S

93202Q

932022

NEW ZEALAND QUALIFICATIONS AUTHORITY
MANA TOHU MĀTAURANGA O AOTEAROA

QUALIFY FOR THE FUTURE WORLD
KIA NOHO TAKATŪ KI TŌ ĀMUA AO!

Scholarship 2017
Calculus

9.30 a.m. Friday 10 November 2017
Time allowed: Three hours
Total marks: 40

QUESTION BOOKLET

There are five questions in this booklet. Answer ALL FIVE questions.

Write your answers in Answer Booklet 93202A.

Pull out Formulae and Tables Booklet S–CALCF from the centre of this booklet.

Show ALL working. Start your answer to each question on a new page. Carefully number each question.

Answers developed using a CAS calculator require **ALL commands to be shown**. Correct answers only will not be sufficient.

Check that this booklet has pages 2–6 in the correct order and that none of these pages is blank.

YOU MAY KEEP THIS BOOKLET AT THE END OF THE EXAMINATION.

QUESTION ONE

(a) Determine ALL integers x and y such that $x^4 - y^2 = 71$.

Show algebraically that you have found ALL of the solutions.

(b) The equation $\dfrac{x^2 - bx}{p-1} = \dfrac{ax+c}{p+1}$ has two real roots of equal magnitude but opposite in sign.

Prove that $\dfrac{bc}{a} < 0$.

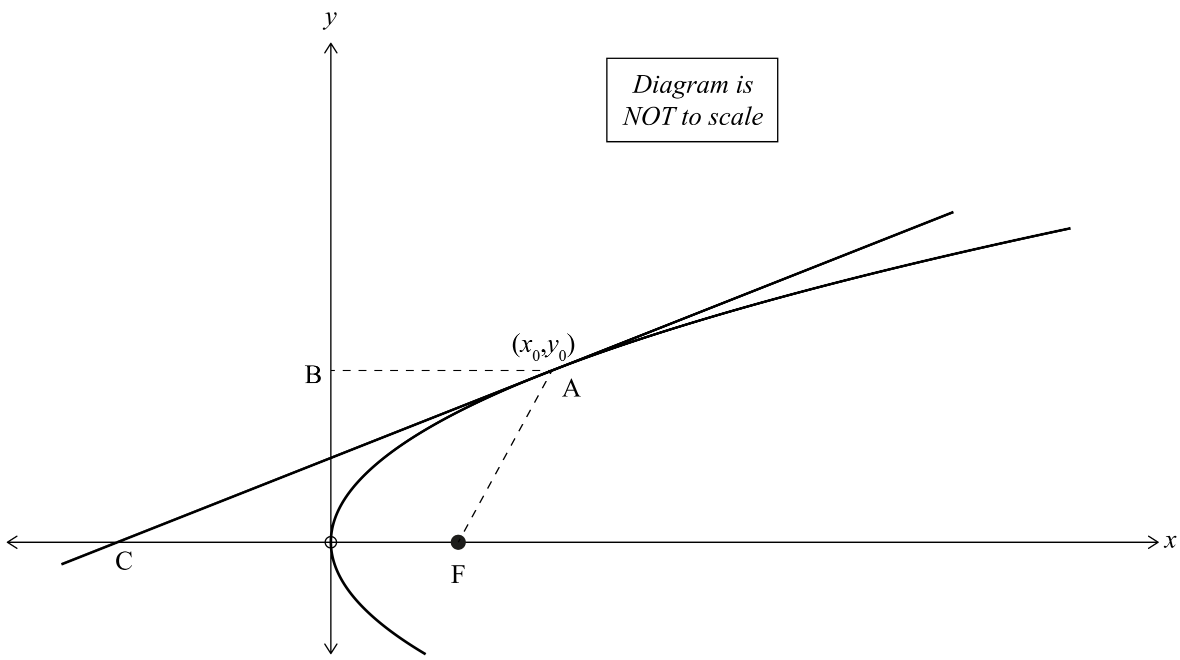
(c) A (x_0, y_0) is a point on the parabola $y^2 = 4ax$, $a > 0$.

AC is the tangent to the parabola at point A, where C is on the x-axis.

F is the focus of the parabola.

