NZQA Scholarship Calculus
2020 1) a)

$$\lim_{x\to\infty}\left(\frac{3x^2+2x-4}{5x^2+8x-1}\right)$$

This seems completely simple, but the answer requires...

$$= \lim_{x\to\infty}\left(\frac{3+\frac{2}{x}-\frac{4}{x^2}}{5+\frac{8}{x}-\frac{1}{x^2}}\right)$$

All the reciprocal terms $\to 0$,

leaving $\lim_{x\to\infty}\left(\frac{3}{5}\right) = \underline{\underline{\frac{3}{5}}}.$

1) b) $I = \int_0^a \frac{x^3\,dx}{\sqrt{a^4+x^4}}$.

This is a '$\frac{d}{dx}$ on the top' question:

Let $u = a^4 + x^4$

$du = 4x^3\,dx \qquad x^3\,dx = \frac{1}{4}du.$

So $I = \frac{1}{4}\int_{a^4}^{2a^4} \frac{1}{\sqrt{u}}\,du = \left[\frac{1}{4}\cdot\frac{2}{1}\,u^{\frac{1}{2}}\right]_{a^4}^{2a^4}$

$$= \frac{1}{2}\left(\sqrt{2}a^2 - a^2\right) = \underline{\underline{\frac{\sqrt{2}-1}{2}\,a^2}}$$

NZQA Scholarship Calculus

2020 1)c)

These are 'standard results', but presumably we have to do the legwork here...

$$ax^4 + bx^3 + cx^2 + dx + e = 0$$

So $$x^4 + \frac{b}{a}x^3 + \frac{c}{a}x^2 + \frac{dx}{a} + \frac{e}{a} = 0 \qquad \text{—— ①}$$

The roots are $\alpha, \beta, \gamma, \delta$ so the equation can be expressed as

$$(x-\alpha)(x-\beta)(x-\gamma)(x-\delta) = 0$$

Multiplying this out:

$$x^4 - (\alpha+\beta+\gamma+\delta)x^3 + (\alpha\beta+\alpha\gamma+\alpha\delta+\beta\gamma+\beta\delta+\gamma\delta)x^2$$
$$-(\beta\gamma\delta+\alpha\gamma\delta+\alpha\beta\delta+\alpha\beta\gamma)x + \alpha\beta\gamma\delta = 0$$
$$\text{—— ②}$$

So equating coefficients between ① and ②:

x^4: $1 = 1$

x^3: $-\frac{b}{a} = \alpha+\beta+\gamma+\delta$

x^2: $\frac{c}{a} = \alpha\beta+\alpha\gamma+\alpha\delta+\beta\gamma+\beta\delta+\gamma\delta$

x: $-\frac{d}{a} = \beta\gamma\delta+\alpha\gamma\delta+\alpha\beta\delta+\alpha\beta\gamma$

1: $\frac{e}{a} = \alpha\beta\gamma\delta$

as required.

NZQA Scholarship Calculus
2020 1)d)

$$x^4 - 8x^3 + 19x^2 + px + 2 = 0$$

we can pick the roots arbitrarily, so assume for the given condition

$$\alpha + \beta = \gamma + \delta$$

and substitute where we can.

Coeff x^3: $8 = \alpha + \beta + \gamma + \delta = 2(\alpha + \beta)$

So $\alpha + \beta = 4$

Coeff x^2: $19 = \alpha\beta + \alpha\gamma + \alpha\delta + \beta\gamma + \beta\delta + \gamma\delta$

$= \alpha\beta + \alpha(\gamma+\delta) + \beta(\gamma+\delta) + \gamma\delta$

$= \alpha\beta + \alpha(\alpha+\beta) + \beta(\alpha+\beta) + \gamma\delta$

$= \alpha\beta + (\alpha+\beta)^2 + \gamma\delta = \alpha\beta + 16 + \gamma\delta$ — ①

Coeff x: $-p = \beta\gamma\delta + \alpha\gamma\delta + \alpha\beta\delta + \alpha\beta\gamma$

