1)a) $|z+a| = \sqrt{a}\,|z+1|$ $\quad z \in \mathbb{C},\ a \in \mathbb{R}$

Let $z = x + iy$

$|z+a|^2 = a|z+1|$

$(x+a)^2 + y^2 = a\left[(x+1)^2 + y^2\right]$

$x^2 + \cancel{2ax} + a^2 + y^2 = ax^2 + \cancel{2ax} + a + ay^2$

$x^2(1-a) + y^2(1-a) + a(a-1) = 0$

$(x^2+y^2)(1-a) = a(1-a)$

$x^2 + y^2 = a.$

So $|z| = \sqrt{a}.$

b) $x + y = \frac{\pi}{4}$ (A) $\quad x, y \in \mathbb{R}$

$\tan x + \tan y = 1$ (B)

using the trig identity:

$1 = \tan(x+y) = \dfrac{\tan x + \tan y}{1 - \tan x \tan y} = \dfrac{1}{1 - \tan x \tan y}$

So $1 - \tan x \tan y = 1$: $\quad \tan x \tan y = 0$

Either $\tan x = 0$: $\quad x = 0,\ y = \frac{\pi}{4}$

or $\tan y = 0$: $\quad x = \frac{\pi}{4},\ y = 0.$

c) $x^4 + x^3 - 4x^2 + x + 1 = 0$ find $x^3 + \frac{1}{x^3}$

$x < 0$

This 'obviously' needs dividing through by x^2:

$$x^2 + x - 4 + \frac{1}{x} + \frac{1}{x^2} = 0 \qquad \text{(A)}$$

and this looks like a couple of quadratics... but how to link them?

Try $u = x + \frac{1}{x}$.

Then $u^2 = x^2 + 2 + \frac{1}{x^2}$

And (A) is $u^2 + u - 6 = 0$

$$(u+3)(u-2) = 0$$

$u = -3$: $x + \frac{1}{x} = -3$ which is feasible

$u = 2$: $x + \frac{1}{x} = 2$ which is not, since $x < 0$

$$-27 = u^3 = \left(x + \frac{1}{x}\right)\left(x^2 + 2 + \frac{1}{x^2}\right)$$

$$= x^3 + x + 2x + \frac{2}{x} + \frac{1}{x} + \frac{1}{x^3}$$

$$= x^3 + 3x + \frac{3}{x} + \frac{1}{x^3} \qquad \text{(correct)}$$

$$= \left(x^3 + \frac{1}{x^3}\right) + 3\left(x + \frac{1}{x}\right) = -9$$

So $x^3 + \frac{1}{x^3} = -27 + 9 = \underline{\underline{-18}}$

2a) $x^2 - 4x + 10 = k(x+1)^2$

$$= kx^2 + 2kx + k$$

$$x^2(1-k) - (4+2k) + (10-k) = 0$$

For this 'to have distinct real roots' is one thing - 'to have the same sign' is another.

'$b^2 - 4ac$' gives $(4+2k)^2 - 4(1-k)(10-k) > 0$

$$16 + 16k + \cancel{4k^2} - 40 + 44k - \cancel{4k^2} > 0$$

$$-24 + 60k > 0$$

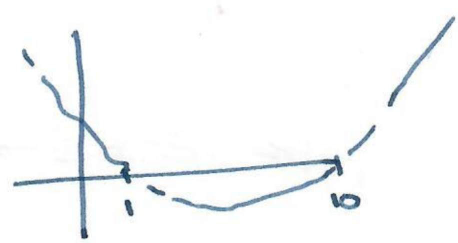
$60k > 24$ $\quad$ $\underline{\underline{k > \frac{2}{5}}}$ —— ①

For the roots to be of the same sign:

" $|b| > \sqrt{b^2 - 4ac}$ "

$$(4+2k)^2 > (4+2k)^2 - 4(1-k)(10-k)^2$$

$$4(1-k)(10-k) > 0$$

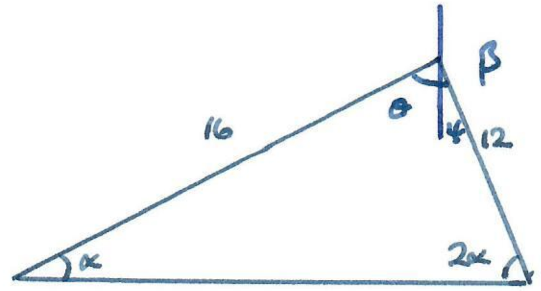


So $k < 1$ or $k > 10$.

Combine this with ①:

$\underline{\underline{\frac{2}{5} < k < 1 \quad \text{or } k > 10}}$

b)



This one's harder than it looks...

