NZQA Scholarship Calculus

2017 1) a)

$$x^4 - y^2 = 71$$

71 is prime and LHS is a difference of squares so

$$(x^2-y)(x^2+y) = 71$$

(x^2-y) and (x^2+y) = 1 and 71
or −1 and −71

Assume $y>0$ (since if $y>0$ is a sol^n, so is $y<0$)

Then $\left.\begin{array}{l} x^2-y=1 \\ x^2+y=71 \end{array}\right\}$ $2x^2 = 72$
$x = \pm 6$ (and both are valid)
$y = 35$

or $\left.\begin{array}{l} x^2-y=-71 \\ x^2+y=-1 \end{array}\right\}$ But this makes $x^2=-36$.

So $x = \pm 6$ and $y = \pm 35$ are the four possible sol^{n}s.

NZQA Scholarship Calculus
2017 1) b)

$$\frac{x^2 - bx}{p-1} = \frac{ax+c}{p+1} \quad \text{has 2 real roots } \pm\alpha.$$

Since it's a quadratic this means 'b' in the quadratic formula is 0.

So multiplying out:

$$(p+1)(x^2-bx) = (p-1)(ax+c)$$

$$(p+1)x^2 - (p+1)bx = (p-1)ax + (p-1)c$$

$$(p+1)x^2 - [(p+1)b + (p-1)a]x - (p-1)c = 0$$

So in the quadratic formula:

$$(p+1)b + (p-1)a = 0$$
$$pb + b + pa - a = 0$$
$$p(b+a) = a-b$$
$$p = \frac{a-b}{b+a}. \qquad p+1 = \frac{a-b+a+b}{a+b} = \frac{2a}{a+b}$$
$$p-1 = \frac{a-b-b-a}{b+a} = \frac{-2b}{a+b}$$

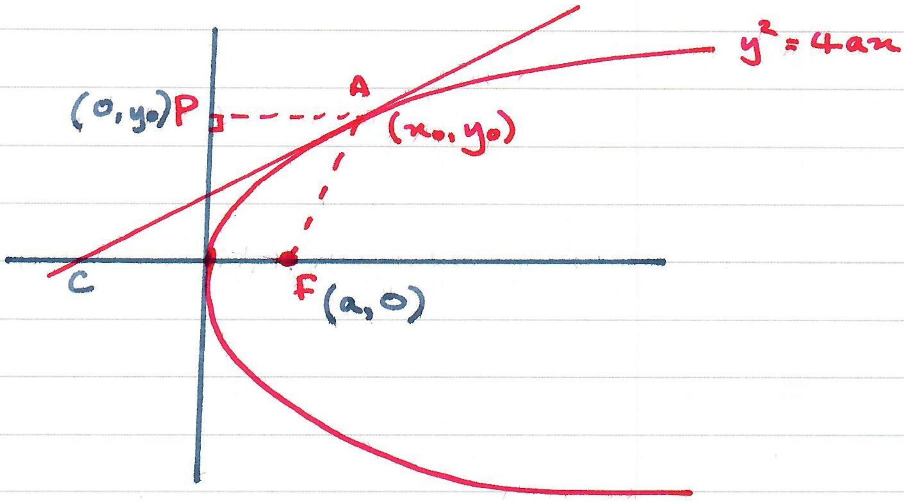
For there to be 2 roots the discriminant > 0:

$$\frac{2a}{a+b}x^2 - \underbrace{\left(\frac{2ab}{a+b} + \left(-\frac{2ab}{a+b}\right)\right)}_{=0} + \frac{2bc}{a+b} = 0$$

'$b^2 - 4ac$' = just $-4ac = \frac{-16abc}{\cancel{a+b}}$ $\quad \therefore abc < 0$?

But / by a^2: $\frac{bc}{a} < 0$ as req.

NZQA Scholarship Calculus
2017 1)c)



Prove $\angle PAC = \angle FAC$.

This is a standard parabola result ... but I can't recall how to prove it geometrically. Aha! Show $\triangle$ AFC is isoceles.

