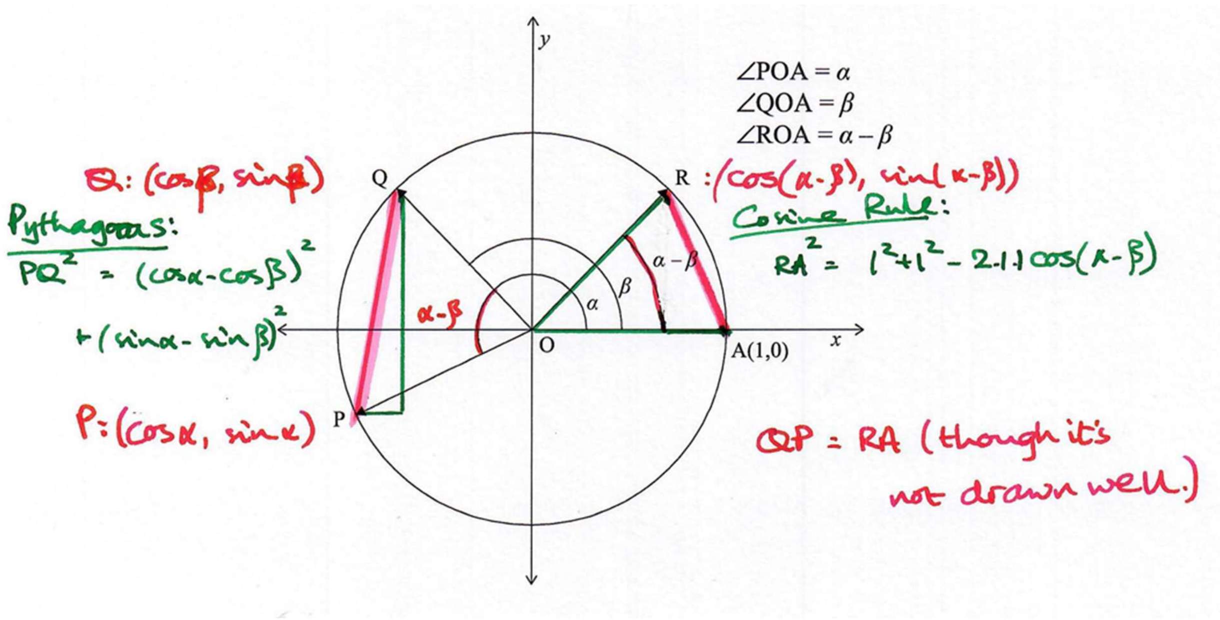
(Questions loosely done by ChatGPT but refined by me to make things properly clear – and in some cases more efficient.)

Question One

(a) Draw PQ and RA, show they're equal, and work out what they are



$$P = (\cos\alpha, \sin\alpha) \text{ and } Q = (\cos\beta, \sin\beta)$$

are points on the unit circle. Rotating both points through angle $-\beta$ takes Q to $A = (1,0)$ and takes P to R (as in the diagram), where R is

$$R = \left(\cos(\alpha-\beta), \sin(\alpha-\beta)\right)$$

A rotation preserves distance and angle between points, so $PQ = RA$

Now

$$PQ^2 = (\cos\alpha - \cos\beta)^2 + (\sin\alpha - \sin\beta)^2$$
$$= \cos^2\alpha + \sin^2\alpha + \cos^2\beta + \sin^2\beta - 2(\cos\alpha\cos\beta + \sin\alpha\sin\beta)$$
$$= 2 - 2(\cos\alpha\cos\beta + \sin\alpha\sin\beta).$$