$= (\alpha+\beta)\gamma\delta + \alpha\beta(\gamma+\delta)$

$= (\alpha+\beta)\gamma\delta + \alpha\beta(\alpha+\beta)$

$= (\alpha+\beta)(\alpha\beta + \gamma\delta) = 4(\alpha\beta + \gamma\delta)$ — ②

Coeff 1: $2 = \alpha\beta\gamma\delta$

so $\gamma\delta = \dfrac{2}{\alpha\beta}$

So in ① and ②: $19 = \alpha\beta + 16 + \dfrac{2}{\alpha\beta}$

$\alpha\beta + \dfrac{2}{\alpha\beta} = 3$ — ③

and $-p = 4\left(\alpha\beta + \dfrac{2}{\alpha\beta}\right)$

So $\left(\alpha\beta + \frac{2}{\alpha\beta}\right) = \frac{-p}{4}$

So in ③: $3 = -\frac{p}{4}$ $\quad \underline{\underline{p = -12.}}$

Subs this back into ②:

$$12 = 4(\alpha\beta + \gamma\delta)$$

$$\alpha\beta + \gamma\delta = 3 \quad \text{———— ④}$$

$$\text{So} \quad \alpha\beta + \frac{2}{\alpha\beta} = 3.$$

$$(\alpha\beta)^2 - 3\alpha\beta + 2 = 0$$

$$(\alpha\beta - 2)(\alpha\beta - 1) = 0$$

$$\underline{\alpha\beta = 2 \text{ or } \alpha\beta = 1.}$$

So from ④,

$$\underline{\gamma\delta = 1 \text{ or } \gamma\delta = 2}$$

i.e. $(\alpha\beta = 2 \text{ and } \gamma\delta = 1)$ or $(\alpha\beta = 1 \text{ and } \gamma\delta = 2)$

But we also know

$$\alpha + \beta = 4.$$

Case ①: $\alpha\beta = 2$ so $\beta = \frac{2}{\alpha}$: $\alpha + \frac{2}{\alpha} = 4$: $\alpha^2 - 4\alpha + 2 = 0$

$\alpha = 2 \pm 2\sqrt{2}$ $\quad \gamma = 2 \pm 2\sqrt{3}$

$\beta = 2 \mp 2\sqrt{2}$ $\quad \delta = 2 \mp 2\sqrt{3}$

Case ②: $\alpha\beta = 1$ so $\beta = \frac{1}{\alpha}$: $\alpha + \frac{1}{\alpha} = 4$: $\alpha^2 - 4\alpha + 1 = 0$

$\alpha = 2 \pm 2\sqrt{3}$ $\quad \gamma = 2 \pm 2\sqrt{2}$

$\beta = 2 \mp 2\sqrt{3}$ $\quad \delta = 2 \mp 2\sqrt{2}$

And since $\gamma + \delta = \alpha + \beta = 4$, we can say

NZQA Scholarship Calculus

2020 2) a)

