq \theta \leq 2\pi.$$

The arc length is

$$L = \int_0^{2\pi} \sqrt{\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2}\, d\theta.$$

Differentiate:

$$\frac{dx}{d\theta} = -4\sin\theta - 4\sin 4\theta,$$

$$\frac{dy}{d\theta} = 4\cos\theta + 4\cos 4\theta\,.$$

So

$$\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2 = 16(\sin\theta + \sin 4\theta)^2 + 16(\cos\theta + \cos 4\theta)^2.$$

Factor out 16:

$$= 16[(\sin\theta + \sin 4\theta)^2 + (\cos\theta + \cos 4\theta)^2].$$

Expand inside:

$$\begin{aligned}(\sin\theta + \sin 4\theta)^2 + (\cos\theta + \cos 4\theta)^2 \\ = (\sin^2\theta + \cos^2\theta) + (\sin^2 4\theta + \cos^2 4\theta) \\ + 2(\sin\theta\sin 4\theta + \cos\theta\cos 4\theta).\end{aligned}$$

So

$$= 1 + 1 + 2\cos(\theta - 4\theta) = 2 + 2\cos 3\theta\,.$$

Thus

$$\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2 = 16(2 + 2\cos 3\theta).$$

Use

$$1 + \cos 3\theta = 2\cos^2\frac{3\theta}{2},$$

so

$$2 + 2\cos 3\theta = 4\cos^2\frac{3\theta}{2}.$$

Hence

$$\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2 = 64\cos^2\frac{3\theta}{2}$$

Therefore

$$\sqrt{\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2} = 8\left|\frac{\cos 3\theta}{2}\right|$$

So

$$L = \int_0^{2\pi} 8\left|\frac{\cos 3\theta}{2}\right| d\theta$$

Let

$$u = \frac{3\theta}{2}, \qquad d\theta = \frac{2}{3}\,du$$

When $\theta = 0,\ u = 0$; when $\theta = 2\pi,\ u = 3\pi$. So

$$\frac{L = 8\cdot\frac{2}{3}\int_0^{3\pi}⟦\ |\cos u|\ du =⟧\ 16}{3}\int_0^{3\pi}⟦\ |\cos u|\ du⟧$$

Now

$$\int_0^{\pi}⟦\ |\cos u|\ du = 2⟧$$

so over $0$to 3π,

$$\int_0^{3\pi}⟦\ |\cos u|\ du = 3\times 2 = 6⟧$$

Hence

$$L = \frac{16}{3}\cdot 6 = 32$$

So the total distance travelled is

$$\boxed{32}$$

3(c)

The "waffle cone" diagram consists of infinitely many similar right-angled triangles inside a large right-angled triangle. We are asked to work through the trig proof. The diagram and the required statements are given on pages 12–13.

(i) Show that

$$c^2 = \frac{2ab}{\sin 2\alpha}.$$

At the top of the diagram, the large isosceles triangle is split into two congruent right triangles. For one of those right triangles:

- hypotenuse $= c$,
- one leg $= a$,
- the other leg $= b$,
- the angle at the top is α.

So by the definitions of sine and cosine in that right triangle,

$$\sin\alpha = \frac{a}{c}, \cos\alpha = \frac{b}{c}$$

Multiply these:

$$\sin\alpha\cos\alpha = \frac{ab}{c^2}$$

Use

$$\sin 2\alpha = 2\sin\alpha\cos\alpha\,.$$

Therefore

$$\sin 2\alpha = 2\cdot\frac{ab}{c^2} = \frac{2ab}{c^2}$$

Rearranging,

$$\boxed{c^2 = \frac{2ab}{\sin 2\alpha}}$$

(ii) Show that

$$u = \frac{2abc}{b^2 - a^2},$$

find a similar expression for v, and hence prove that

$$a^2 + b^2 = c^2.$$

A clean way to do this is to place the figure in coordinates.

Let the left-hand corner of the horizontal segment be

$$L = (0,0),$$

the top vertex be

$$T = (a, b),$$

and the right-hand end of the top horizontal segment be

$$R = (2a, 0).$$

This matches the diagram: the top isosceles triangle has base $2a$ and height b.