Also, by the cosine rule

$$RA^2 = 1^2 + 1^2 - 2.1.1.\cos(\alpha-\beta)$$
$$= 2 - 2\cos(\alpha-\beta)$$

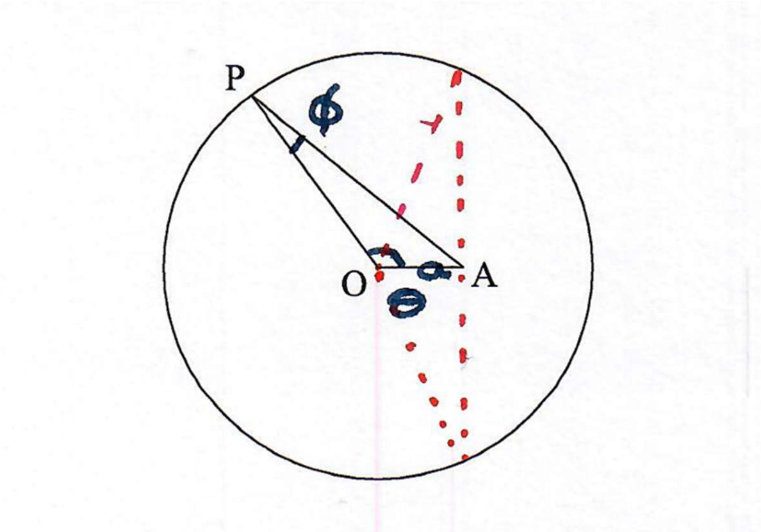
So since $PQ^2 = RA^2$

$$2 - 2(\cos\alpha\cos\beta + \sin\alpha\sin\beta) = 2 - 2\cos(\alpha-\beta)$$

So

$$\boxed{\cos(\alpha-\beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta.}$$

(b) Use the sine rule in triangle AOP



Let the radius of the circle be R, and let $OA = a$, where $0 < a < R$. Write

$$\theta = \angle AOP \qquad \phi = \angle APO$$

By the sine rule in triangle AOP,

$$\frac{\sin\phi}{a} = \frac{\sin\angle OAP}{R}.$$

Hence

$$\sin\phi = \frac{a}{R}\sin\angle OAP.$$

Since $\frac{a}{R}$ is fixed, ϕ is maximised when $\sin\angle OAP$ is maximised. Its maximum value is 1, occurring when

$$\angle OAP = 90°.$$

Thus AP must be perpendicular to AO

Equivalently, draw the line through A perpendicular to OA; its two intersections with the circumference give the two possible positions of P.

$$\boxed{P \text{ is located so that } AP \perp AO}$$

(c) Multiply the LHS out and find what cancels

Begin with the left-hand side:

$$\frac{\cos\theta}{1+\sin\theta}-\frac{\sin\theta}{1+\cos\theta}=\frac{\cos\theta(1+\cos\theta)-\sin\theta(1+\sin\theta)}{(1+\sin\theta)(1+\cos\theta)}$$

$$\frac{\cos\theta-\sin\theta+\cos^2\theta-\sin^2\theta}{(1+\sin\theta)(1+\cos\theta)}$$

$$\frac{(\cos\theta-\sin\theta)(1+\sin\theta+\cos\theta)}{(1+\sin\theta)(1+\cos\theta)}$$

Also

$$(1+\sin\theta)(1+\cos\theta)=1+\sin\theta+\cos\theta+\sin\theta\cos\theta$$

$$\frac{1}{2}(1+\sin\theta+\cos\theta)^2=\frac{1}{2}(1+\sin^2\theta+\cos^2\theta+2\sin\theta+2\cos\theta+2\sin\theta\cos\theta)$$

$$1+\sin\theta+\cos\theta+\sin\theta\cos\theta$$

Therefore

$$(1+\sin\theta)(1+\cos\theta)=\frac{1}{2}(1+\sin\theta+\cos\theta)^2$$

Substituting this into the earlier expression gives

$$\frac{\cos\theta}{1+\sin\theta}-\frac{\sin\theta}{1+\cos\theta}=\boxed{\frac{2(\cos\theta-\sin\theta)}{1+\sin\theta+\cos\theta}}$$

Question Two

(a) (i) Factorise the denominators to find where the function goes to ∞

The function is

$$f(x) = \frac{2x^2 - 1}{(3x+2)(5x-3)} - \frac{2-3x}{x^2-5x+3}$$

It's discontinuous wherever any denominator is zero

From

$$(3x+2)(5x-3) = 0$$

we get

$$x = -\frac{2}{3} \text{ and } x = \frac{3}{5}$$

Also from

$$x^2 - 5x + 3 = 0$$

we get

$$x = \frac{5 \pm \sqrt{25-12}}{2} = \frac{5 \pm \sqrt{13}}{2}$$

So the discontinuities occur at

$$\boxed{x = -\frac{2}{3}, \frac{3}{5}, \frac{5-\sqrt{13}}{2}, \frac{5+\sqrt{13}}{2}}$$