$$x^4 + \frac{1}{x^4} = 7.$$

So $\left(x^4 + \frac{1}{x^4}\right)^2 = 49$

$$= x^8 + 2 \cdot x^4 \cdot \frac{1}{x^4} + \frac{1}{x^8}$$

$$= x^8 + 2 + \frac{1}{x^8}$$

So $x^8 + \frac{1}{x^8} = 49 - 2 = \underline{\underline{47.}}$

b) $f(x) = \dfrac{\cos x}{2 + \sin x}$

$$\frac{df}{dx} = \frac{(2+\sin x)(-\sin x) - \cos x(\cos x)}{(2+\sin x)^2}$$

$$= \frac{-2\sin x - \sin^2 x - \cos^2 x}{(2+\sin x)^2}$$

$$= \frac{-(1+2\sin x)}{(2+\sin x)^2}$$

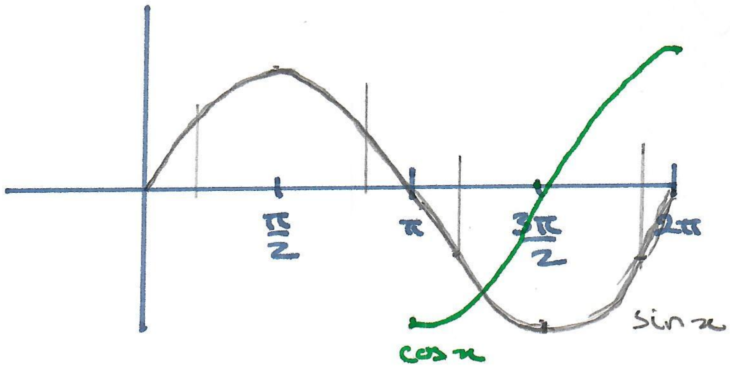
The denominator is always +ve (strictly) so for a turning point

$$1 + 2\sin x = 0$$

$$\sin x = \frac{-1}{2}$$

$\sin x = -\frac{1}{2}$

so $x = \frac{7\pi}{12}$ or $\frac{11\pi}{6}$



So the turning points are:

$x = \frac{7\pi}{12}$ $\quad f\left(\frac{7\pi}{12}\right) = -\frac{\sqrt{3}}{2}\,\frac{1}{3/2} = -\frac{1}{\sqrt{3}}$

so $\left(\frac{7\pi}{12}, -\frac{1}{\sqrt{3}}\right)$

$x = \frac{11\pi}{12}$ $\quad f\left(\frac{11\pi}{12}\right) = \frac{\sqrt{3}}{2}\cdot\frac{1}{3/2} = \frac{1}{\sqrt{3}}$

so $\left(\frac{11\pi}{12}, \frac{1}{\sqrt{3}}\right)$

So far so good... but finding $f''(x)$ is tedious, and it would be nice to have a graph of $f(x)$.

$$\frac{d^2 f}{dx^2} = \frac{(2+\sin x)^2(-2\cos x) + (1+2\sin x).2.\cos x(2+\sin x)}{(2+\sin x)^4}$$

$$= \frac{-8\cos x - 8\sin x\cos x - 2\sin^2 x\cos x + 10\cos x\sin x + 4\cos x + 4\sin^2 x\cos x}{(2+\sin x)^4}$$

$$= \frac{2\sin^2 x\cos x + 2\cos x\sin x - 4\cos x}{(2+\sin x)^4}$$

$$= \frac{2\cos x\,(\sin^2 x + \sin x - 2)}{(2+\sin x)^4}$$

$$= \frac{2\cos x\,(\sin x + 2)(\sin x - 1)}{(2+\sin x)^4}$$

$$= \frac{2\cos x\,(1-\sin x)}{(2+\sin x)^3}$$

This is 0 (inflexion points) when $\sin x = 1$

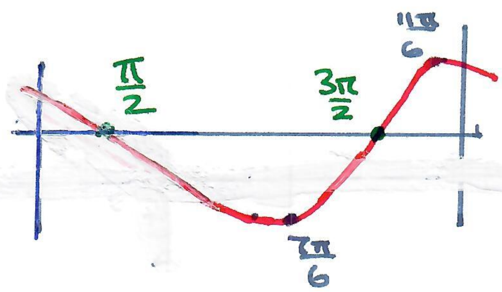
$$\left(\frac{\pi}{2}, 0\right)$$

or $\cos x = 0$

$$\left(\frac{\pi}{2}, 0\right), \left(\frac{3\pi}{2}, 0\right)$$

Beyond this it gets esoteric, not least because $x = \frac{\pi}{2}$ is a double root of $f''(x)$.

what does help, though it's not asked for, is a graph of $f(x)$:



From this it's clear the concavity is

upward on $\left(\frac{\pi}{2}, \frac{3\pi}{2}\right)$

and downward on $\left[0, \frac{\pi}{2}\right)$ and $\left(\frac{3\pi}{2}; \pi\right]$

but doing this by algebra is tedious.