We know F is the focus, ie point $(a, 0)$.

$$\text{So } AF = \sqrt{y_0^2 + (x_0 - a)^2} = \sqrt{4ax_0 + (x_0 - a)^2}$$

$$= \sqrt{4ax_0 + x_0^2 - 2ax_0 + a^2} = \sqrt{x_0^2 + 2ax_0 + a^2}$$

$$= \sqrt{(x_0 + a)^2} = \underline{x_0 + a}.$$

For the parabola, $y^2 = 4ax$

$$2y \frac{dy}{dx} = 4a$$

So $\dfrac{dy}{dx} = \dfrac{4a}{2y} = \dfrac{2a}{y_0}$ at (x_0, y_0)

So the tangent at A is

$$y = \frac{2a}{y_0}x + c$$

Subs: $y_0 = \dfrac{2a}{y_0}x_0 + c$ $\therefore$ $c = y_0 - 2a\dfrac{x_0}{y_0}$

$$= y_0 - 2a \cdot \frac{y_0^2}{4a\, y_0}$$

$$= y_0 - \frac{y_0}{2}$$

$$= \frac{y_0}{2}$$

(ooh, this is useful).

So the tangent is $y = \dfrac{2a}{y_0}x + \dfrac{y_0}{2}$

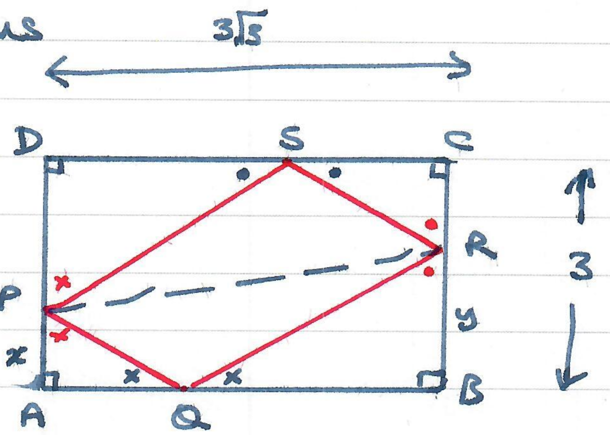
$= 0$ when $\dfrac{2ax}{y_0} = -\dfrac{y_0}{2}$ or $x = \dfrac{-y_0^2}{4a}$.

and this equals $-x_0$.

So $CF = x_0 + a$ – which equals AF.

So $\triangle$ AFC is isosceles: $\angle ACF = \angle FAC$, and
$\angle ACF = \angle PAE$ (|| lines).
So $\underline{\underline{AC \text{ bisects } \angle PAF}}$ as required.

NZQA Scholarship Calculus
2017 2) a)



The Diagram is indeed not drawn to scale - in fact it's drawn misleadingly badly.

i) Equal angles are marked here, and it's clear that because the corners are right-angles, • = x and • = x.

So PQRS is a parallelogram. (and $x = \theta$)

$$\frac{x}{PQ} = \sin\theta$$

So $PQ = \dfrac{x}{\sin\theta}$ $\qquad RS = \dfrac{(3-y)}{\sin\theta}$

Similarly $QR = \dfrac{y}{\sin\theta}$ $\qquad SP = \dfrac{(3-x)}{\sin\theta}$.

So $PQ+QR+RS+SP = \dfrac{1}{\sin\theta}\left(x+y+3-y+3-x\right)$

$= \dfrac{6}{\sin\theta}$ which is ind^t of x, as required.

ii) $\theta = \frac{\pi}{3}$: $x = 2$ So $AQ = 2\tan\frac{\pi}{3} = 2\sqrt{3}$.

So $QB = 3\sqrt{3} - 2\sqrt{3} = \sqrt{3}$ so $RB = QB\tan\frac{\pi}{6} = \dfrac{\sqrt{3}}{\sqrt{3}} = 1$

So $PR = \sqrt{(RB-PA)^2 + AB^2} = \sqrt{1+27} = \underline{\underline{2\sqrt{7}}}$. (units)

NZQA Scholarship Calculus
2017 2) b)

$$x + y - z = 1 \quad ①$$
$$x^2 + y^2 - z^2 = 5 - 2xy \quad ②$$
$$x^3 + y^3 - z^3 = 43 - 3xy \quad ③$$