(a)(ii) $x = 1/2$ isn't a discontinuity, so just do straight substitution

None of the denominators is zero at $x = 1/2$, so direct substitution may be used

$$\lim_{x \to 1/2} f(x) = \frac{2(1/2)^2 - 1}{(3/2+2)(5/2-3)} - \frac{2-3/2}{(1/2)^2 - 5/2 + 3}$$

$$\frac{-1/2}{(7/2)(-1/2)} - \frac{1/2}{3/4}$$

$$\frac{2}{7} - \frac{2}{3}$$

$$\boxed{-\frac{8}{21}}.$$

(b) Differentiate and show the reciprocals multiply to -1

For the curve $xy = 2$, implicit differentiation gives

$$x\frac{dy}{dx} + y = 0$$

so

$$m_1 = \frac{dy}{dx} = -\frac{y}{x}$$

For the curve $x^2 - y^2 = 1$

$$2x - 2y\frac{dy}{dx} = 0$$

so

$$m_2 = \frac{dy}{dx} = \frac{x}{y}$$

Therefore

$$m_1 m_2 = \left(-\frac{y}{x}\right)\left(\frac{x}{y}\right) = -1$$

and at any point of intersection, neither x nor y is zero because $xy = 2$

So the tangents are perpendicular at every point of intersection (and we don't actually need to find the points of intersection.)

$$\boxed{m_1 m_2 = -1}$$

(c) Do the basic geometry, then differentiate. (The hardest thing is bothering to read the question)

Let M be the midpoint of BC. Since $AB = AC = 50$ km and $BC = 60$ km

$$BM = CM = 30 \text{ km}$$

By Pythagoras,

$$AM = \sqrt{50^2 - 30^2} = 40 \text{ km}$$

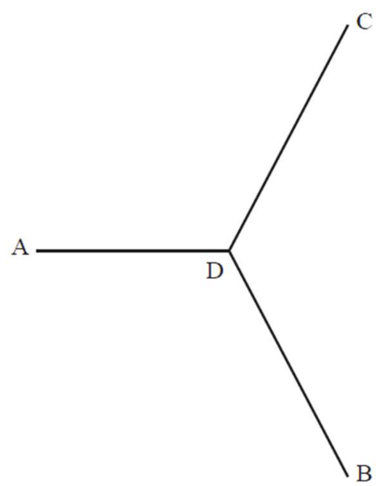
By symmetry, the optimal point D lies on AM. Let

$$AD = x$$

Then

$$DM = 40 - x$$

and



$$DB = DC = \sqrt{(40 - x)^2 + 30^2}$$

The total road length is

$$L(x) = x + 2\sqrt{(40-x)^2 + 900}, \qquad 0 \le x \le 40$$

Differentiate

$$L'^{(x)} = 1 - \frac{2(40-x)}{\sqrt{(40-x)^2 + 900}}$$

At a stationary point

$$1 = \frac{2(40-x)}{\sqrt{(40-x)^2 + 900}}$$

Squaring

$$(40-x)^2 + 900 = 4(40-x)^2$$

so

$$3(40-x)^2 = 900$$

Since $40 - x \ge 0$

$$40 - x = 10\sqrt{3}$$

and therefore

$$\boxed{x = 40 - 10\sqrt{3}}$$

Thus D is $40 - 10\sqrt{3}$ km from A towards BC, or $10\sqrt{3}$ km from the midpoint of BC

At this position

$$DB = \sqrt{\left(10\sqrt{3}\right)^2 + 30^2} = 20\sqrt{3}$$

Hence

$$L_{\min} = \left(40 - 10\sqrt{3}\right) + 2\left(20\sqrt{3}\right)$$

$$\boxed{40 + 30\sqrt{3}\text{ km}}$$

Finally,

$$L''^{(x)} = \frac{1800}{((40-x)^2 + 900)^{3/2}} > 0$$

so the stationary point is indeed a minimum.