NZQA Scholarship Calculus

2020 2) c)

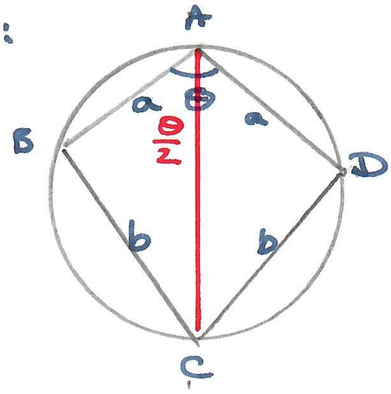
Part (b) was a nightmare: Part (c) seems rather too easy.

ABCD is a kite and so is symmetrical:

so AC is clearly a diameter

of the circle and $\hat{ABC} = \hat{ADC}$

$= 90°.$



So $\hat{BAC} = \hat{CAD} = \frac{\theta}{2}$

and $\frac{a}{b} = \cot\frac{\theta}{2}$: $\frac{b}{a} = \tan\frac{\theta}{2}$

Using the standard double-angle formula:

$$\tan\theta = \frac{\tan\frac{\theta}{2} + \tan\frac{\theta}{2}}{1 - \tan\frac{\theta}{2}\tan\frac{\theta}{2}} = \frac{2b}{a}\Big/1 - \frac{b^2}{a^2}$$

$$\frac{\sin\theta}{\cos\theta} = \frac{2b}{a}\cdot\frac{a^2}{a^2-b^2} = \frac{2ab}{a^2-b^2}$$

$$(a^2-b^2)\sin\theta = 2ab\cos\theta$$

$$\left(\frac{a^2}{b^2}-1\right)\sin\theta = \frac{2a}{b}\cos\theta$$

$$\frac{a^2}{b^2}\sin\theta - \frac{2a}{b}\cos\theta - \sin\theta = 0$$

$$\frac{a}{b} = \frac{2\cos\theta \pm \sqrt{4\cos^2\theta + 4\sin^2\theta}}{2\sin\theta}$$

$$= \frac{\cos\theta \pm 1}{\sin\theta}$$

(discard −ve)

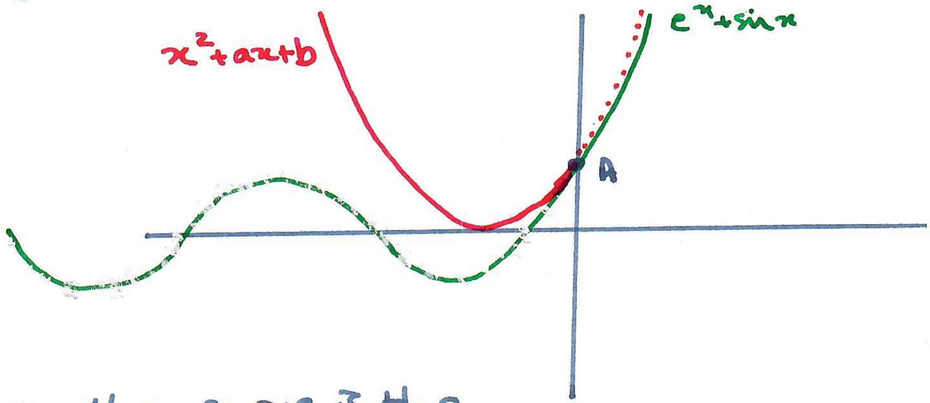
$$= \frac{1+\cos\theta}{\sin\theta}$$ as req'd.

and if $\frac{b}{a} = \frac{1}{\sqrt{3}}$, $\tan\frac{\theta}{2} = 30°$: $\underline{\theta = 60°}$

NZQA Scholarship Calculus

2020 3)a)

Another one that feels like it's been cooked up on a computer with Geogebra... and benefits a lot from a picture.



The point A (0,1) is where it all happens.

For $x > 0$ we know the curve is the green one: $e^x + \sin x$.

And we know

- the function is continuous through $x=0$ —— ①
- $f'(x)$ is continuous through $x=0$ —— ②
- $f''(x)$ exists at $x=0$ - even though the green curve 'stops' at 0^+.

