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93202Q

932022

NEW ZEALAND QUALIFICATIONS AUTHORITY
MANA TOHU MĀTAURANGA O AOTEAROA

QUALIFY FOR THE FUTURE WORLD
KIA NOHO TAKATŪ KI TŌ ĀMUA AO!

Scholarship 2020
Calculus

9.30 a.m. Monday 16 November 2020
Time allowed: Three hours
Total score: 40

QUESTION BOOKLET

There are five questions in this booklet. Answer ALL FIVE questions.

Write your answers in Answer Booklet 93202A.

Pull out Formulae and Tables Booklet S–CALCF from the centre of this booklet.

Show ALL working. Start your answer to each question on a new page. Carefully number each question.

Answers developed using a CAS calculator require **ALL commands to be shown**. Correct answers only will not be sufficient.

Check that this booklet has pages 2–6 in the correct order and that none of these pages is blank.

YOU MAY KEEP THIS BOOKLET AT THE END OF THE EXAMINATION.

QUESTION ONE

(a) Evaluate $\lim_{x\to\infty}\left(\dfrac{3x^2+2x-4}{5x^2+8x-1}\right)$.

(b) Evaluate the definite integral:

$$\int_0^a \frac{x^3}{\sqrt{a^4+x^4}}\,\mathrm{d}x$$

(c) Given the quartic equation:

$$\mathrm{a}x^4+\mathrm{b}x^3+\mathrm{c}x^2+\mathrm{d}x+\mathrm{e}=0$$

with roots α, β, γ, and δ.

(i) Show that:

$$\alpha+\beta+\gamma+\delta=-\frac{\mathrm{b}}{\mathrm{a}}$$

$$\beta\gamma+\gamma\alpha+\alpha\beta+\alpha\delta+\beta\delta+\gamma\delta=\frac{\mathrm{c}}{\mathrm{a}}$$

$$\beta\gamma\delta+\gamma\alpha\delta+\alpha\beta\delta+\alpha\beta\gamma=-\frac{\mathrm{d}}{\mathrm{a}}$$

$$\alpha\beta\gamma\delta=\frac{\mathrm{e}}{\mathrm{a}}$$

(ii) Now consider the equation:

$$x^4-8x^3+19x^2+px+2=0$$

Given that there are two roots whose sum is equal to the sum of the other two roots, find the exact value of p, and exact values for the roots.

QUESTION TWO

(a) If $x^4 + \dfrac{1}{x^4} = 7$, find the exact value of $x^8 + \dfrac{1}{x^8}$.

(b) Given $f(x) = \dfrac{\cos x}{2 + \sin x}$ on the domain $0 \le x \le 2\pi$:

(i) Find the exact co-ordinates of the turning points of $f(x)$.

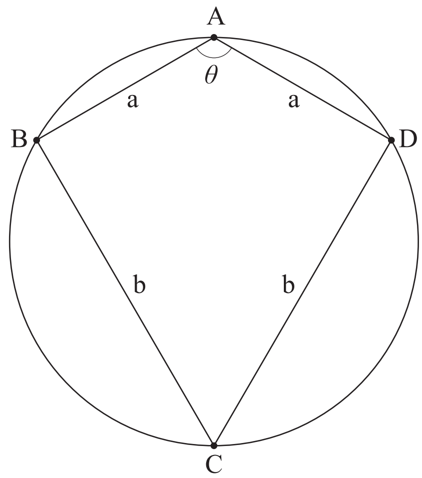
(ii) Discuss the concavity of $f(x)$, giving any point(s) of inflection in exact form.

(c) In the accompanying diagram, ABCD is a cyclic quadrilateral with:

AB = AD = a

BC = DC = b

$\angle BAD = \theta$



(i) Prove that $\dfrac{a}{b} = \dfrac{1 + \cos\theta}{\sin\theta}$

(ii) Calculate the magnitude of θ if:

$$b = \frac{a}{\sqrt{3}}$$

QUESTION THREE

(a) Given that $f(x) = \begin{cases} x^2 + ax + b & x \leq 0 \\ e^x + \sin x & x > 0 \end{cases}$

and $f''(0)$ exists, find the values of a and b.

(b) $y = f(x)$ is a smooth curve over the interval [–2,2] that passes through the point $\left(\sqrt{2}, -\sqrt{2}\right)$.