Question Three

(a)(i) Use the obvious substitution to make the √ easier

Evaluate

$$I = \int_0^5 (2x-5)\sqrt{2x+5}\,dx$$

Use the substitution

$$u = 2x+5 \qquad du = 2\,dx \qquad dx = \frac{1}{2}\,du$$

Also

$$2x-5 = u-10$$

When $x=0$, $u=5$; when $x=5$, $u=15$. Therefore

$$I = \frac{1}{2}\int_5^{15} (u-10)\,u^{1/2}\,du$$

$$\frac{1}{2}\int_5^{15} \left(u^{3/2} - 10u^{1/2}\right) du$$

$$\frac{1}{2}\left[\frac{2}{5}u^{5/2} - \frac{20}{3}u^{3/2}\right]_5^{15}$$

$$\left[\frac{1}{5}u^{5/2} - \frac{10}{3}u^{3/2}\right]_5^{15}$$

$$-5\sqrt{15} + \frac{35}{3}\sqrt{5}$$

So

$$\boxed{I = \frac{35\sqrt{5}}{3} - 5\sqrt{15}}$$

(a)(ii) Check for oddities in the set-up

The second integral is

$$\int_0^5 (2x+5)\sqrt{2x-5}\,dx$$

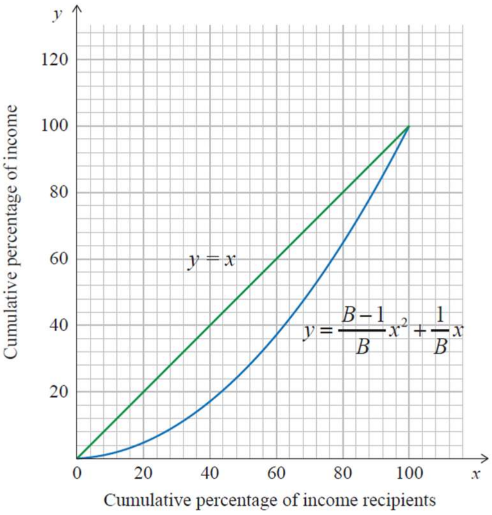
For $0 \le x < 5/2$, we have $2x-5<0$, so $\sqrt{2x-5}$ is not real. Therefore the integrand is not a real-valued function throughout the stated interval.

The real substitution method used in part (i) can't be applied to evaluate this as an ordinary real definite integral over [0,5].

(b) Change to proportions from %s, and bash through it (it's easier than it looks)

It's simplest to use proportions rather than percentages, so take $0 \le x \le 1$. The Lorenz curve is

$$y = \frac{B-1}{B}x^2 + \frac{1}{B}x$$



The equality diagonal is $y = x$. The vertical difference between the diagonal and the Lorenz curve is

$$x - y = x - \left(\frac{B-1}{B}x^2 + \frac{1}{B}x\right)$$
$$= \frac{B-1}{B}x(1-x)$$

The area between the curves is therefore

$$A = \int_0^1 \frac{B-1}{B}x(1-x)\,dx$$
$$\frac{B-1}{B}\left[\frac{x^2}{2} - \frac{x^3}{3}\right]_0^1$$
$$\frac{B-1}{6B}$$