① gives: $0+0+b = e^0+0 = 1$: $\underline{\underline{b=1}}$

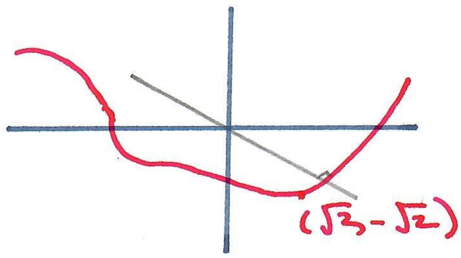
② gives: $f'(x) = 2x+a$ and $e^x+\cos x$

$0+a = e^0+1 = 1+1$: $\underline{\underline{a=2.}}$

Oddly, although $f''(0)$ exists, we don't need to use $f''(x) = 2$ or $f''(x) = 1$. Which is just as well because this is a mathematical error. If $f''(0)$ does not

NZQA Scholarship Calculus

2020 3) b).



Given that every normal passes through the origin, a circle $x^2 + y^2 = 4$ (lower half) would provide an answer...

Consider the point (h, k): the normal to the curve has gradient $\frac{k}{h}$

This means the curve's gradient is $-\frac{h}{k}$.

i.e. $\frac{dy}{dx} = \frac{-x}{y}$

$y\,dy = -x\,dx$

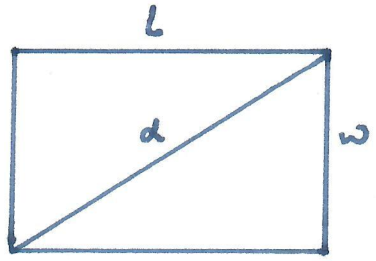
$\frac{y^2}{2} = \frac{-x^2}{2} + c$

So $x^2 + y^2 = A$: $(\sqrt{2})^2 + (\sqrt{2})^2 = 4 = A$

So the curve is indeed $\underline{\underline{x^2 + y^2 = 4.}}$

NZQA Scholarship Calculus

2020 3) c)



$\frac{dl}{dt} = 2 \qquad \frac{dw}{dt} = 3$

$d = \sqrt{l^2 + w^2}$ More usefully: $d^2 = l^2 + w^2$

So using implicit differentiation:

$$\cancel{2}d\frac{dd}{dt} = \cancel{2}l\frac{dl}{dt} + \cancel{2}w\frac{dw}{dt}$$

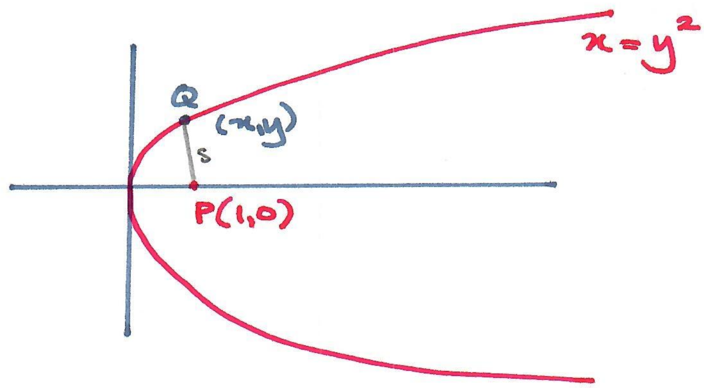
So when $l = 12$ and $w = 9$ – and so $d = 15$,

$$15\frac{dd}{dt} = 12.2 + 9.3 = 24 + 27 = 51.$$

$$\text{So } \frac{dd}{dt} = \frac{51}{15} = 3.4 \text{ cm/s}$$

NZQA Scholarship Calculus

2020 3)d)



The distance under consideration is

$$s = QP = \sqrt{(x-1)^2 + y^2}$$

$$= \sqrt{x^2 - 2x + 1 + x}$$

$$= \sqrt{x^2 - x + 1}$$

Differentiating wrt x (and resolving to use only +ve values for y)

This gives $\frac{ds}{dx} = \frac{1}{2}\left(\frac{1}{\sqrt{x^2-x+1}}\right)(2x-1)$