Step 1: Equations of the two outer sloping lines

The **lower** outer side $u$passes through $L = (0,0)$, and makes angle αbelow the horizontal. Since in the small right triangle

$$\tan\alpha = \frac{a}{b},$$

its slope is

$$-\frac{a}{b}.$$

So its equation is

$$y = -\frac{a}{b}x.$$

The **upper** outer side $v$passes through $T = (a, b)$and also through $R = (2a, 0)$. Its slope is

$$\frac{0-b}{2a-a} = -\frac{b}{a}.$$

So its equation is

$$y - b = -\frac{b}{a(x-a)},$$

that is,

$$y = -\frac{b}{a}x + 2b.$$

Step 2: Find the tip where the two lines meet

Set the equations equal:

$$-\frac{a}{b}x = -\frac{b}{a}x + 2b.$$

Multiply through by ab:

$$-a^2x = -b^2x + 2ab^2.$$

So

$$x(b^2 - a^2) = 2ab^2,$$

hence

$$x = \frac{2ab^2}{b^2 - a^2}.$$

Call this point P.

Step 3: Find $\boldsymbol{u}$

The segment u is the length along the lower sloping line from $L = (0,0)$to P.

Its horizontal projection is

$$\frac{2ab^2}{b^2 - a^2}.$$

Now on the lower sloping line, the angle to the horizontal is α, so

$$\cos\alpha = \frac{b}{c}.$$

Thus

$$u = \frac{\text{horizontal projection}}{\cos\alpha} = \frac{2ab^2}{b^2 - a^2} \cdot \frac{c}{b}$$

So

$$u = \frac{2abc}{b^2 - a^2}$$

This is exactly what was required:

$$\boxed{u = \frac{2abc}{b^2 - a^2}}.$$

Step 4: Find a similar expression for $\boldsymbol{v}$

The segment $v$is the length along the upper sloping line from $T = (a, b)$to P.

The horizontal change is

$$\frac{2ab^2}{b^2 - a^2} - a = \frac{2ab^2 - a(b^2 - a^2)}{b^2 - a^2} = \frac{ab^2 + a^3}{b^2 - a^2} = \frac{a(a^2 + b^2)}{b^2 - a^2}$$

The upper line makes angle β to the horizontal, and in the right triangle

$$\cos\beta = \frac{a}{c}.$$

So

$$v = \frac{\text{horizontal projection}}{\cos\beta} = \frac{a(a^2+b^2)}{b^2-a^2} \cdot \frac{c}{a}$$

Therefore

$$\boxed{v = \frac{c(a^2+b^2)}{b^2-a^2}}$$

Step 5: Relate u, v, and α

Now look at the **outer** triangle formed by sides c, u, and v.

- The angle at the top is 2α.
- The angle at the left corner is $90°$.
- Therefore the side v is opposite the right angle.

So by the sine rule,

$$\frac{u}{\sin 2\alpha} = \frac{v}{\sin 90°} = v.$$

Hence

$$u = v \sin 2\alpha\,.$$

Substitute the expressions found above:

$$\frac{2abc}{b^2-a^2} = \frac{c(a^2+b^2)}{b^2-a^2} \sin 2\alpha$$

Cancel the common factor $\frac{c}{b^2-a^2}$:

$$2ab = (a^2+b^2)\sin 2\alpha\,.$$

So

$$\sin 2\alpha = \frac{2ab}{a^2+b^2}.$$

Step 6: Use part (i)

From part (i),

$$c^2 = \frac{2ab}{\sin 2\alpha}$$

Substitute

$$\sin 2\alpha = \frac{2ab}{a^2+b^2}$$

Then

$$c^2 = \frac{2ab}{\frac{2ab}{(a^2 + b^2)}} = a^2 + b^2$$

So we have proved

$$\boxed{a^2 + b^2 = c^2}$$

Final answers for Question Three

$$\boxed{\int_0^{\frac{\pi}{4}} \frac{\cos 2x}{\cos^2 x}\, dx = \frac{\pi}{2} - 1}$$

$$\boxed{\text{Closest first-quadrant point is } \left(\frac{3}{2}, \frac{3\sqrt{3}}{2}\right)}$$

$$\boxed{\text{Distance travelled in one full rotation is } 32}$$

$$\boxed{c^2 = \frac{2ab}{\sin 2\alpha}}$$

$$\boxed{u = \frac{2abc}{b^2 - a^2}}$$

$$\boxed{v = \frac{c(a^2 + b^2)}{b^2 - a^2}}$$

$$\boxed{a^2 + b^2 = c^2}$$

Question Four — Full Worked Solutions

4(a)

Consider all complex numbers $z$satisfying

$$-\frac{\pi}{3} \leq \arg(z) \leq \frac{\pi}{3}, \qquad z + z^{-} \leq 4, \qquad |\, z \,| \geq 2.$$

Find the exact area of the region in the Argand diagram represented by z.