Usually with questions like this we consider $(x+y+z)^{2,3}$ and knock terms out - but this one has a $-z$ and the $2xy$ term in ②. So...

from ① : $x + y = 1 + z$ ——— ④

in ② : $x^2 + y^2 + 2xy = 5 + z^2$

So from ④ : $(1+z)^2 = 5 + z^2$

$$1 + 2z + \cancel{z^2} = 5 + \cancel{z^2}$$

$$2z = 4$$
$$\underline{\underline{z = 2.}}$$

So we now know $x + y = 3$

$x^2 + y^2 = 9 - 2xy$ (basically the same)

$x^3 + y^3 = 51 - 3xy$

Subs. for 3: $x^3 + y^3 = 51 - xy(x+y)$

$$x^3 + x^2y + xy^2 + y^3 = 51$$

$$(x+y)(x^2+y^2) = 51$$

$$3(x^2+y^2) = 51$$

$$x^2+y^2 = 17$$

using $x^2+y^2 = 9-2xy$

this gives $0 = 8+2xy$

or $xy = -4$: $y = -\frac{4}{x}$

using this in $x+y=3$:

$$x-\frac{4}{x} = 3$$

$$(x^2-3x-4) = 0$$

$$(x-4)(x+1) = 0$$

$$x = -1 \text{ or } 4.$$

Since $xy = -4$ and x, y are interchangable,

this gives solutions $x = -1,\ y = 4,\ z = 2$

or $x = 4,\ y = -1,\ z = 2.$

NZQA Scholarship Calculus

2017 3)a)

$$y = x^{(x^x)}$$

Let $x = e^{\ln x}$

Then $$y = e^{(x^x)\ln x}$$

$$= e^{(e^{\ln x})^x \ln x}$$

$$= e^{e^{x\ln x} \cdot \ln x}$$

So $$\frac{dy}{dx} = e^{e^{x\ln x} \cdot \ln x} \times \frac{d}{dx}\left(e^{x\ln x} \cdot (\ln x)\right)$$

$$= \underset{\text{(again)}}{x^{x^x}} \left(e^{x\ln x} \cdot \frac{1}{x} + e^{x\ln x} \cdot \ln x \cdot \frac{d}{dx}(x \cdot \ln x)\right)$$

$$= x^{x^x}\left(\frac{x^x}{x} + x^x \ln x \left(x \cdot \frac{1}{x} + \ln x\right)\right)$$

$$= x^{x^x}\left(x^{x-1} + x^x \ln x \,(1 + \ln x)\right)$$

At $x = 2$: $$= 2^{2^2}\left(2^1 + 2^2 \ln 2\,(1 + \ln 2)\right)$$

$$= 16\left(2 + 4\ln 2 + 4(\ln 2)^2\right)$$

$$= \boxed{32\left(1 + 2\ln 2 + 2(\ln 2)^2\right)}$$

NZQA Scholarship Calculus

2017 3) b)

$$y = e^x \sin x$$

i) $\dfrac{dy}{dx} = e^x \cos x + e^x \sin x$ (by product rule)

$$= e^x(\cos x + \sin x)$$

$$= \sqrt{2}e^x\left(\frac{1}{\sqrt{2}}\cos x + \frac{1}{\sqrt{2}}\sin x\right)$$

$$= \sqrt{2}e^x\left(\sin\frac{\pi}{4}\cos x + \cos\frac{\pi}{4}\sin x\right)$$

$= \underline{\underline{\sqrt{2}e^x \sin\left(x+\frac{\pi}{4}\right)}}$ by standard trig identities.

ii) $\dfrac{d^2y}{dx^2} = \sqrt{2}\left(e^x\cos\left(x+\frac{\pi}{4}\right) + e^x \sin\left(x+\frac{\pi}{4}\right)\right)$