Sine rule:

$$\frac{\cancel{\sin\alpha}}{12} = \frac{\sin 2\alpha}{16} = \frac{2\cancel{\sin\alpha}\cos\alpha}{16}$$

So $\cos\alpha = \frac{8}{12} = \frac{2}{3}.$

$$\sin\alpha = \sqrt{1-\frac{4}{9}} = \frac{\sqrt{5}}{3}$$

$$\cos 2\alpha = \cos^2\alpha - \sin^2\alpha = \frac{4}{9} - \frac{5}{9} = -\frac{1}{9}$$

and $\sin 2\alpha = 2\sin\alpha\cos\alpha = 2\cdot\frac{2}{3}\cdot\frac{\sqrt{5}}{3} = \frac{4\sqrt{5}}{9}$

Now consider $\beta = \theta + \psi.$

$$\sin\beta = \sin\theta\cos\psi + \cos\theta\sin\psi$$

$$\left.\begin{aligned} \sin\theta &= \cos\alpha = \frac{2}{3} \\ \cos\theta &= \sin\alpha = \frac{\sqrt{5}}{3} \\ \sin\psi &= \cos 2\alpha = -\frac{1}{9} \\ \cos\psi &= \sin 2\alpha = \frac{4\sqrt{5}}{9} \end{aligned}\right\}$$

So $\sin\beta = \frac{2}{3}\cdot\frac{4\sqrt{5}}{9} - \frac{1}{9}\frac{\sqrt{5}}{3}$

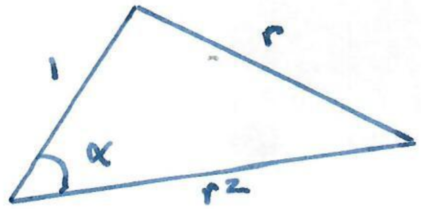
$= \frac{7\sqrt{5}}{27}$

So Area of the Δ

$= \left(\frac{1}{2}ab\sin C\right)$:

$\frac{1}{2}\cdot 12\cdot 16\cdot 7\frac{\sqrt{5}}{27} = \underline{\underline{\frac{224\sqrt{5}}{9}}}$

c)



(diagram assumes $r>1$, which may not be the case.)

Scale everything to 1.

α certainly can't be 180° since this would imply

$$1+r^2 \geqslant r \quad (r \geqslant 1)$$

$$1 \leqslant r \quad (r \leq 1)$$

Use the cosine rule: $\left(\cos A = \dfrac{b^2+c^2-a^2}{2bc}\right)$

$$\cos\alpha = \frac{1+r^4-r^2}{2r^2}$$

$$= \frac{1}{2}\left(\frac{1}{r^2}+r^2-1\right)$$

$$\cancel{= \frac{1}{2r^3}\left(1+r^4-r^2\right)}$$

$$\cancel{= \frac{1}{2r^3}\left(\left(r^2-\frac{1}{2}\right)^2-\frac{5}{4}\right)}$$

Bit of a jump.

$$= \frac{1}{2}\left(\left(r-\frac{1}{r}\right)^2+1\right)$$

This has a minimum value when $r=1$ (bit of another jump), which makes an

equilateral $\triangle$: sides 1,1,1.

And (to confirm) $\cos\alpha = \frac{1}{2}$ so $\alpha = 60°$.

3) a) Using the identity provided:

$$f(x) = \frac{e^{5x}\sqrt{x+1}}{e^{\sqrt{x+1}}}$$

$$\ln f(x) = \ln(e^{5x}) + \ln\sqrt{x+1} - \ln e^{\sqrt{x+1}}$$

$$= 5x - \sqrt{x+1} + \cancel{\ln\sqrt{x+1}}. \quad \frac{1}{2}\ln(x+1)$$

actually it's ok.

So $f'(x) =$

And $$\frac{d}{dx}(\ln f(x)) = 5 - \frac{1}{2\sqrt{x+1}} + \frac{1}{\sqrt{x+1}} \cdot \frac{1}{2\sqrt{x+1}}$$

$$= 5 - \frac{1}{2\sqrt{x+1}} + \frac{1}{2(x+1)}. \quad *$$

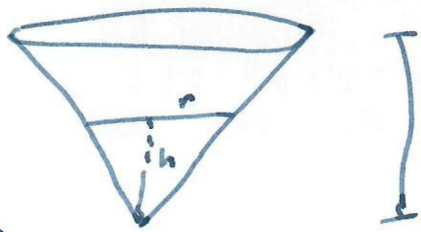
So $$f'(x) = * \times f(x)$$

$$= \left(5 - \frac{1}{2\sqrt{x+1}} + \frac{1}{2(x+1)}\right) \frac{e^{5x}\sqrt{x+1}}{e^{\sqrt{x+1}}}$$

$$f'(0) = \left(5 - \frac{1}{2} + \frac{1}{2}\right) \frac{e^{0}\sqrt{1}}{e^{1}}$$

$$= \frac{5}{e}$$

3.2 b)