Let

$$z = x + iy.$$

We interpret each condition geometrically.

Step 1: Interpret the three conditions

Condition 1:

$$-\frac{\pi}{3} \leq \arg(z) \leq \frac{\pi}{3}.$$

This is the sector between the two rays making angles $-\frac{\pi}{3}$and $+\frac{\pi}{3}$with the positive real axis. So it is a wedge opening to the right.

Condition 2:

$$z + z^{-} \leq 4.$$

Since

$$z + z^{-} = (x + iy) + (x - iy) = 2x,$$

this becomes

$$2x \leq 4 \Rightarrow x \leq 2.$$

So the region lies on or to the **left** of the vertical line $x = 2$.

Condition 3:

$$|\, z \,| \geq 2.$$

This means the point lies **on or outside** the circle of radius 2 centred at the origin.

Step 2: Describe the bounded region

Inside the wedge $-\frac{\pi}{3} \leq \theta \leq \frac{\pi}{3}$, the line $x = 2$ becomes, in polar form,

$$\mathrm{r}\cos\theta = 2 \Rightarrow r = 2\sec\theta\,.$$

So for each angle $\theta \in \left[-\frac{\pi}{3}, \frac{\pi}{3}\right]$, the allowed region runs from the circle

$$r = 2$$

out to the line

$$r = 2\sec\theta\,.$$

Therefore the area is

$$A = \frac{1}{2}\int_{-\frac{\pi}{3}}^{\frac{\pi}{3}} [(2\sec\theta)^2 - 2^2]\ d\theta.$$

So

$$A = \frac{1}{2}\int_{-\frac{\pi}{3}}^{\frac{\pi}{3}} [\![(4]\!] \sec^2\theta - 4)\, d\theta = 2\int_{-\frac{\pi}{3}}^{\frac{\pi}{3}} (\sec^2\theta - 1)\, d\theta\,.$$

Now

$$\int (\sec^2\theta - 1)\, d\theta = \tan\theta - \theta.$$

Hence

$$A = 2[\tan\theta - \theta]_{-\frac{\pi}{3}}^{\frac{\pi}{3}}.$$

Evaluate:

$$\tan\frac{\pi}{3} = \sqrt{3}, \tan\left(-\frac{\pi}{3}\right) = -\sqrt{3}.$$

So

$$A = 2\left[\left(\sqrt{3} - \frac{\pi}{3}\right) - \left(-\sqrt{3} + \frac{\pi}{3}\right)\right].$$

Thus

$$A = 2\left(2\sqrt{3} - \frac{2\pi}{3}\right) = 4\sqrt{3} - \frac{4\pi}{3}.$$

Answer to 4(a):

$$\boxed{4\sqrt{3} - \frac{4\pi}{3}}$$

4(b)

If

$$z = \cos\theta + \mathrm{i}\sin\theta \text{and} 0 \le \theta \le \frac{\pi}{2},$$

prove the following.

(i) Prove that

$$\frac{1}{1 - \text{zcos}\,\theta} = 1 + \text{icot}\,\theta\,.$$

Start with

$$z = \cos\theta + \text{isin}\,\theta\,.$$

Then

$$\text{zcos}\,\theta = \cos^2\theta + \text{isin}\,\theta\cos\theta\,.$$

So

$$1 - \text{zcos}\,\theta = 1 - \cos^2\theta - \text{isin}\,\theta\cos\theta\,.$$

Since

$$1 - \cos^2\theta = \sin^2\theta,$$

we get

$$1 - \text{zcos}\,\theta = \sin^2\theta - \text{isin}\,\theta\cos\theta = \sin\theta(\sin\theta - \text{icos}\,\theta)\,.$$

