NZ NCEA Scholarship Calculus 2018

Worked Solutions

These solutions give full working for the 2018 Scholarship Calculus examination. They are intended as teaching solutions rather than a marking schedule.

Question One

(a)

Let

$$P = (\cos\alpha, \sin\alpha), \qquad Q = (\cos\beta, \sin\beta)$$

be points on the unit circle. Rotating both points through angle $-\beta$ takes Q to $A = (1,0)$ and takes P to

$$R = \big(\cos(\alpha-\beta), \sin(\alpha-\beta)\big).$$

A rotation preserves distance, so $PQ = RA$.

Now

$$\begin{aligned} PQ^2 &= (\cos\alpha - \cos\beta)^2 + (\sin\alpha - \sin\beta)^2 \\ &\cos^2\alpha + \sin^2\alpha + \cos^2\beta + \sin^2\beta \\ &-2(\cos\alpha\cos\beta + \sin\alpha\sin\beta) \\ &2 - 2(\cos\alpha\cos\beta + \sin\alpha\sin\beta). \end{aligned}$$

Also,

$$\begin{aligned} RA^2 &= (\cos(\alpha-\beta) - 1)^2 + \sin^2(\alpha-\beta) \\ &2 - 2\cos(\alpha-\beta). \end{aligned}$$

Since $PQ^2 = RA^2$,

$$2 - 2(\cos\alpha\cos\beta + \sin\alpha\sin\beta) = 2 - 2\cos(\alpha-\beta).$$

Therefore,

$$\boxed{\cos(\alpha-\beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta.}$$

(b)

Let the radius of the circle be R, and let $OA = a$, where $0 < a < R$. Write

$$\theta = \angle AOP, \qquad \phi = \angle APO.$$

By the sine rule in triangle AOP,

$$\frac{\sin\phi}{a} = \frac{\sin\angle OAP}{R}.$$

Hence

$$\sin\phi = \frac{a}{R}\sin\angle OAP.$$

Since a/R is fixed, ϕ is maximised when $\sin\angle OAP$ is maximised. Its maximum value is 1, occurring when

$$\angle OAP = 90°.$$

Thus AP must be perpendicular to AO.

Equivalently, draw the line through A perpendicular to OA; its two intersections with the circumference give the two possible positions of P.

$$\boxed{P \text{ is located so that } AP \perp AO.}$$

(c)

Begin with the left-hand side:

$$\frac{\cos\theta}{1+\sin\theta} - \frac{\sin\theta}{1+\cos\theta} = \frac{\cos\theta(1+\cos\theta) - \sin\theta(1+\sin\theta)}{(1+\sin\theta)(1+\cos\theta)}$$

$$\frac{\cos\theta - \sin\theta + \cos^2\theta - \sin^2\theta}{(1+\sin\theta)(1+\cos\theta)}$$

$$\frac{(\cos\theta - \sin\theta)(1+\sin\theta+\cos\theta)}{(1+\sin\theta)(1+\cos\theta)}.$$

Also,

$$(1+\sin\theta)(1+\cos\theta) = 1 + \sin\theta + \cos\theta + \sin\theta\cos\theta,$$

$$\frac{1}{2}(1+\sin\theta+\cos\theta)^2 = \frac{1}{2}(1+\sin^2\theta+\cos^2\theta+2\sin\theta+2\cos\theta+2\sin\theta\cos\theta)$$

$$1 + \sin\theta + \cos\theta + \sin\theta\cos\theta.$$

Therefore,

$$(1+\sin\theta)(1+\cos\theta) = \frac{1}{2}(1+\sin\theta+\cos\theta)^2.$$

Substituting this into the earlier expression gives

$$\frac{\cos\theta}{1+\sin\theta} - \frac{\sin\theta}{1+\cos\theta} = \boxed{\frac{2(\cos\theta - \sin\theta)}{1+\sin\theta+\cos\theta}}.$$

Question Two

(a)(i)

The function is

$$f(x) = \frac{2x^2 - 1}{(3x+2)(5x-3)} - \frac{2-3x}{x^2-5x+3}.$$

It is discontinuous wherever any denominator is zero.

