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93202Q

932022

NEW ZEALAND QUALIFICATIONS AUTHORITY
MANA TOHU MĀTAURANGA O AOTEAROA

QUALIFY FOR THE FUTURE WORLD
KIA NOHO TAKATŪ KI TŌ ĀMUA AO!

Scholarship 2019
Calculus

9.30 a.m. Friday 8 November 2019
Time allowed: Three hours
Total score: 40

QUESTION BOOKLET

There are five questions in this booklet. Answer ALL FIVE questions.

Write your answers in Answer Booklet 93202A.

Pull out Formulae and Tables Booklet S–CALCF from the centre of this booklet.

Show ALL working. Start your answer to each question on a new page. Carefully number each question.

Answers developed using a CAS calculator require **ALL commands to be shown**. Correct answers only will not be sufficient.

Check that this booklet has pages 2–6 in the correct order and that none of these pages is blank.

YOU MAY KEEP THIS BOOKLET AT THE END OF THE EXAMINATION.

QUESTION ONE

(a) Consider $f(x) = \dfrac{x^2 - 2x + 1}{x - 1} + \dfrac{x^2 + 2x + 1}{x + 1}$ where x is real.

State the value(s) of x for which $f(x)$ is differentiable.

(b) Consider $f(x) = \ln(-(x - 2)^2)$ where x is real.

State the value(s) of x for which $f(x)$ is real.

(c) We have 6 books to distribute to three students A, B, and C.

How many different ways are there of distributing these 6 books if:

(i) A is given 1 book, B is given 2 books, and C is given 3 books?

(ii) Each student is given 2 books?

(d) Solve: $|x + 1| - |x - 4| \geq 1$ where x is real.

(e) Given $90° < A < 180°$ and $\sin^4 A + \cos^4 A = \dfrac{2}{3}$, find the value of $\sin 2A$.

QUESTION TWO

(a) In the following equation k is a real number constant.

$$\sqrt{\frac{x-2}{x}}-\sqrt{\frac{x}{x-2}}=\frac{k}{4}$$

For what value(s) of k will the equation have imaginary roots?

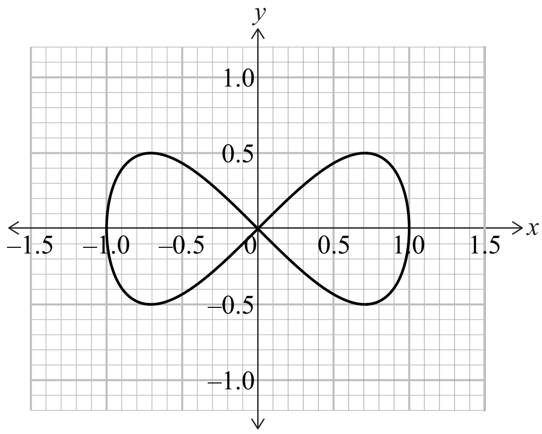
(b) Solve the system:

$$\log_{10}(x^2+y^2)=1+\log_{10}13$$
$$\log_{10}(x+y)-\log_{10}(x-y)=3\log_{10}2$$

where x and y are real.

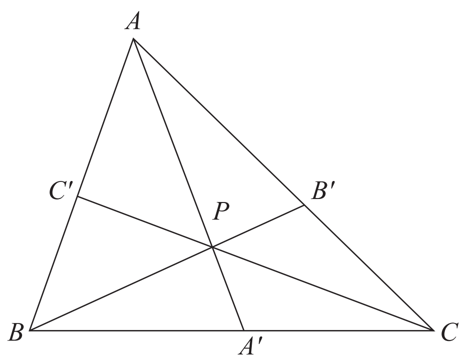
(c) Find the area enclosed by $y^2=x^2-x^4$ where x and y are both real.

The graph of this relation has been provided below.



(d) In triangle ABC, points A', B', and C' lie on sides BC, CA, and AB respectively, with AA', BB', and CC' meeting at P.

Prove that $\dfrac{PA'}{AA'}+\dfrac{PB'}{BB'}+\dfrac{PC'}{CC'}=1$.



QUESTION THREE

(a) Using the method of first principles $f'(x)$ is found by:

$$\lim_{h \to 0}\left[\frac{f(x+h)-f(x)}{h}\right]$$

When $f(x) = (x^2 - 4x + 3)^2$, find the value of $f'(4)$ using first principles.

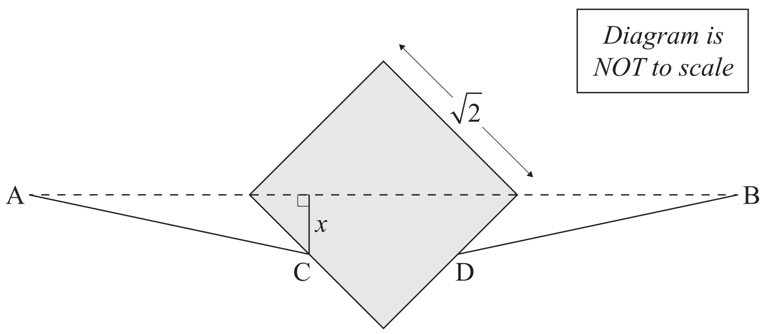
(b) A point P is moving around the circle $x^2 + y^2 = 5^2$.