$r(h) = \frac{h}{2}$

$V(h) = \frac{\pi r^2 h}{3 \cdot \cancel{2}} = \frac{\pi h^3}{4 \cdot 3}$

$\frac{dV}{dt} = -50$ (given)

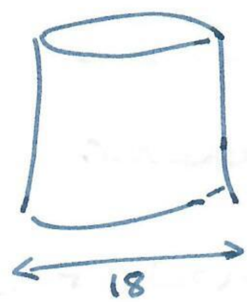
$\frac{dV}{dh} = \frac{3\pi h^2}{4 \cdot 3} = \frac{\pi h^2}{4}$.

So $\frac{dh}{dt} = \frac{dV}{dt} \Big/ \frac{dV}{dh} = \frac{-50 \cdot 4}{\pi h^2}$

when $h = 9$ this gives $\frac{dh}{dt} = \frac{-50 \cdot 4}{81\pi}$.

$= -\frac{200}{81\pi}$. (cm/m)

Flask:



$r = 9$

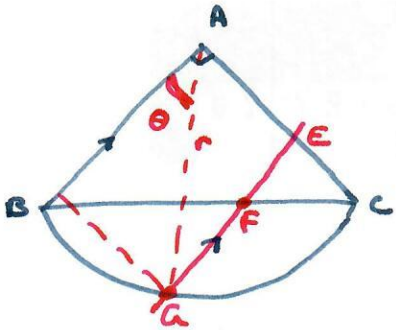
Area $= 81\pi$.

So rate of increase in height $= \frac{50}{81\pi}$. (cm/m)

So the ratio required is $\frac{200}{81\pi} : \frac{50}{81\pi} = 4:1$

(which makes sense since 9 is $\frac{1}{2}$ of 18).

c).



$$EG = r\cos\theta$$

$$AE = r\sin\theta$$

$$\frac{EF}{r} = \frac{EF}{AB} = \frac{EC}{AC} = \frac{r - r\sin\theta}{r}$$

So $EF = r - r\sin\theta$

$$FG = r\cos\theta - r(1-\sin\theta)$$

$$= r(\cos\theta + \sin\theta - 1)$$

So $\dfrac{d\,(FG)}{d\theta} = r(-\sin\theta + \cos\theta)$

$= 0$ when $\sin\theta = \cos\theta$

i.e. $\underline{\underline{\theta = \frac{\pi}{4}}}$

And at this point

$$FG = r\left(2 \cdot \frac{1}{\sqrt{2}} - 1\right) = \underline{\underline{(\sqrt{2}-1)r}}$$

4) a) Tempting to do this using De Moivre, but in fact we can just use Binomial expansion:

$$\cos^6\theta = \frac{\left(e^{i\theta}+e^{-i\theta}\right)^6}{64}$$

$$= \frac{1}{64}\Big(\left(e^{6i\theta}+e^{-6i\theta}\right)$$

$$+6\left(e^{5i\theta}e^{-i\theta}+e^{i\theta}e^{-5i\theta}\right)$$

$$+15\left(e^{4i\theta}e^{-2i\theta}+e^{2i\theta}e^{-4i\theta}\right)$$

$$+20\,e^{3i\theta}e^{-3i\theta}\Big)$$

= (using the expression for cos)

$$\frac{1}{32}\cos 6\theta + \frac{3}{16}\cos 4\theta + \frac{15}{32}\cos 2\theta$$

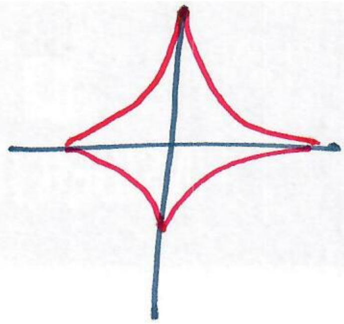
$$+\frac{5}{16}$$

as required.