From

$$(3x+2)(5x-3) = 0,$$

we obtain

$$x = -\frac{2}{3}, \qquad x = \frac{3}{5}.$$

Also,

$$x^2 - 5x + 3 = 0$$

gives

$$x = \frac{5 \pm \sqrt{25 - 12}}{2} = \frac{5 \pm \sqrt{13}}{2}.$$

Thus the discontinuities occur at

$$\boxed{x = -\frac{2}{3}, \frac{3}{5}, \frac{5 - \sqrt{13}}{2}, \frac{5 + \sqrt{13}}{2}.}$$

(a)(ii)

None of the denominators is zero at $x = 1/2$, so direct substitution may be used:

$$\lim_{x \to 1/2} f(x) = \frac{2(1/2)^2 - 1}{(3/2 + 2)(5/2 - 3)} - \frac{2 - 3/2}{(1/2)^2 - 5/2 + 3}$$

$$\frac{-1/2}{(7/2)(-1/2)} - \frac{1/2}{3/4}$$

$$\frac{2}{7} - \frac{2}{3}$$

$$\boxed{-\frac{8}{21}}.$$

(b)

For the curve $xy = 2$, implicit differentiation gives

$$x\frac{dy}{dx} + y = 0,$$

so

$$m_1 = \frac{dy}{dx} = -\frac{y}{x}.$$

For the curve $x^2 - y^2 = 1$,

$$2x - 2y\frac{dy}{dx} = 0,$$

so

$$m_2 = \frac{dy}{dx} = \frac{x}{y}.$$

At any point of intersection, neither x nor y is zero because $xy = 2$. Therefore,

$$m_1 m_2 = \left(-\frac{y}{x}\right)\left(\frac{x}{y}\right) = -1.$$

Hence the tangents are perpendicular at every point of intersection.

$$\boxed{m_1 m_2 = -1.}$$

(c)

Let M be the midpoint of BC. Since $AB = AC = 50$ km and $BC = 60$ km,

$$BM = CM = 30 \text{ km}.$$

By Pythagoras,

$$AM = \sqrt{50^2 - 30^2} = 40 \text{ km}.$$

By symmetry, the optimal point D lies on AM. Let

$$AD = x.$$

Then

$$DM = 40 - x,$$

and

$$DB = DC = \sqrt{(40 - x)^2 + 30^2}.$$

The total road length is

$$L(x) = x + 2\sqrt{(40 - x)^2 + 900}, \qquad 0 \le x \le 40.$$

Differentiate:

$$L'(x) = 1 - \frac{2(40 - x)}{\sqrt{(40 - x)^2 + 900}}.$$

At a stationary point,

$$1 = \frac{2(40 - x)}{\sqrt{(40 - x)^2 + 900}}.$$

Squaring,

$$(40 - x)^2 + 900 = 4(40 - x)^2,$$

so

$$3(40 - x)^2 = 900.$$

Since $40 - x \ge 0$,

$$40 - x = 10\sqrt{3},$$

and therefore

$$\boxed{x = 40 - 10\sqrt{3}.}$$

Thus D is $40 - 10\sqrt{3}$ km from A towards BC, or $10\sqrt{3}$ km from the midpoint of BC.

At this position,

$$DB = \sqrt{\left(10\sqrt{3}\right)^2 + 30^2} = 20\sqrt{3}.$$

Hence

$$L_{\min} = \left(40 - 10\sqrt{3}\right) + 2\left(20\sqrt{3}\right)$$

$$\boxed{40 + 30\sqrt{3}\text{ km}}.$$

Finally,

$$L''(x) = \frac{1800}{((40-x)^2 + 900)^{3/2}} > 0,$$

so the stationary point is indeed a minimum.

Question Three

(a)(i)

Evaluate

$$I = \int_0^5 (2x-5)\sqrt{2x+5}\,dx.$$

Use the substitution

$$u = 2x + 5, \qquad du = 2\,dx, \qquad dx = \frac{1}{2}\,du.$$

Also,

$$2x - 5 = u - 10.$$

When $x = 0$, $u = 5$; when $x = 5$, $u = 15$. Therefore,

$$I = \frac{1}{2}\int_5^{15} (u-10)\,u^{1/2}\,du$$

$$\frac{1}{2}\int_5^{15} \left(u^{3/2} - 10u^{1/2}\right) du$$

$$\frac{1}{2}\left[\frac{2}{5}u^{5/2} - \frac{20}{3}u^{3/2}\right]_5^{15}$$

$$\left[\frac{1}{5}u^{5/2} - \frac{10}{3}u^{3/2}\right]_5^{15}$$

$$-5\sqrt{15} + \frac{35}{3}\sqrt{5}.$$

Thus

$$\boxed{I = \frac{35\sqrt{5}}{3} - 5\sqrt{15}.}$$

(a)(ii)

The second integral is

$$\int_0^5 (2x+5)\sqrt{2x-5}\,dx.$$

For $0 \le x < 5/2$, we have $2x - 5 < 0$, so $\sqrt{2x-5}$ is not real. Therefore the integrand is not a real-valued function throughout the stated interval.

The real substitution method used in part (i) cannot be applied to evaluate this as an ordinary real definite integral over $[0,5]$.