(by a very similar process)

$$= \sqrt{2}.\sqrt{2}e^x \sin\left(x+\frac{\pi}{2}\right)$$

$$= 2e^x \sin\left(x+\frac{\pi}{2}\right).$$

iii) Clearly (we could prove it by induction)

$$\frac{d^ny}{dx^n} = (\sqrt{2})^n \sin\left(x+\frac{n\pi}{4}\right)$$

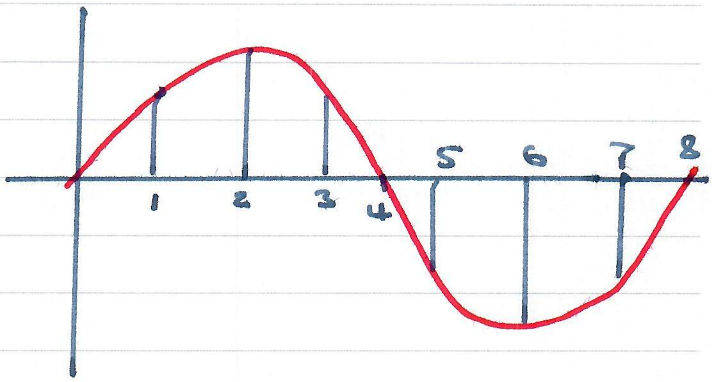
So when $n = 4m$, $\frac{d^n y}{dx^n} = 0$

$n = 8m+1$ or $8m+3$, $(\sqrt{2})^n \frac{1}{\sqrt{2}}$

$n = 8m+2$ $(\sqrt{2})^n$

$n = 8m+5$ or $8m+7$, $-(\sqrt{2})^n \frac{1}{\sqrt{2}}$

$n = 8m-2$, $-(\sqrt{2})^n$



NZQA Scholarship Calculus

2017 3) a)

$$\sinh x = \frac{1}{2}(e^{x} - e^{-x})$$

$$\cosh x = \frac{1}{2}(e^{x} + e^{-x})$$

Show $\frac{d}{dx}(\sinh^{-1} x) = \frac{1}{\sqrt{x^2+1}}$

Let $\sinh y = x$

Then $y = \sinh^{-1}(x)$

So $x = \frac{1}{2}(e^{y} - e^{-y})$

Differentiating:

$$1 = \frac{1}{2}\left(e^{y}\frac{dy}{dx} + e^{-y}\frac{dy}{dx}\right) = \frac{1}{2}\frac{dy}{dx}(e^{y} + e^{-y})$$

$$= \frac{dy}{dx}\cosh y$$

$$\frac{dy}{dx} = \frac{1}{\cosh y} = \frac{1}{\sqrt{\sinh^2 y + 1}} = \frac{1}{\sqrt{x^2+1}}$$

(we know $\cosh^2 y - \sinh^2 y = 1$ – or we could show it.)

NZQA Scholarship Calculus
2017

4)a) Find $I = \int \tan x \tan 2x \tan 3x \; dx$

$$\cancel{= \frac{\sin x}{\cos x} \cdot \frac{\sin 2x}{\cos 2x} \cdot \frac{\sin 3x}{\cos 3x}}$$

Expand: $\tan 3x = \dfrac{\tan 2x + \tan x}{1 - \tan x \tan 2x}$

So $\tan 3x - \tan x \tan 2x \tan 3x = \tan x + \tan 2x$

So $\tan x \tan 2x \tan 3x = -\tan x - \tan 2x + \tan 3x$

$$I = \int -\tan x - \tan 2x + \tan 3x \; dx$$

Now $\int \tan kx \; dx = \frac{1}{k} \ln|\sec kx| = \frac{1}{k} \ln\left|\frac{1}{\cos kx}\right|$

(standard result)

So $I = -\ln\left|\frac{1}{\cos x}\right| - \frac{1}{2}\ln\left|\frac{1}{\cos 2x}\right| + \frac{1}{3}\ln\left|\frac{1}{\cos 3x}\right|$

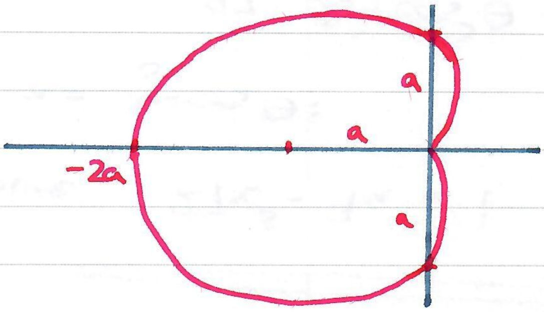
and reversing the signs for reciprocals:

$$I = \ln|\cos x| + \frac{1}{2}\ln|\cos 2x| - \frac{1}{3}\ln|\cos 3x|$$

as required.