4) b) $x = \cos^3 t$

$y = \sin^3 t$

$0 \leq t \leq 2\pi.$



$$\frac{dx}{dt} = -3\cos^2 t \sin t.$$

$$\text{So } \quad y\,dx = (-3\cos^2 t \sin t)\sin^3 t \, dt$$

$$= -3(\cos^2 t)\sin^4 t \, dt$$

$$= \frac{-3}{2^2}\left(2\cos t \sin t\right)^2 \sin^2 t \, dt$$

$$= -\frac{3}{4} \sin^2 2t \sin^2 t \, dt$$

$$\text{And } \quad \cos 2\theta = \cos^2\theta - \sin^2\theta$$

$$= 1 - 2\sin^2\theta$$

$$\sin^2\theta = \frac{1 - \cos 2\theta}{2}$$

$$\text{So } \quad y\,dx = -\frac{3}{4}\cdot\frac{1}{2}\sin^2 2t\left(1 - \cos 2\cancel{\theta}t\right) d\cancel{\theta}\, t$$

$$= \frac{-3}{8} \quad \frac{1}{2}\left(1 - \cos 4\theta\right)\left(1 - \cos 2t\right)$$

$$= -\frac{3}{16}\left(1 - \cos 2t - \cos 4t + \cos 2t \cos 4t\right)$$

Now $\cos 4t = 1 - \sin^2 2t$, so this gives:

$$= -\frac{3}{16}\left(1 - \cancel{\cos 2t} - \cos 4t + \cancel{\cos 2t} - 2\sin^2 2t \cos 2t\right)$$

And this is integrable, to ...

$$\frac{-3}{16}\left(t - \frac{1}{4}\sin 4t - \frac{2}{3}\sin^3 2t\right)$$

For the limits: at $\left.\begin{matrix} x=0 \\ y=1 \end{matrix}\right\} t = \frac{\pi}{2}$

$\left.\begin{matrix} x=1 \\ y=0 \end{matrix}\right\} t = 0$

So the integral is $\int_{\frac{\pi}{2}}^{0}$ (wierdly)

which gives

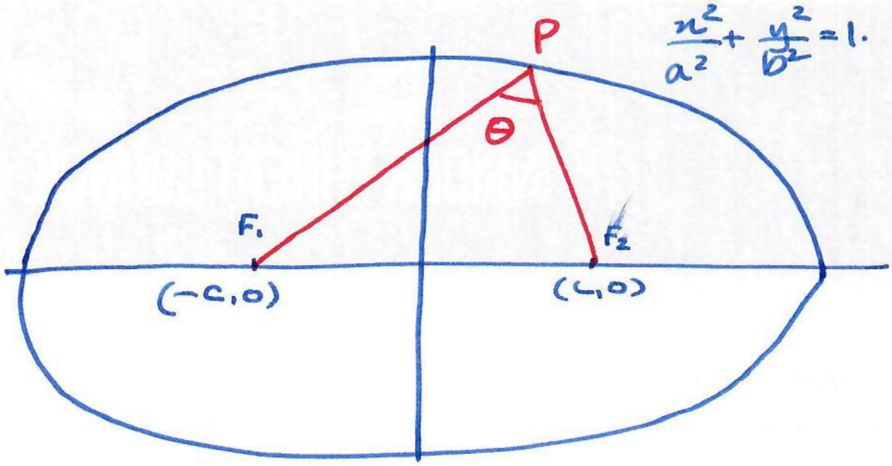
$$-\frac{3}{16}\left(-\frac{\pi}{2} - \frac{1}{4}.0 - \frac{2}{3}0 \right.$$
$$\left. + 0 + \frac{1}{4}.0 + \frac{2}{3}.0\right)$$

$$= \frac{3}{32}\pi.$$

So the area of the whole asteroid

is $4 \times \frac{3}{32}\pi = \underline{\underline{\frac{3}{8}\pi}}$

4)c)



Prove $\Delta F_1PF_2 = b^2 \tan\frac{\theta}{2}$

This looks to me like a $\tan 2A = \frac{2\tan A}{1-\tan^2 A}$ problem,

but that gets into a mess.

First, note that $a^2 = b^2 + c^2$ (property of ellipses)
Second, " " " " $F_1P + F_2P = 2a$ " "
Third
Then apply the cosine rule to F_1PF_2:

$$\cos\theta = \frac{F_1P^2 + F_2P^2 - F_1F_2^2}{2\, F_1P.F_2P}$$

$$= \frac{(F_1P+F_2P)^2 - 2F_1P.F_2P - F_1F_2^2}{2\, F_1P.F_2P}$$

$$= \frac{4a^2-4c^2}{2F_1P.F_2P} - 1$$

So $1+\cos\theta = \frac{2(a^2-c^2)}{F_1P.F_2P}$: $F_1P.F_2P = \frac{2(a^2-c^2)}{1+\cos\theta}$