Given that $\sqrt{x^2-x+1}$ is always > 0 when Q is on the curve, this can only be 0 when $2x-1 = 0$ i.e. $x = \frac{1}{2}$. This clearly produces two points $\underline{\underline{\left(\frac{1}{2}, +\frac{1}{\sqrt{2}}\right)}}$ and $\underline{\underline{\left(\frac{1}{2}, -\frac{1}{\sqrt{2}}\right)}}$

But are these minima? (if one is, the other is by symmetry)

$$\frac{d^2s}{dx^2} = (\text{ugh}) \ \frac{1}{2} \cdot \frac{2\sqrt{x^2-x+1} - (2x-1) \cdot \frac{1}{2} \cdot \frac{1}{\sqrt{x^2-x+1}} (2x-1)}{x^2-x+1}$$

And this is always +ve ... so the points are indeed minima.

NZQA Scholarship Calculus
2020 4) a)

$$\frac{d}{dx}\left(f(x)g(x)\right) = \lim_{h\to 0} \frac{f(x+h)g(x+h) - f(x)g(x)}{h}$$

$$= \lim_{h\to 0} \frac{f(x+h)g(x+h) - f(x+h)g(x)}{h} + \frac{f(x+h)g(x) - f(x)g(x)}{h}$$

$$= \lim_{h\to 0} f(x+h)\left(\frac{g(x+h)-g(x)}{h}\right) + \left(\frac{f(x+h)-f(x)}{h}\right)g(x)$$

$$= f(x)\frac{dg}{dx} + \frac{df}{dx}g(x) \quad \text{as required.}$$

4) b) Using

$$\int f'g\,dx = fg - \int fg'\,dx$$

$$\int e^{-x}\cos x\,dx = (-e^{-x})\cos x + \int \sin x\,(-e^{-x})\,dx$$

$$= -e^{-x}\cos x - \int \sin x\, e^{-x}\,dx$$

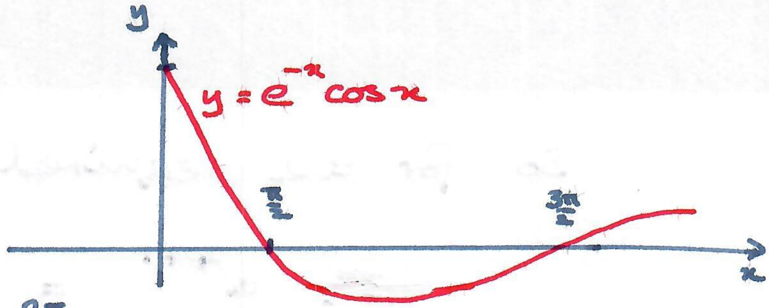
And $\int e^{-x}\sin x\,dx = (-e^{-x})\sin x - \int(-e^{-x})\cos x\,dx$

$$= -e^{-x}\sin x + \int e^{-x}\cos x\,dx$$

so $\int e^{-x}\cos x\,dx = e^{-x}\cos x + e^{-x}\sin x - \int e^{-x}\cos x\,dx$

So $\int e^{-x}\cos x\,dx = \underline{\underline{\frac{1}{2}e^{-x}(-\cos x + \sin x)}}$

ii) Using part (i),



$$\int_0^{2\pi} e^{-x}\cos x \, dx$$

$$= \frac{1}{2}\left[e^{-x}(\sin x - \cos x)\right]_0^{2\pi}$$

$$= \frac{1}{2}\left[e^{-2\pi}(\sin 2\pi - \cos 2\pi) - e^{0}(\sin 0 - \cos 0)\right]$$

$$= \frac{1}{2}\left[-e^{-2\pi} + 1\right] \qquad = \underline{\underline{\frac{1}{2}\left[1 - e^{-2\pi}\right]}}$$

Except that... the question deviously wants all the areas separately, as positives...