(b)

It is simplest to use proportions rather than percentages, so take $0 \le x \le 1$. The Lorenz curve is

$$y = \frac{B-1}{B}x^2 + \frac{1}{B}x.$$

The equality diagonal is $y = x$. The vertical difference between the diagonal and the Lorenz curve is

$$x - y = x - \left(\frac{B-1}{B}x^2 + \frac{1}{B}x\right)$$

$$\frac{B-1}{B}x(1-x).$$

The area between the curves is therefore

$$A = \int_0^1 \frac{B-1}{B}x(1-x)\,dx$$

$$\frac{B-1}{B}\left[\frac{x^2}{2} - \frac{x^3}{3}\right]_0^1$$

$$\frac{B-1}{6B}.$$

The area beneath the equality diagonal is

$$\int_0^1 x\ dx = \frac{1}{2}.$$

Hence the coefficient of inequality is

$$\frac{A}{1/2} = \frac{B-1}{3B}.$$

We are told that this is $20/63$, so

$$\frac{B-1}{3B} = \frac{20}{63}.$$

Thus

$$63(B-1) = 60B,$$

so

$$3B = 63$$

and therefore

$$\boxed{B = 21.}$$

(c)

The cross-sectional area of the cylindrical tank is

$$A_T = \pi \left(\frac{D}{2}\right)^2 = \frac{\pi D^2}{4}.$$

The cross-sectional area of the spout is

$$A_S = \pi \left(\frac{d}{2}\right)^2 = \frac{\pi d^2}{4}.$$

The volume flow rate out of the tank is

$$A_S v = A_S \sqrt{2gh}.$$

Since the tank volume is $A_T h$,

$$A_T \frac{dh}{dt} = -A_S \sqrt{2gh}.$$

Therefore,

$$\frac{dh}{dt} = -\frac{d^2}{D^2} \sqrt{2gh}.$$

Rearranging,

$$dt = -\left(\frac{D}{d}\right)^2 \frac{dh}{\sqrt{2gh}}.$$

If the water level falls from h_1 to h_2, the elapsed time is

$$T = -\left(\frac{D}{d}\right)^2 \frac{1}{\sqrt{2g}} \int_{h_1}^{h_2} h^{-1/2}\, dh$$

$$-\left(\frac{D}{d}\right)^2 \frac{1}{\sqrt{2g}} \left[2\sqrt{h}\right]_{h_1}^{h_2}$$

$$\left(\frac{D}{d}\right)^2 \sqrt{\frac{2}{g}} \left(\sqrt{h_1} - \sqrt{h_2}\right).$$

Hence

$$\boxed{T = \left(\frac{D}{d}\right)^2 \sqrt{\frac{2}{g}} \left(\sqrt{h_1} - \sqrt{h_2}\right).}$$

Question Four

(a)

The logarithms require

$$x > 0$$

and

$$4 - 2x > 0,$$

so the domain is

$$0 < x < 2.$$

Within this domain,

$$\log_3(4 - 2x) + \log_3 x = \log_3\big(x(4 - 2x)\big).$$

Since the base 3 is greater than 1,

$$\log_3\big(x(4 - 2x)\big) \leq 2$$

is equivalent to

$$x(4 - 2x) \leq 9.$$

But

$$x(4 - 2x) = 4x - 2x^2 = 2 - 2(x - 1)^2 \leq 2 < 9.$$

Thus the inequality is true for every x in the domain.

$$\boxed{0 < x < 2.}$$

(b)

Complete the square for the first circle:

$$x^2 + y^2 - 16x - 20y + 115 = 0,$$

so

$$(x - 8)^2 + (y - 10)^2 = 49.$$

Its centre and radius are

$$C_1 = (8,10), \qquad r_1 = 7.$$

For the second circle,

$$x^2 + y^2 + 8x - 10y + 5 = 0,$$

so

$$(x + 4)^2 + (y - 5)^2 = 36.$$

Its centre and radius are

$$C_2 = (-4,5), \qquad r_2 = 6.$$

The distance between the centres is

$$C_1C_2 = \sqrt{(8+4)^2 + (10-5)^2} = \sqrt{12^2 + 5^2} = 13.$$

Since

$$C_1C_2 = r_1 + r_2 = 7 + 6,$$

the circles are externally tangent.