4) b) $$S = \int_{\theta_1}^{\theta_2} \sqrt{\left(r^2 + \left(\frac{dr}{d\theta}\right)^2\right)}\; d\theta$$

Curve $r = a(1 - \cos\theta)$ (Cardioid = heart-shaped)



It's not entirely clear from the diagram given, but θ runs from 0 $(r=0)$
through π $(r=2a)$
to 2π $(r=0)$

$$\frac{dr}{d\theta} = a\sin\theta$$

$$r^2 + \left(\frac{dr}{d\theta}\right)^2 = a^2(1-\cos\theta)^2 + a^2\sin^2\theta$$

$$= a^2\left[1 - 2\cos\theta + \cos^2\theta + \sin^2\theta\right]$$

$$= 2a^2(1 - \cos\theta) \longrightarrow ①$$

Now $\cos\theta = \cos^2\frac{\theta}{2} - \sin^2\frac{\theta}{2} = 1 - 2\sin^2\frac{\theta}{2}$

So $1 - \cos\theta = 2\sin^2\frac{\theta}{2}$.

So ① is $4a^2 \sin^2 \frac{\theta}{2}$

and $S = \int_0^{2\pi} 2a \sin \frac{\theta}{2} \, d\theta$

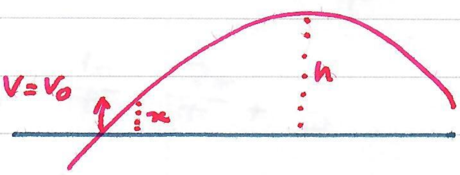
$= -\frac{2a}{1/2} \left[\cos \frac{\theta}{2} \right]_0^{2\pi}$

$= -4a \left(\cos \pi - \cos 0 \right)$

$= \underline{\underline{8a}}.$

NZQA Scholarship Calculus
2017

4)c)
$$\cancel{m}a = \frac{-\cancel{m}gR^2}{(x+R)^2}$$



Getting through the 'words' ...

$a =$ acceleration $= \frac{dv}{dt}$
$v =$ velocity upwards $= \frac{dx}{dt}$
$x =$ vertical height.

This would be easy without the $\underline{x}$ term in the acceleration formula. But since it's there, it's a clue to do something (as is the lack of t)

$$a = \frac{dv}{dt} = \frac{dv}{dx} \cdot \frac{dx}{dt} = v\frac{dv}{dx}$$

So
$$v\frac{dv}{dx} = \frac{-gR^2}{(x+R)^2}$$

$$v\,dv = \frac{-gR^2}{(x+R)^2}\,dx$$

$$\frac{v^2}{2} = \frac{+gR^2}{x+R} + c$$

But $v = 0$ when $x = h$:

$$c = \frac{-gR^2}{h+R}$$

So when $x = 0$ and $v = v_0$:

$$\frac{v_0^2}{2} = \frac{gR^2}{R} - \frac{gR^2}{h+R}$$

$$= gR^2\left(\frac{h+R-R}{R(h+R)}\right)$$

$$= \frac{gR^{\cancel{2}}h}{\cancel{R}(R+h)}$$

$$v_0 = \sqrt{\frac{2gRh}{R+h}}$$ as required.

NZQA Scholarship Calculus

2017 5) a)

i) $\cos(5\theta) = \cos(4\theta + \theta) = \cos(4\theta)\cos\theta - \sin(4\theta)\sin\theta$

$$= (\cos^2 2\theta - \sin^2 2\theta)\cos\theta - 2\sin 2\theta\cos 2\theta\sin\theta$$

$$= (2\cos^2 2\theta - 1)\cos\theta - 4\sin^2\theta\cos\theta\cos 2\theta$$

$$= \left(2(\cos^2\theta - \sin^2\theta)^2 - 1\right)\cos\theta - 4\sin^2\theta\cos\theta(\cos^2\theta - \sin^2\theta)$$

$$= \left(2(2\cos^2\theta - 1)^2 - 1\right)\cos\theta - 4(1 - \cos^2\theta)\cos\theta(2\cos^2\theta - 1)$$