The area beneath the equality diagonal is

$$\int_0^1 x\ dx = \frac{1}{2}$$

So the coefficient of inequality is

$$\frac{A}{1/2} = \frac{B-1}{3B}$$

We know this is $\frac{20}{63}$, so

$$\frac{B-1}{3B} = \frac{20}{63}$$

So

$$63(B-1) = 60B$$

so

$$3B = 63$$

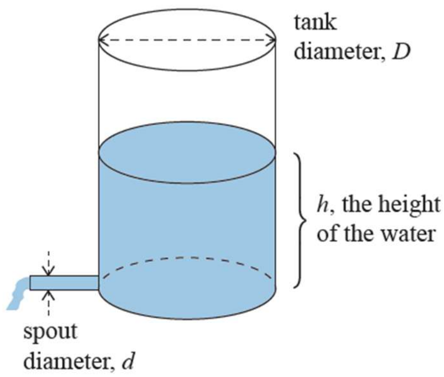
and therefore

$$\boxed{B = 21}$$

(c) A simple problem formulated oddly to make it come out tidily: just do it

The cross-sectional area of the cylindrical tank is

$$A_T = \pi \left(\frac{D}{2}\right)^2 = \frac{\pi D^2}{4}$$



The cross-sectional area of the spout is

$$A_S = \pi \left(\frac{d}{2}\right)^2 = \frac{\pi d^2}{4}$$

The volume flow rate out of the tank is

$$A_S v = A_S\sqrt{2gh}$$

Since the tank volume is $A_T h$

$$A_T \frac{dh}{dt} = -A_S\sqrt{2gh}$$

So

$$\frac{dh}{dt} = -\frac{d^2}{D^2}\sqrt{2gh}$$

Rearranging

$$dt = -\left(\frac{D}{d}\right)^2 \frac{dh}{\sqrt{2gh}}.$$

If the water level falls from h_1 to h_2, the elapsed time is

$$T = -\left(\frac{D}{d}\right)^2 \frac{1}{\sqrt{2g}} \int_{h_1}^{h_2} h^{-1/2}\ dh$$

$$-\left(\frac{D}{d}\right)^2 \frac{1}{\sqrt{2g}} \left[2\sqrt{h}\right]_{h_1}^{h_2}$$

$$\left(\frac{D}{d}\right)^2 \sqrt{\frac{2}{g}}\left(\sqrt{h_1} - \sqrt{h_2}\right)$$

Hence

$$\boxed{T = \left(\frac{D}{d}\right)^2 \sqrt{\frac{2}{g}}\left(\sqrt{h_1} - \sqrt{h_2}\right)}$$

Question Four

(a) Just undo the logarithms – but always check for validity

The logarithms require

$$x > 0$$

and

$$4 - 2x > 0$$

so the domain is

$$0 < x < 2$$

Within this domain,

$$\log_3(4 - 2x) + \log_3 x = \log_3\big(x(4 - 2x)\big)$$

Since the base 3 is greater than 1

$$\log_3\big(x(4 - 2x)\big) \le 2$$

is equivalent to

$$x(4 - 2x) \le 9$$

But

$$x(4 - 2x) = 4x - 2x^2 = 2 - 2(x - 1)^2 \le 2 < 9$$

So the inequality is true for every x in the domain

$$\boxed{0 < x < 2}$$

(b) Find the circle basics then bash out the geometry – a simple (5,12,13) triangle

Complete the square for the first circle (or just do it by inspection of the x, y coefficients)