Therefore

$$\frac{1}{1 - \text{zcos}\,\theta} = \frac{1}{\sin\theta(\sin\theta - \text{icos}\,\theta)}.$$

Now note that

$$(\sin\theta - \text{icos}\,\theta)(\sin\theta + \text{icos}\,\theta) = \sin^2\theta + \cos^2\theta = 1.$$

So

$$\frac{1}{\sin\theta - \text{icos}\,\theta} = \sin\theta + \text{icos}\,\theta\,.$$

Hence

$$\frac{1}{1 - \text{zcos}\,\theta} = \frac{\sin\theta + \text{icos}\,\theta}{\sin\theta} = 1 + i\frac{\cos\theta}{\sin\theta}.$$

Thus

$$\boxed{\frac{1}{1 - \text{zcos}\,\theta} = 1 + \text{icot}\,\theta}.$$

(ii) Prove that

$$2\arg(z + 1) = \arg(z)\,.$$

Since

$$z = \cos\theta + \text{isin}\,\theta = e^{i\theta},$$

we have

$$z + 1 = 1 + \cos\theta + \text{isin}\,\theta\,.$$

Use the half-angle identities:

$$\frac{1+\cos\theta = 2\cos^{2}\frac{\theta}{2}, \sin\theta = \frac{2\sin\theta}{2}\cos\theta}{2}.$$

So

$$z+1 = \frac{2\cos\theta}{2}\left(\frac{\cos\theta}{2} + \frac{i\sin\theta}{2}\right).$$

That is,

$$z+1 = \frac{2\cos\theta}{2}\, e^{\frac{i\theta}{2}}.$$

Now because

$$0 \le \theta \le \frac{\pi}{2},$$

we have

$$0 \le \frac{\theta}{2} \le \frac{\pi}{4},$$

so

$$\frac{\cos\theta}{2} > 0.$$

Therefore the positive real factor $2\cos\left(\frac{\theta}{2}\right)$ does not affect the argument, and hence

$$\arg(z+1) = \frac{\theta}{2}.$$

But

$$\arg(z) = \theta.$$

So

$$2\arg(z+1) = 2\cdot\frac{\theta}{2} = \theta = \arg(z).$$

Therefore

$$\boxed{2\arg(z+1) = \arg(z)}.$$

4(c)

The point $P(2p, p)$ lies on the line

$$y = \frac{1}{2}x, p \ge 0.$$

Another point $Q(q, -2q)$ lies on the line

$$y = -2x, q \le 0.$$

(i) Explain why the locus of points equidistant from $P$and the line $y = -2x$is a parabola, and describe its key features.

A parabola is defined as the locus of points that are equidistant from:

- a fixed point (the **focus**), and
- a fixed line (the **directrix**).

Here the fixed point is

$$P = (2p, p),$$

and the fixed line is

$$y = -2x.$$

So by definition, this locus is a **parabola**.

Now describe its key features.

Focus

The focus is simply

$$\boxed{(2p, p)}.$$

Directrix

The directrix is

$$\boxed{y = -2x}.$$

Axis of symmetry

The axis of a parabola is perpendicular to the directrix and passes through the focus.

The line $y = -2x$has slope -2, so a perpendicular line has slope

$$\frac{1}{2}.$$

The focus $P(2p, p)$already lies on the line

$$y = \frac{1}{2}x,$$

since

$$p = \frac{1}{2(2p)}.$$

So the axis is

$$\boxed{y = \frac{1}{2}x}.$$

Vertex

The vertex is halfway between the focus and the directrix, measured along the axis.

The axis $y = \frac{1}{2}x$ meets the directrix $y = -2x$ at the origin $(0, 0)$, and that is the foot of the perpendicular from P to the directrix.

So the vertex is the midpoint of

$$(0,0) \text{ and } (2p, p),$$

which is

$$\left(\frac{0+2p}{2}, \frac{0+p}{2}\right) = \boxed{\left(p, \frac{p}{2}\right)}.$$

Direction of opening

The parabola opens away from the directrix and towards the focus, so it opens along the line $y = \frac{1}{2}x$in the positive direction.