$= \frac{2b^2}{1+\cos\theta}.$

And the area of ΔF_1PF_2 is

$$\frac{1}{2} F_1P \cdot F_2P \sin\theta$$

$$= \frac{2b^2}{2} \frac{\sin\theta}{1+\cos\theta}$$

$$\cos\theta = \cos^2\frac{\theta}{2} - \sin^2\frac{\theta}{2}$$

$$= 2\cos^2\frac{\theta}{2} - 1$$

$$= b^2 \frac{\cancel{2}\sin\frac{\theta}{2}\cancel{\cos\frac{\theta}{2}}}{\cancel{2}\cos^{\cancel{2}}\frac{\theta}{2}}$$

$$= b^2 \tan\frac{\theta}{2}.$$

5) Part (a) is a trick question - (b) is nasty, and (c) is just horrid.

a) Note that $\tan 89 = \cot 1 = \frac{1}{\tan 1}$, so the whole expression collapses to $\tan 45^\circ$ which is 1 anyway.

So $\int_0^a 1\,dx = \underline{\underline{a.}}$

b) This is a standard identity, but here we're asked to <u>prove</u> it. Which isn't trivial... unless you use <u>hyperbolic</u> functions.

For $\int \frac{1}{\sqrt{1+x^2}}\,dx$

We know for any θ $\quad \cosh^2\theta - \sinh^2\theta = 1$

Let $x = \sinh u$

then $dx = \cosh u\,du$.

and $1+x^2 = \cosh^2 u$

So the integral is $\int \frac{\cosh u\,du}{\cosh u} = \int 1\,du$

$= u + c$ $\qquad$ But what is this in terms of x?

Standard expression: $\sinh^{-1} x = \ln\left|x+\sqrt{x^2+1}\right|$

To prove this:

$u = \sinh^{-1} x$: $\sinh u = x$ $\cosh u = \sqrt{1+x^2}$

And $x = \sinh u = \dfrac{e^u - e^{-u}}{2}$

$$\cosh u = \frac{e^u + e^{-u}}{2}$$

So $\sinh u + \cosh u = \dfrac{2e^u}{2} = x + \sqrt{1+x^2}$

$$e^u = x + \sqrt{1+x^2}$$

$$u = \ln\left|x + \sqrt{1+x^2}\right|$$

So the integral is $\underline{\underline{\ln\left|x + \sqrt{1+x^2}\right| + C.}}$ as required.

(I first tried this by putting $x = \tan\theta$ ... which got me some way; and working backwards did the rest – but it was a hard slog.)

c) Let $\dfrac{dy}{dx} = A$ and this becomes:

$$x\frac{dA}{dx} = \frac{V_1}{V_2}\sqrt{1+A^2}$$

Separating variables:

$$\frac{dA}{\sqrt{1+A^2}} = \frac{V_1}{V_2}\frac{dx}{x}$$

So using (b)

$$\ln\left|\sqrt{1+A^2}+A\right| + c = \frac{v_1}{v_2}\ln x \quad \left(\begin{array}{l}\text{no need for a}\\ \text{second constant}\end{array}\right)$$

And $A(1) = y(1) = 0$

So $$\ln\left|\sqrt{1+0}+0\right| + c = \frac{v_1}{v_2}.0$$
$$= 0$$

so $c = 0$

$$\ln\left|\sqrt{1+A^2}+A\right| = \frac{v_1}{v_2}\ln x = \ln x^{\frac{v_1}{v_2}}$$

So $$\sqrt{1+A^2}+A = x^{\frac{v_1}{v_2}}$$

Beyond this the algebra gets hideous but eventually

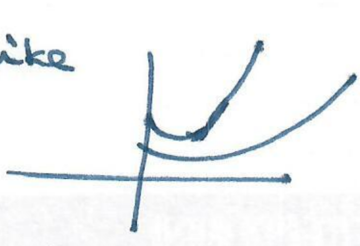
$$\frac{dy}{dx} = \frac{1}{2}\left[x^{\frac{v_1}{v_2}} - x^{-\frac{v_1}{v_2}}\right]$$

(which looks rather sinh-y).

Then if $v_1 = v_2$ $$y = \frac{x^2}{4} - \frac{1}{2}\ln x - \frac{1}{4}$$

if $v_1 \neq v_2$ $$y = \frac{1}{2}\left[\frac{v_2}{v_1+v_2}x^{\left(\frac{v_1}{v_2}+1\right)} + \frac{v_2}{v_1-v_2}x^{\left(1-\frac{v_1}{v_2}\right)}\right] + \frac{v_1 v_2}{v_2^2 - v_1^2}.$$

These curves are both like



but by now I've lost the will to carry on.