B is the point $(0, y_0)$ on the y-axis. Angle BAF is formed by joining points B and F to the given point A.

Show that AC bisects angle BAF.



QUESTION TWO

(a) ABCD is a rectangle with

$AB = 3\sqrt{3}$ units

$AD = 3$ units

PQRS is a quadrilateral with

$\angle BRQ = \angle CRS$

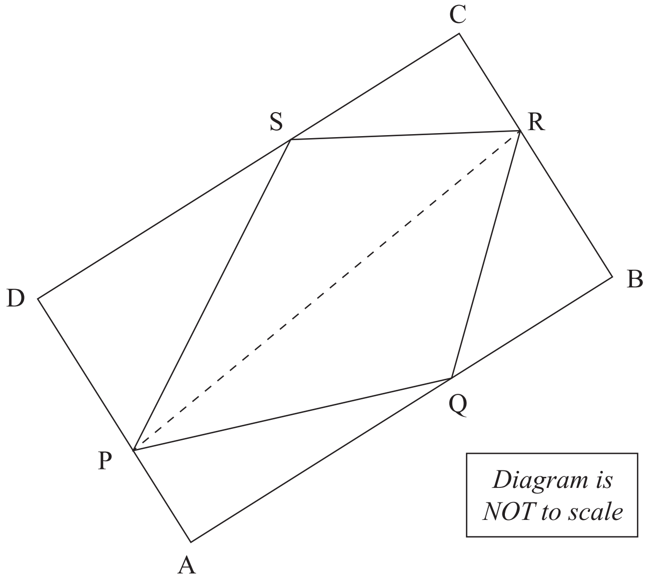
$\angle CSR = \angle DSP$

$\angle SPD = \angle APQ$

$\angle AQP = \angle BQR$

$\angle APQ = \theta$ and $AP = x$.

P is joined to R.



(i) Show that the perimeter of PQRS is not dependent on x.

(ii) If $\theta = \dfrac{\pi}{3}$ radians and $x = 2$, calculate the exact length of PR.

(b) Solve algebraically the system of equations:

$x + y - z = 1$ (1)

$x^2 + y^2 - z^2 = 5 - 2xy$ (2)

$x^3 + y^3 - z^3 = 43 - 3xy$ (3)

QUESTION THREE

(a) For $y = x^{(x^x)}$, find $\dfrac{\mathrm{d}y}{\mathrm{d}x}$ where $x = 2$.

You may use any valid algebraic techniques such as implicit differentiation and/or logarithms.

(b) Let $y = \mathrm{e}^x \sin x$.

(i) Show that $\dfrac{\mathrm{d}y}{\mathrm{d}x} = 2^{\frac{1}{2}}\mathrm{e}^x \sin\left(x + \dfrac{\pi}{4}\right)$.

(ii) Express $\dfrac{\mathrm{d}^2y}{\mathrm{d}x^2}$ in terms of exponential and sine functions only.

(iii) Find an expression for $\dfrac{\mathrm{d}^ny}{\mathrm{d}x^n}$ and evaluate this expression at $x = 0$.

(c) The hyperbolic functions $\sinh x$ and $\cosh x$ are defined as follows:

$$\sinh x = \frac{1}{2}\left(\mathrm{e}^x - \mathrm{e}^{-x}\right)$$

$$\cosh x = \frac{1}{2}\left(\mathrm{e}^x + \mathrm{e}^{-x}\right)$$

The inverse function of $\sinh x$ is denoted by $\sinh^{-1} x$.

By implicit differentiation, or otherwise, show that $\dfrac{\mathrm{d}\left(\sinh^{-1} x\right)}{\mathrm{d}x} = \dfrac{1}{\sqrt{x^2+1}}$.