$$x^2 + y^2 - 16x - 20y + 115 = 0$$

so

$$(x - 8)^2 + (y - 10)^2 = 49$$

Its centre and radius are

$$C_1 = (8,10) \qquad r_1 = 7$$

For the second circle

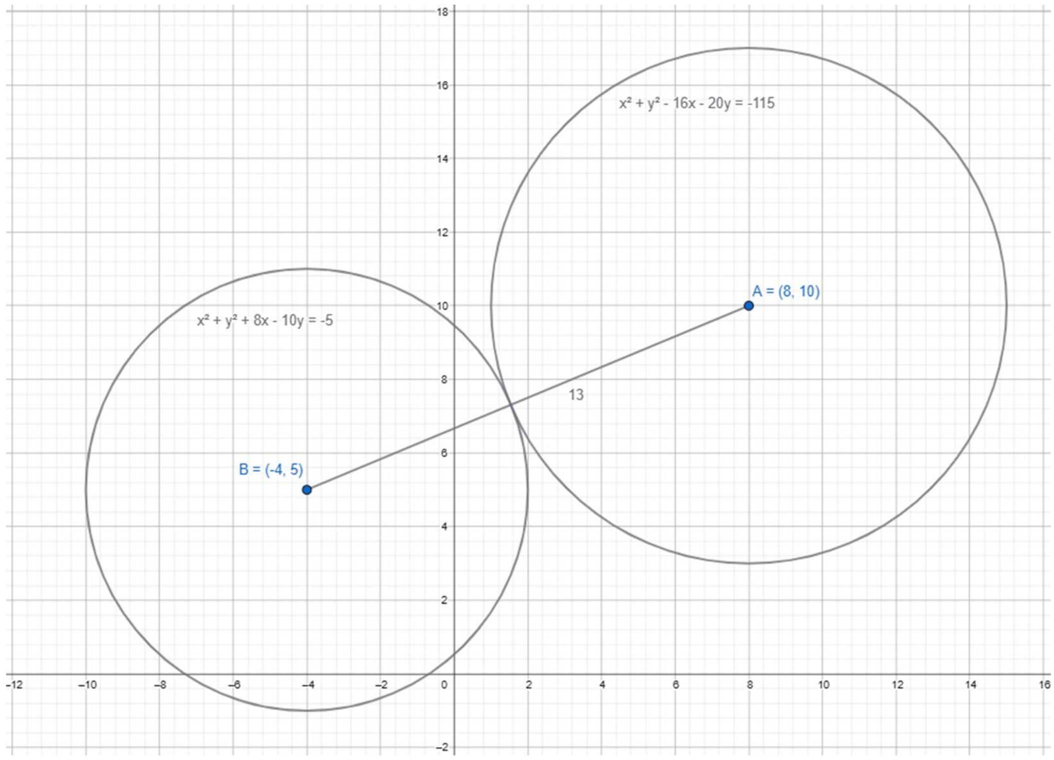
$$x^2 + y^2 + 8x - 10y + 5 = 0$$

so

$$(x + 4)^2 + (y - 5)^2 = 36$$

Its centre and radius are

$$C_2 = (-4,5) \qquad r_2 = 6$$



The distance between the centres is

$$C_1C_2 = \sqrt{(8+4)^2 + (10-5)^2} = \sqrt{12^2 + 5^2} = 13$$

Since

$$C_1C_2 = r_1 + r_2 = 7 + 6,$$

the circles are externally tangent

The point of tangency lies on the line joining the centres, at distance 7 from C_1. So

$$T = C_1 + \frac{7}{13}(C_2 - C_1)$$

$$(8,10) + \frac{7}{13}(-12, -5)$$

$$\left(\frac{20}{13}, \frac{95}{13}\right)$$

Therefore the circles are tangent at

$$\boxed{\left(\frac{20}{13}, \frac{95}{13}\right)}$$

(c) Notice the equations fit together nicely, then just bash it out

We've got

$$x^2 + y^2 + 3x + 3y = 12 \qquad (1)$$

$$xy + 4x + 4y = 2 \qquad (2)$$

(1) + 2 × (2) gives

$$x^2 + 2xy + \ y^2 + 11x + 11y = 12$$

$$(x + y)^2 + 11(x + y) = 12$$

$$(x + y)^2 + 11(x + y) - 12 = 0$$

$$x + y = 1 \quad \text{or} \quad x + y = -12$$

If $x + y \ = 1$, then $y = 1 - x$, so in (2)

$$x - x^2 + 4x + 4 - 4x - 2 = 0$$

$$-x^2 + x + 2 = 0$$

$$x^2 - x - 2 = 0$$

$$(x + 1)(x - 2) = 0$$

$$x = -1 \ \text{ or } \ x = 2$$

And since the equations are symmetric in x and y

$$y = 2 \ \text{ or } \ y = -1$$

If $x + y \ = -12$, then similarly in (2)

$$-12x - x^2 + 4x - 48 - 4x - 2 = 0$$

$$-x^2 - 12x - 50 = 0$$

$$x^2 + 12x + 50 = 0$$

$$\frac{-12 \pm \sqrt{144 - 200}}{2}$$

$$x = -6 + i\sqrt{14} \ \text{ or } \ x == -6 + i\sqrt{14}$$

And since the equations are symmetric in x and y

$$y = -6 - i\sqrt{14} \ \text{ or } \ x = -6 + i\sqrt{14}$$

So the complete set of solutions is

$$\boxed{\begin{array}{c} (2, -1) \\ (-1, 2) \\ \left(-6 + i\sqrt{14}, -6 - i\sqrt{14}\right) \\ \left(-6 - i\sqrt{14}, -6 + i\sqrt{14}\right) \end{array}}$$

Question Five

(a) Basically just bash it – though there's a trick which helps

$$z = x + iy$$
$$z^{-1} = (a + ib)^{-1} + (a + ic)^{-1}$$

where a, b and c are real and non-zero. We require the real part of z, namely x

Write the second expression in a more friendly form

$$\frac{1}{z} = \frac{1}{a + ib} + \frac{1}{a + ic}$$

We could clear out the complex denominators but a cleaner trick is to combine the two reciprocals before rationalizing and leave everything complex

$$\frac{1}{z} = \frac{(a + ib)(a + ic)}{(a + ib)(a + ic)}$$
$$= \frac{2a + i(b + c)}{a^2 - bc + ia(b + c)}$$