$$\frac{1}{2}\Big[\left|e^{-2\pi}(\sin 2\pi - \cos 2\pi) - e^{-\frac{3}{2}\pi}(\sin \tfrac{3}{2}\pi - \cos \tfrac{3}{2}\pi)\right|$$
$$+ \left|e^{-\frac{3}{2}\pi}(\sin \tfrac{3}{2}\pi - \cos \tfrac{3}{2}\pi) - e^{-\frac{\pi}{2}}(\sin \tfrac{\pi}{2} - \cos \tfrac{\pi}{2})\right|$$
$$+ \left|e^{-\frac{\pi}{2}}(\sin \tfrac{\pi}{2} - \cos \tfrac{\pi}{2}) - e^{0}(\sin 0 - \cos 0)\right|\Big]$$

$$= \frac{1}{2}\left[\left|e^{-2\pi}(-1) + e^{-\frac{3}{2}\pi}\right| + \left|-e^{-\frac{3}{2}\pi} - e^{-\frac{\pi}{2}}\right| + \left|e^{-\frac{\pi}{2}} + 1\right|\right]$$

$$= \frac{1}{2}\left(1 + e^{-\frac{\pi}{2}} + e^{-\frac{\pi}{2}} + e^{-\frac{3\pi}{2}} + e^{-\frac{3\pi}{2}} - e^{-2\pi}\right)$$

$$= \underline{\underline{\frac{1}{2}\left(1 + 2e^{-\frac{\pi}{2}} + 2e^{-\frac{3\pi}{2}} - e^{-2\pi}\right)}} \qquad \text{(a nasty twist)}$$

c) $xy + e^y = 2x+1 \longrightarrow ①$

$\frac{d}{dx}$: $xy' + y + e^y y' = 2$

$y'(x+e^y) = 2-y \longrightarrow ②$

From ① : $e^y = 2x+1-xy$

So in ②:

$y'(x+2x+1-xy) = 2-y \longrightarrow ③$

$y'(3x+1-xy) = 2-y$

$\frac{d}{dx}$: $y'(3-xy'-y) + y''(3x+1-xy) = -y'$

use $x=0$:

$y'(3-y) + y''\cdot 1 = -y'$

So $y'' = y'(-1+y-3) = y'(y-4) \longrightarrow ④$

And from ③: $y' \times (1) = 2-y$

So in ④: $y'' = (2-y)(y-4) \longrightarrow ⑤$

From ①, $0+e^y = 1$ So $y=0$.

So from ⑤ $y'' = (2-0)(0-4) = -8$

NZQA Scholarship Calculus

2020 5)a) $e^{i\theta} = \cos\theta + i\sin\theta$ ①

i) Subs $-\theta$ for θ:

$$e^{-i\theta} = \cos(-\theta) + i\sin(-\theta)$$

and use the properties that cos is an even f^n;
sin is an odd f^n:

$$e^{-i\theta} = \cos\theta - i\sin\theta \quad ②$$

①+②: $e^{i\theta} + e^{-i\theta} = 2\cos\theta + \cancel{0.i\sin\theta}$

So $\dfrac{e^{i\theta}+e^{-i\theta}}{2} = \cos\theta$ as required.

ii) In ①, subs. 3θ for θ:

$$e^{3i\theta} = \cos 3\theta + i\sin 3\theta. \text{ —— } ③$$

but $e^{3i\theta} = (e^{i\theta})^3 = (\cos\theta + i\sin\theta)^3$

$$= \cos^3\theta + 3i\cos^2\theta\sin\theta + 3i^2\cos\theta\sin^2\theta + i^3\sin^3\theta$$

$$= \cos^3\theta - 3\cos\theta\sin^2\theta + i(3\cos^2\theta\sin\theta - \sin^3\theta) \text{ —— } ④$$

Comparing imaginary parts in ③ and ④:

$$\sin 3\theta = 3\cos^2\theta\sin\theta - \sin^3\theta$$
$$= 3(1-\sin^2\theta)\sin\theta - \sin^3\theta$$
$$= 3\sin\theta - 3\sin^3\theta - \sin^3\theta$$
$$= 3\sin\theta - 4\sin^3\theta$$

Rearranging: $\sin^3\theta = \frac{3}{4}\sin\theta - \frac{1}{4}\sin 3\theta$ as required.