The point of tangency lies on the line joining the centres, at distance 7 from C_1. Thus

$$T = C_1 + \frac{7}{13}(C_2 - C_1)$$

$$(8,10) + \frac{7}{13}(-12,-5)$$

$$\left(\frac{20}{13}, \frac{95}{13}\right).$$

Therefore the circles are tangent at

$$\boxed{\left(\frac{20}{13}, \frac{95}{13}\right).}$$

(c)

Let

$$s = x + y, \qquad p = xy.$$

Since

$$x^2 + y^2 = s^2 - 2p,$$

the first equation becomes

$$s^2 - 2p + 3s = 8.$$

The second equation gives

$$p + 4s = 2,$$

so

$$p = 2 - 4s.$$

Substitution gives

$$s^2 - 2(2 - 4s) + 3s = 8,$$

hence

$$s^2 + 11s - 12 = 0.$$

Therefore,

$$(s-1)(s+12) = 0,$$

so

$$s = 1 \quad \text{or} \quad s = -12.$$

If $s = 1$, then

$$p = 2 - 4 = -2.$$

Thus x and y are the roots of

$$t^2 - t - 2 = 0,$$

so

$$t = 2 \quad \text{or} \quad t = -1.$$

This gives

$$(x, y) = (2, -1), \quad (-1, 2).$$

If $s = -12$, then

$$p = 2 + 48 = 50.$$

Now x and y are the roots of

$$t^2 + 12t + 50 = 0.$$

Hence

$$t = \frac{-12 \pm \sqrt{144 - 200}}{2} = -6 \pm i\sqrt{14}.$$

Thus the complex solutions are the two orders of these conjugate roots.

The complete set of ordered pairs is

$$\boxed{(2, -1),\ (-1, 2),\ \left(-6 + i\sqrt{14}, -6 - i\sqrt{14}\right),\ \left(-6 - i\sqrt{14}, -6 + i\sqrt{14}\right).}$$

Question Five

(a)

We are given

$$z - 1 = (a + ib)^{-1} + (a + ic)^{-1}.$$

Now

$$\frac{1}{a + ib} = \frac{a - ib}{a^2 + b^2}$$

and

$$\frac{1}{a + ic} = \frac{a - ic}{a^2 + c^2}.$$

Therefore,

$$z = 1 + \frac{a - ib}{a^2 + b^2} + \frac{a - ic}{a^2 + c^2}.$$

The real part is

$$\boxed{\mathrm{Re}(z) = 1 + \frac{a}{a^2 + b^2} + \frac{a}{a^2 + c^2}.}$$

(b)(i)

Let the semi-vertical angle of the cone be α. For any sphere touching both sides of the cone, its radius is the perpendicular distance from its centre to either side. If its centre is a distance s from the vertex along the axis, then

$$r = s\sin\alpha.$$

For the first ball-bearing,

$$20 = 120\sin\alpha,$$

so

$$\sin\alpha = \frac{1}{6}.$$

Let the first and second centres be distances s_1 and s_2 from the vertex. Since the balls touch one another,

$$s_1 - s_2 = r_1 + r_2.$$

But

$$r_1 = s_1\sin\alpha, \qquad r_2 = s_2\sin\alpha.$$

Thus

$$s_1 - s_2 = (s_1 + s_2)\sin\alpha.$$

Using $\sin\alpha = 1/6$,

$$6s_1 - 6s_2 = s_1 + s_2,$$

so

$$5s_1 = 7s_2.$$

Therefore,

$$\frac{s_2}{s_1} = \frac{5}{7}.$$

Since radius is proportional to distance from the vertex,

$$\frac{r_2}{r_1} = \frac{5}{7}.$$

Hence

$$\boxed{r_2 = 20\left(\frac{5}{7}\right) = \frac{100}{7}\text{ mm.}}$$

(b)(ii)

The radii form a geometric sequence with first term 20 and common ratio $5/7$. Therefore,

$$\boxed{r_n = 20\left(\frac{5}{7}\right)^{n-1}\text{ mm.}}$$

(b)(iii)

The volume of the nth ball-bearing is

$$V_n = \frac{4}{3}\pi r_n^3.$$

Therefore,

$$\begin{aligned} V_n &= \frac{4}{3}\pi\left[20\left(\frac{5}{7}\right)^{n-1}\right]^3 \\ &= \frac{32000\pi}{3}\left(\frac{125}{343}\right)^{n-1}. \end{aligned}$$

Thus the volumes form an infinite geometric series with first term $32000\pi/3$ and common ratio $125/343$.

The total volume is

$$\begin{aligned} V &= \frac{32000\pi/3}{1 - 125/343} \\ &= \frac{32000\pi}{3}\cdot\frac{343}{218} \\ &= \frac{5\,488\,000\pi}{327}\ \text{mm}^3. \end{aligned}$$

Therefore,

$$\boxed{V_{\max} = \frac{5.488 \times 10^6\pi}{327}\ \text{mm}^3.}$$