When the coordinates of P are (3,4), the y-coordinate is decreasing at the rate of 2 units per second.

At what rate is the x-coordinate changing at this time?

(c) An ecologist is doing research in a national park. She must frequently travel between two huts, A and B, which are 4 km apart.

Half-way between A and B is a square lake of edge $\sqrt{2}$ km, as shown in the following diagram:



The ecologist can tramp across land at 3 km per hour, and there is a boat with a very small outboard motor that she can use to cross the lake if she wishes. The boat goes at 2.5 km per hour.

Let x be the perpendicular distance of a point C from the line AB, as shown in the diagram above.

The boat's path, CD, is parallel to AB.

Where should the boat be positioned so that her journey between the huts takes the minimum time to complete?

You must establish that your solution is indeed a minimum.

QUESTION FOUR

(a) Ewes often have twin lambs during lambing season. On a New Zealand sheep farm it has been observed that over a time interval $[0,t_p]$ the mean probability, T, that both twins have the same sex is given by

$$T = \frac{1}{t_p}\int_0^{t_p}\left\{\left[p(m)\right]^2 + \left[p(f)\right]^2\right\}\mathrm{d}t$$

where:

- $p(m)$ is the probability that one of the twins will be male and is given by

  $$p(m) = a - \frac{a-b}{t_p}t$$

- $p(f)$ is the probability of one twin being a female, and
- both a and b are constants.

Show that $T = 1 - a + b(2a-1) + \frac{2}{3}(a-b)^2$.

(b) The function $y = f(x)$ satisfies the differential equation

$$4x^2\frac{\mathrm{d}y}{\mathrm{d}x} = y^2 - 2xy$$

Find the value of $f(4)$, given that $f(1) = -6$.

You may use any valid method, however, the following techniques may prove helpful:

- A substitution for y could be made, for example $y(x) = u(x) \times x$.
- A rational expression such as $\frac{w}{x(x+p)}$, where w and p are constants,

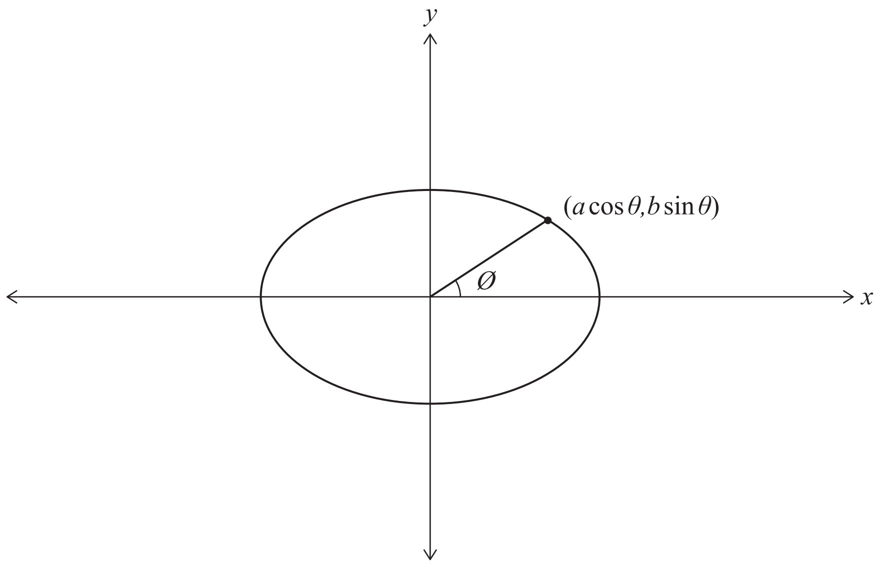
  can be written as $\frac{w}{x(x+p)} = \frac{A}{x} + \frac{B}{(x+p)}$, where A and B are also constants.

QUESTION FIVE

(a) If $w = \cos\theta + \mathrm{i}\sin\theta$, show that $\dfrac{w-1}{w+1} = \mathrm{i}\tan\dfrac{\theta}{2}$.

(b) In the parametric equation $(x,y) = (\mathrm{a}\cos\theta, b\sin\theta)$ of an ellipse ($a \neq b$), the parameter θ is **not** the same as the angle $\varnothing$ at the origin, although it is often mistaken for this.

The two angles θ and $\varnothing$ are, however, closely related. In particular, they are the same at the x and y intercepts.



(i) Write an equation relating θ and $\varnothing$.

(ii) At what value(s) of $\varnothing$ is the difference between the angles θ and $\varnothing$ the greatest?

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