QUESTION FOUR

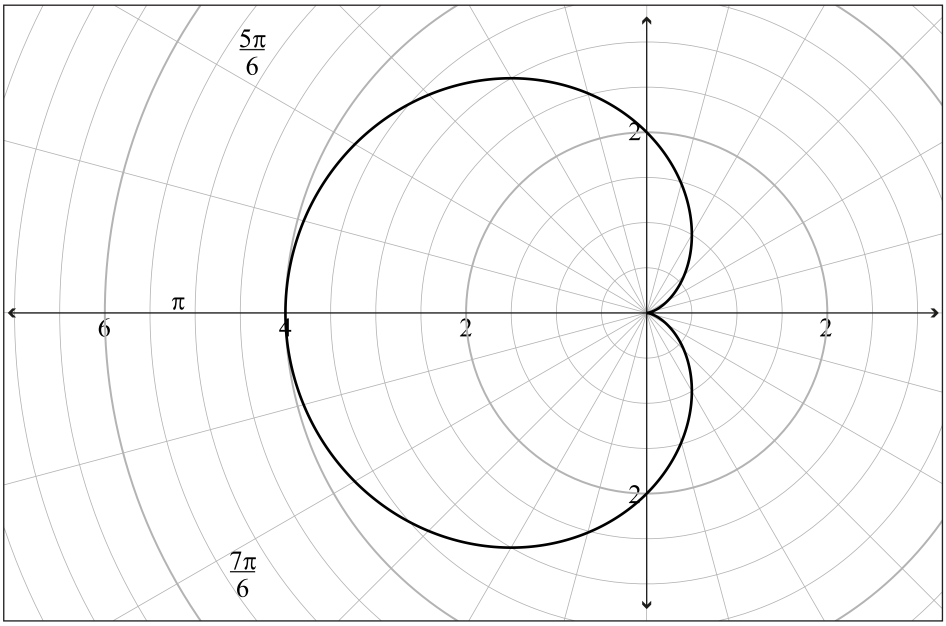
(a) Prove that $\int \tan x \tan(2x) \tan(3x)\,dx = \ln|\cos x| + \frac{1}{2}\ln|\cos 2x| - \frac{1}{3}\ln|\cos 3x| + \mathrm{c}$

(b) The length of a curve S expressed in polar coordinates is given by

$$S = \int_{\theta_1}^{\theta_2} \sqrt{\left\{ r^2 + \left(\frac{\mathrm{d}r}{\mathrm{d}\theta}\right)^2 \right\}}\,\mathrm{d}\theta$$

Find the length of the entire curve $r = a(1 - \cos\theta)$ in terms of the constant a.

In order to demonstrate shape, the curve for $a = 2$ is drawn below.



(c) According to Newton's Law of Universal Gravitation, the gravitational force F on an object of mass m that has been projected vertically upward from the Earth's surface is given by

$$F = \frac{-mg\mathrm{R}^2}{(x+\mathrm{R})^2}$$

where $x = x(t)$ is the object's distance above the Earth's surface at time t, R is the earth's radius, and g is the acceleration due to gravity. Also, by Newton's second Law, $F = ma$, and so

$$ma = -\frac{mg\mathrm{R}^2}{(x+\mathrm{R})^2}$$

Suppose a rocket is fired vertically upward with an initial velocity of v_0. Let h be the maximum height above the Earth's surface reached by the object.

Show that $v_0 = \sqrt{\dfrac{2g\mathrm{R}h}{\mathrm{R}+h}}$

QUESTION FIVE

(a) (i) Prove the identity: $\cos(5\theta) = 16\cos^5\theta - 20\cos^3\theta + 5\cos\theta$

(ii) Find a polynomial, in terms of $\cos\theta$, with roots $\cos\frac{2\pi}{9}$, $\cos\frac{4\pi}{9}$, and $\cos\frac{8\pi}{9}$.

Hint: First find the general solution for $\cos(5\theta) = \cos(4\theta)$.

(b) The sequence $\{a_n\}$ is defined as follows:

$\{a_1\} = 2$

$\{a_2\} = 7$

$\{a_{n+1}\} = \frac{1}{2}(a_n + a_{n-1})$ where $n \geq 2$

(i) Find an exact formula for the nth term of the sequence.

(ii) What is the limit of a_n as $n \to +\infty$?

You may find the following formulae useful:

$$T_n = T_1 + (n-1)d$$
$$T_n = T_1 r^{n-1}$$
$$S_n = \frac{n}{2}\left[2T_1 + (n-1)d\right]$$
$$S_n = \frac{T_1(1-r^n)}{1-r}$$

— *End of examination* —

93202Q