And this is 1/z, not z – so invert the whole expression

$$z = \frac{a^2 - bc + ia(b + c)}{2a + i(b + c)}$$

Rationalise the denominator by multiplying numerator and denominator by $2a - i(b + c)$

$$z = \frac{\left(a^2 - bc + ia(b + c)\right)\left(2a - i(b + c)\right)}{4a^2 + (b + c)^2}$$

Happily we only want the real part of the numerator. Expanding:

$$\left(a^2 - bc + ia(b + c)\right)\left(2a - i(b + c)\right)$$
$$= 2a(a^2 - bc) - i(a^2 - bc)(b + c) + 2ia^2(b + c) + a(b + c)^2$$

So the real part of the numerator is

$$2a(a^2 - bc) + a(b + c)^2$$
$$= 2a^3 - 2abc + ab^2 + 2abc + ac^2$$
$$= a(2a^2 + b^2 + c^2)$$

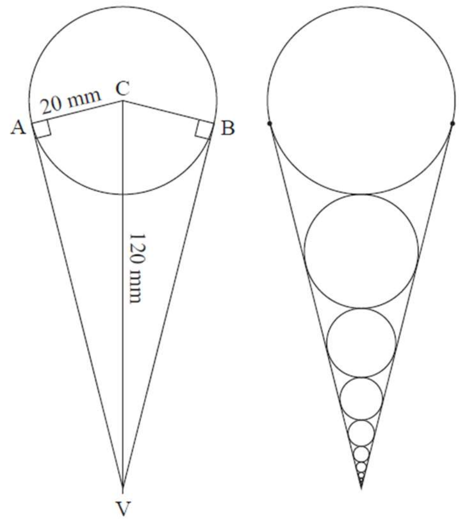
So

$$x = Re(z) = \frac{a(2a^2 + b^2 + c^2)}{4a^2 + (b + c)^2}$$

Combining the two original reciprocals first is substantially cleaner than rationalising each one separately.

(b)(i) Some very basic geometry

Let the semi-vertical angle of the cone be α. For any sphere touching both sides of the cone, its radius is the perpendicular distance from its centre to either side. If its centre is a distance s from the vertex along the axis, then



$$r = s\ \sin\alpha$$

For the first ball-bearing

$$20 = 120\ \sin\alpha$$

so

$$\sin\alpha = \frac{1}{6}$$

Let the first and second centres be distances s_1 and s_2 from the vertex. Since the balls touch one another

$$s_1 - s_2 = r_1 + r_2$$

But

$$r_1 = s_1 \sin\alpha \qquad r_2 = s_2 \sin\alpha$$

So

$$s_1 - s_2 = (s_1 + s_2)\sin\alpha$$

Using $\sin\alpha = \frac{1}{6}$

$$6s_1 - 6s_2 = s_1 + s_2$$

So

$$5s_1 = 7s_2$$

So

$$\frac{s_2}{s_1} = \frac{5}{7}$$

Since radius is proportional to distance from the vertex

$$\frac{r_2}{r_1} = \frac{5}{7}$$

So

$$\boxed{r_2 = 20\left(\frac{5}{7}\right) = \frac{100}{7}\ \text{mm}}$$

(b)(ii) Notice a geometric sequence

The radii form a geometric sequence with first term 20 and common ratio $\frac{5}{7}$. Therefore

$$\boxed{r_n = 20\left(\frac{5}{7}\right)^{n-1} \text{ mm}}$$

(b)(iii) Follow the geometric sequence result through

The volume of the nth ball-bearing is

$$V_n = \frac{4}{3}\pi r_n^3$$

So

$$V_n = \frac{4}{3}\pi\left[20\left(\frac{5}{7}\right)^{n-1}\right]^3$$
$$\frac{32000\pi}{3}\left(\frac{125}{343}\right)^{n-1}$$

So the volumes form an infinite geometric series with first term $\frac{32000}{3}$ and common ratio $\frac{125}{343}$

The total volume is

$$V = \frac{32000\pi/3}{1 - 125/343}$$
$$\frac{32000\pi}{3} \cdot \frac{343}{218}$$
$$\frac{5\,488\,000\pi}{327} \text{ mm}^3$$

So

$$\boxed{V_{\max} = \frac{5.488 \times 10^6\pi}{327} \text{ mm}^3}$$