NZQA Scholarship Calculus

2020 5) b)

This looks like a 'real and imaginary parts' question.

$$\int e^{ix}.e^{x}\,dx = \frac{1}{i+1}e^{(i+1)x}$$

$e^{ix} = \cos x + i\sin x$, so

$$\text{LHS} = \int (\cos x + i\sin x).e^{x}\,dx$$

$$= \underbrace{\int (\cos x . e^{x})\,dx}_{\text{real}} + i\underbrace{\int (\sin x . e^{x})\,dx}_{\text{imaginary}}$$

$$\text{RHS} = \frac{1}{i+1}e^{(i+1)x}$$

$$= \frac{1(-i+1)}{(i+1)(-i+1)}e^{(i+1)x}$$

$$= \frac{1-i}{2}e^{ix}.e^{x}$$

$$= \frac{1}{2}(1-i)e^{x}(\cos x + i\sin x)$$

$$= \underbrace{\frac{1}{2}e^{x}(\cos x + \sin x)}_{\text{real}} + \underbrace{\frac{1}{2}e^{x}i(-\cos x + \sin x)}_{\text{imaginary}}$$

Equating real parts:

$$\underline{\underline{\int (\cos x . e^{x})\,dx = \frac{1}{2}e^{x}(\cos x + \sin x)}}$$

NZQA Scholarship Calculus

2020 5) c)

$|z| = 1$ so if $z = x + iy$,

$x^2 + y^2 = 1$

z is on the unit circle

Method 1 (bash it with algebra)

$$w = \frac{i+z}{i-z} = \frac{x + i(1+y)}{-x + i(1-y)}$$

$$= \frac{(x + i(1+y))(-x - i(1-y))}{(-x + i(1-y))(-x - i(1-y))}$$

$$= \frac{-x^2 + (1+y)(1-y) + i[-x(1+y) - x(1-y)]}{x^2 + (1-y)^2}$$

$$= \frac{-x^2 - y^2 + 1 - 2xi}{x^2 + y^2 + 1 - 2y} = \frac{-2xi}{2-2y}$$

$$= \left(\frac{x}{y-1}\right)i.$$

This is the y-axis and the allowable values for x and y mean it's the whole y-axis.

Method 2 (Geometric)

Rearrange $z = \frac{iw - i}{1+w}$ and since $|z| = 1$, $|iw - i| = |1+w|$

Remove i: $|w-1| = |w+1|$ so w is the perp bisector between -1 and 1 - the y-axis.

NZQA Scholarship Calculus
2020 5) d)

$$x^2 - yz = 1 \quad ①$$
$$y^2 - zx = 2 \quad ②$$
$$z^2 - xy = 3 \quad ③$$

This one is bizarre and relies on a lot of tedious 'wandering about'. In the end I looked at the published answer...

② - ①: $y^2 - x^2 - zx + yz = 1$

$$(y-x)(y+x) + z(y-x) = 1$$
$$(y-x)(x+y+z) = 1. \quad ③$$

③ - ②: $z^2 - y^2 - xy + zx = 1$

$$(z-y)(z+y) + x(z-y) = 1$$
$$(z-y)(x+y+z) = 1. \quad ④$$

Comparing ③ and ④,

$$(y-x) = (z-y) = \frac{1}{x+y+z}$$

This means x, y, z form an arithmetic progression with $d = \frac{1}{x+y+z}$.

So in ②: $y^2 - (y+d)(y-d) = 2 \quad \therefore \quad d^2 = 2 \therefore d = \pm\sqrt{2}$

Also, since x, y, z are an arithmetic progression,

$$y = \frac{x+y+z}{3} = \frac{d}{3} = \frac{\pm 1}{\sqrt{2}}$$

Bash, bash ... $x = \frac{-5}{3\sqrt{2}}, y = \frac{1}{3\sqrt{2}}, z = \frac{7}{3\sqrt{2}}$ or $x = \frac{5}{3\sqrt{2}}, y = \frac{-1}{3\sqrt{2}}, z = \frac{-7}{3\sqrt{2}}$