$8\cos^5\theta$

$$= -8\cos^3\theta + \cos\theta - 4\cos\theta(3\cos^2\theta - 1 - 2\cos^4\theta)$$

$8\cos^5\theta$

$$= -8\cos^3\theta + \cos\theta - 12\cos^3\theta + 8\cos^5\theta + 4\cos\theta$$

$$= \underline{\underline{16\cos^5\theta - 20\cos^3\theta + 5\cos\theta}}$$ as required.

ii) On the face of it, getting to $\frac{\pi}{9}$ from here looks hard - but perhaps $\cos 5\theta - \cos 4\theta$ will help

$$\cos 5\theta - \cos 4\theta = -2\sin\frac{9\theta}{2}\sin\frac{\theta}{2}$$

and this is 0 when $\sin\frac{\theta}{2} = 0$ or $\sin\frac{9\theta}{2} = \theta$

$\theta = 0,\ 2\pi \ldots$ $\theta = 0,\ \frac{2\pi}{9},\ \frac{4\pi}{9},\ \frac{6\pi}{9},\ \frac{8\pi}{9} \ldots$

So if $\cos 5\theta - \cos 4\theta$ can be expressed as a polynomial in $\cos\theta$ it will have these as roots (ie solutions).

$$\cos 4\theta = \cos^2 2\theta - \sin^2 2\theta = 2\cos^2 2\theta - 1$$

$$= 2(\cos 2\theta)^2 - 1$$

$$= 2(2\cos^2\theta - 1)^2 - 1$$

$$= 8\cos^4\theta - 8\cos^2\theta + 2 - 1$$

$$= 8\cos^4\theta - 8\cos^2\theta + 1.$$

So $\cos(5\theta) - \cos(4\theta)$

$$= 16\cos^5\theta - 8\cos^4\theta - 20\cos^3\theta + 8\cos^2\theta + 5\cos\theta - 1$$

This is a quintic and we know 5 roots:

$$\theta = 0, \frac{2\pi}{9}, \frac{4\pi}{9}, \frac{6\pi}{9}, \frac{8\pi}{9}$$

and we note $\cos 0 = 1$, $\cos\frac{6\pi}{9} = \cos\frac{2\pi}{3} = -\frac{1}{2}$

So it must have factors $(\cos\theta - 1)$ and $(\cos\theta + \frac{1}{2})$

(After a lot of bashing ...) $(\cos\theta - 1)(\cos\theta + \frac{1}{2})(8\cos^3\theta - 6\cos\theta + 1) = 0$

So the required polynomial is $\underline{\underline{8\cos^3\theta - 6\cos\theta + 1}}$

NZQA Scholarship Calculus
2017 5) b)

$a_1 = 2$
$a_2 = 7$
$a_{n+1} = \frac{1}{2}(a_n + a_{n-1})$ ——— ①

i) By inspection: $a_3 = \frac{1}{2}(2+7) = 4.5$
$a_4 = \frac{1}{2}(7+4.5) = \frac{11.5}{2} = 5.75$

Hmm. Are we allowed to use generating functions?

Use ① to solve $x^2 = \frac{1}{2}(x+1)$

i.e. $x^2 - \frac{x}{2} - \frac{1}{2} = 0$

So $(2x+1)(x-1) = 0$

So $x = -\frac{1}{2}$ or 1.

This points to a general expression for an as:

$a_n = a(1)^n + b\left(-\frac{1}{2}\right)^n$

And $a_1 = a - \frac{b}{2} = 2$
$a_2 = a + \frac{b}{4} = 7$

$a = \frac{16}{3}$ $b = \frac{20}{3}$

and $a_n = \frac{16}{3} + \frac{20}{3}\left(-\frac{1}{2}\right)^n$.

ii) Clearly as $n \to \infty$ the $\left(-\frac{1}{2}\right)^n$ term $\to 0$, oscillating
So $\lim_{n\to\infty} a_n = \frac{16}{3}$