Answer to 4(c)(i):

It is a parabola because it is the locus of points equidistant from a fixed point and a fixed line. Its key features are:

$$\boxed{\text{Focus } (2p, p)}$$
$$\boxed{\text{Directrix } y = -2x}$$
$$\boxed{\text{Axis } y = \frac{1}{2}x}$$
$$\boxed{\text{Vertex } \left(p, \frac{p}{2}\right)}$$

and it opens along the axis away from the directrix.

(ii) Let R be the region enclosed by the locus of points that are three times as far from P as they are from Q. If p increases at $3\ cm\ s^{-1}$and q decreases at $2\ cm\ s^{-1}$, at what rate is the area of R increasing when $P = (6, 3)$and $Q = (-4, 8)$?

Let a general point in the plane be (x, y). The locus is defined by

$$\text{distance to } P = 3 \times \text{distance to } Q$$

So

So

$$\sqrt{(x-2p)^2+(y-p)^2}=3\sqrt{(x-q)^2+(y+2q)^2}$$

Square both sides:

$$(x-2p)^2+(y-p)^2=9((x-q)^2+(y+2q)^2)$$

Expand:

$$x^2-4px+4p^2+y^2-2py+p^2=9x^2-18qx+9q^2+9y^2+36qy+36q^2$$

Bring everything to one side:

$$8x^2+8y^2+(4p-18q)x+(2p+36q)y+(45q^2-5p^2)=0$$

Divide by 8:

$$x^2+y^2+\frac{2p-9q}{4}x+\frac{p+18q}{4}y+\frac{45q^2-5p^2}{8}=0$$

This is the equation of a **circle**.

Step 1: Find the radius

For

$$x^2+y^2+Bx+Cy+D=0$$

the radius satisfies

$$r^2=\frac{B^2+C^2}{4}-D$$

Here

$$B=\frac{2p-9q}{4},C=\frac{p+18q}{4},D=\frac{45q^2-5p^2}{8}$$

So

$$r^2=\frac{1}{4}\left(\frac{2p-9q}{4}\right)^2+\frac{1}{4}\left(\frac{p+18q}{4}\right)^2-\frac{45q^2-5p^2}{8}$$

This simplifies to

$$r^2=\frac{45}{64(p^2+q^2)}$$

So

$$A=\pi r^2=\frac{45\pi}{64(p^2+q^2)}$$

Step 2: Differentiate with respect to time

Differentiate:

$$\frac{dA}{dt} = \frac{45\pi}{64} \cdot 2\left(p\frac{dp}{dt} + q\frac{dq}{dt}\right)$$

So

$$\frac{dA}{dt} = \frac{45\pi}{32}\left(p\frac{dp}{dt} + q\frac{dq}{dt}\right)$$

We are told:

- p is increasing at 3, so

$$\frac{dp}{dt} = 3$$

- q is decreasing at 2, so

$$\frac{dq}{dt} = -2$$

When

$$P = (6,3)$$

we have

$$2p = 6, p = 3$$

When

$$Q = (-4,8)$$

we have

$$q = -4$$

since

$$(q, -2q) = (-4,8)$$

Substitute:

$$\frac{dA}{dt} = \frac{45\pi}{32}\left(3 \cdot 3 + (-4)(-2)\right)$$

So

$$\frac{dA}{dt} = \frac{45\pi}{32(9+8)} = \frac{45\pi}{32} \cdot 17$$

Hence

$$\boxed{\frac{dA}{dt} = \frac{765\pi}{32}\,\mathrm{cm}^2\mathrm{s}^{-1}}$$

Final answers for Question Four

$$\boxed{\text{Area in 4(a)} = 4\sqrt{3} - \frac{4\pi}{3}}$$

$$\boxed{\frac{1}{1 - \mathrm{z}\cos\theta} = 1 + \mathrm{i}\cot\theta}$$

$$\boxed{2\arg(z+1) = \arg(z)}$$

For 4(c)(i), the locus is a parabola with

$$\boxed{\text{focus } (2p, p)\text{, directrix } y = -2x\text{, axis } y = \frac{1}{2}x\text{, vertex } \left(p, \frac{p}{2}\right)}$$

and for 4(c)(ii),

$$\boxed{\frac{dA}{dt} = \frac{765\pi}{32}\,\mathrm{cm}^2\mathrm{s}^{-1}}$$