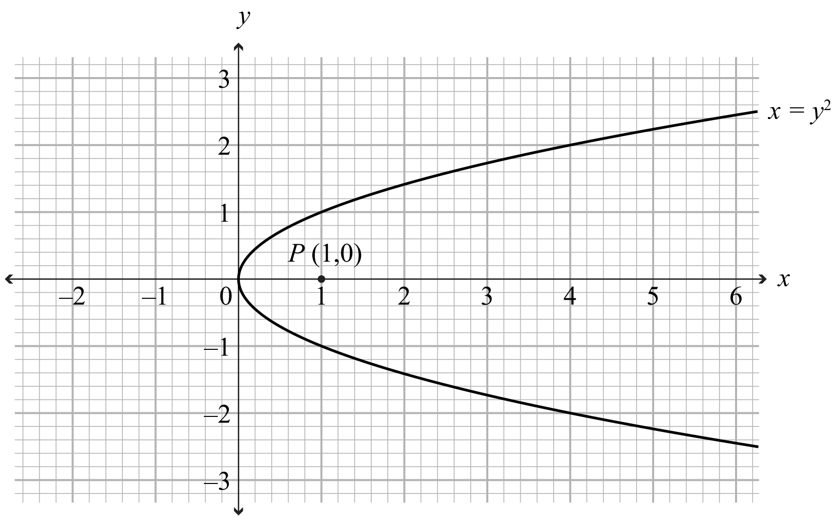
Given that the normal to the curve, at any point on the curve, passes through the origin, find the equation of the curve.

(c) The rates of increase of the length and width of a rectangle are 2 cm s^{-1}, and 3 cm s^{-1} respectively.

Find the rate of change of its diagonal when the rectangle has length 12 cm and width 9 cm.

(d) Find the point(s) on the parabola $x = y^2$ that is (are) the shortest distance from the point P (1,0)

(You must show that your result(s) give the minimum distance.)



QUESTION FOUR

(a) Given two functions, $f(x)$ and $g(x)$, use first principles to prove:

$$\frac{\mathrm{d}}{\mathrm{d}x}\big(f(x)\cdot g(x)\big)=\frac{\mathrm{d}f(x)}{\mathrm{d}x}\cdot g(x)+f(x)\cdot\frac{\mathrm{d}g(x)}{\mathrm{d}x}$$

(b) The product rule, which you were asked to prove in part (a) above, leads to a very useful integration formula.

$(f\cdot g)'=f'\cdot g+f\cdot g'$ implies $f\cdot g=\int(f'\cdot g)\mathrm{d}x+\int(f\cdot g')\mathrm{d}x$

which can be arranged as $\int(f'\cdot g)\mathrm{d}x=f\cdot g-\int(f\cdot g')\mathrm{d}x$

Using this result:

(i) Find $\int\left(\mathrm{e}^{-x}\cos x\right)\mathrm{d}x$

(ii) Find, in exact form, the area between the curve $y=\mathrm{e}^{-x}\cos(x)$ and the x-axis, where $0\le x\le 2\pi$.

(c) $y=f(x)$ is defined implicitly by the following: $xy+\mathrm{e}^{y}=2x+1$.

Evaluate $\frac{\mathrm{d}^2y}{\mathrm{d}x^2}$ at $x=0$.

QUESTION FIVE

(a) Euler discovered a very useful formula shown below.

$$e^{i\theta} = \cos\theta + i\sin\theta$$

Use this formula, or otherwise, to show:

(i) $\cos\theta = \dfrac{e^{i\theta} + e^{-i\theta}}{2}$

(ii) $\sin^3\theta = \dfrac{3}{4}\sin\theta - \dfrac{1}{4}\sin(3\theta)$

(b) We saw in part (a) above that $\cos x + i\sin x$ can also be written as e^{ix}.

It can be shown that $\int \left(e^{ix} \cdot e^{x}\right) dx = \dfrac{1}{i+1} e^{(i+1)x}$

Find $\int (\cos x \cdot e^{x})\, dx$

(c) Suppose z is a complex number, and $|z| = 1$.

Describe the locus of w, where $w = \dfrac{i+z}{i-z}$.

(d) Solve the system of equations

$$x^2 - yz = 1$$
$$y^2 - zx = 2$$
$$z^2 - xy = 3$$

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