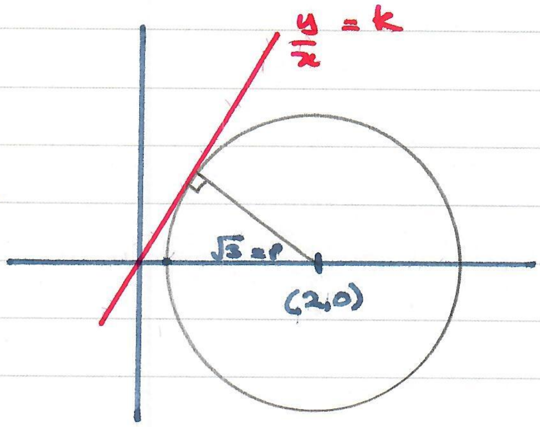
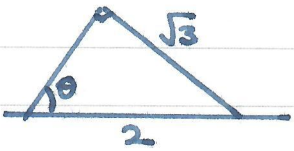
NZQA Scholarship Calculus
2016 1)a)




Given that what's required is a tangent to the ⊙ I suspect there's an easy geometric answer:




$\sin\theta = \frac{\sqrt{3}}{2}$

So $\tan\theta = \sqrt{3}$ ∴ $\theta = 60^\circ \; \left(\frac{\pi}{3}\right)$ and $\frac{y}{x} = \sqrt{3}$

But we should probably do it with algebra:

$$(x-2)^2 + y^2 = 3$$

So $$2(x-2) + 2y\frac{dy}{dx} = 0$$

$$\frac{dy}{dx} = \frac{2-x}{y}$$

But we want the max. of $\frac{y}{x}$.

And $$\frac{y}{x} = \frac{\sqrt{3-(x-2)^2}}{x}$$

So $$\frac{d}{dx}\left(\frac{y}{x}\right) = \frac{x\,\frac{1}{2}\,\frac{1}{\sqrt{3-(x-2)^2}}\cdot(-2)(x-2) - \sqrt{3-(x-2)^2}}{x^2}$$

$$= \frac{\dfrac{2x - x^2 - 3 + (x-2)^2}{\sqrt{3 - (x-2)^2}}}{x^2}$$

This is 0 when $\quad 2x - x^2 - 3 + (x-2)^2 = 0$

$$2x - \cancel{x^2} - 3 + \cancel{x^2} - 4x + 4 = 0$$

$$-2x + 1 = 0$$

$$x = \frac{1}{2}.$$

So $\quad y^2 = 3 - \left(\frac{-3}{2}\right)^2 = 3 - \frac{9}{4} = \frac{3}{4}$

$$y = \frac{\sqrt{3}}{2}$$

So $\frac{y}{x} = \frac{\sqrt{3}}{2} \times \frac{2}{1} = \sqrt{3}$

... which is exactly what we got by geometry

NZQA Scholarship Calculus

2016 1) b)

$$f(x) = x^2 \ln(x+1)$$

i) $$f'(x) = \frac{x^2}{x+1} + 2x \ln(x+1)$$

$$f''(x) = \frac{2(x+1)x - x^2}{(x+1)^2} + \frac{2x}{x+1} + 2\ln(x+1)$$

$$f''(0) = 0 + 0 + 2\ln(0+1) = 0.$$

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~~Tidying ① up:~~

$$\cancel{f''(x) = \frac{2x^2 - x^2 + 2x + 2x^2 + 2x}{(x+1)^2} + 2\ln(x+1)}$$

$$\cancel{= \frac{3x^2 + 4x}{(x+1)^2} + 2\ln(x+1)}$$

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Tidying up:

$$f''(x) = \frac{2x^2 + 2x - x^2}{(x+1)^2} + \frac{2x}{x+1} + 2\ln(x+1)$$

$$= \frac{x^2 + 2x + 1 - 1}{(x+1)^2} + \cdots\cdots$$

$$= 1 - \frac{1}{(x+1)^2} + \frac{2x}{x+1} + 2\ln(x+1)$$

NZQA Scholarship Calculus

2016 2)a)

$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (\sin^5 x + \cos^5 x)\,dx$$

Two useful tricks here:

a) $\sin^5 x$ is an odd function and the interval is $[-\frac{\pi}{2}, \frac{\pi}{2}]$ so it contributes precisely 0.

b) replace $(\cos^4 x)$ by $(1-\sin^2 x)^2$ and we get $\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (1-2\sin^2 x+\sin^4 x)\cos x\,dx$,

which is nicely integrable:

$$\left[\sin x - \frac{2}{3}\sin^3 x + \frac{1}{5}\sin^5 x\right]_{-\frac{\pi}{2}}^{\frac{\pi}{2}}$$

$$= 1-\frac{2}{3}+\frac{1}{5}+1-\frac{2}{3}+\frac{1}{5}$$

$$= \frac{15-10+3}{15}\times 2 = \underline{\underline{\frac{16}{15}}}$$

NZQA Scholarship Calculus

2016 2) b)

$$I_n = \int_0^{\frac{\pi}{2}} \frac{\sin 2nx}{\sin x}\, dx$$

$$I_n - I_{n-1} = \int_0^{\frac{\pi}{2}} \frac{\sin 2nx - \sin 2(n-1)x}{\sin x}\, dx$$

$$= \int_0^{\frac{\pi}{2}} \frac{2 \sin x \cos(2n-1)x}{\sin x}\, dx$$

$$= 2\int_0^{\frac{\pi}{2}} \cos(2n-1)x \, dx$$

$$= \left. \frac{2\sin(2n-1)}{2n-1} \right|_0^{\frac{\pi}{2}}$$

$$= \frac{2(-1)^{n-1}}{2n-1} - 0$$

as required.

NZQA Scholarship Calculus

2016 2)c)

Looking at the answer to this it's not clear how any student is supposed to get there... it seems to be just some examiner's 'neat little result' and therefore not a question of much teaching merit. But here goes.

It clearly wants to see the usual formula:

$$\log_a x = \frac{\log_b x}{\log_b a}$$

and it's good to observe $(\tan\theta + \cot\theta)$ is useful:

$$\tan\theta + \cot\theta = \frac{\sin\theta}{\cos\theta} + \frac{\cos\theta}{\sin\theta}$$

$$= \frac{\sin^2\theta + \cos^2\theta}{\cos\theta\sin\theta}$$

$$= \frac{1}{\cos\theta\sin\theta}.$$

(note the reciprocal)

~~also, because $\tan(\frac{\pi}{2} - \theta) = \cot\theta$ etc.~~

~~$\log_{(\tan\theta + \cot\theta)}(\cot\theta) = k$ also.~~

~~(which seems extremely odd)~~

Having got this far we might well stop, but...

$$\log_{(\tan\theta+\cot\theta)}(\cos\theta) = \frac{\log \cos\theta}{\log(\tan\theta+\cot\theta)}$$

$$= \frac{\log\cos\theta}{-\log(\cos\theta\sin\theta)}$$

$$= \frac{-\log\cos\theta}{\log\sin\theta+\log\cos\theta}$$

$$= \frac{-1}{\frac{\log\sin\theta}{\log\cos\theta}+1} = k$$

So $\dfrac{\log\sin\theta}{\log\cos\theta} = -\dfrac{1}{k} - 1$

Meanwhile

$$\log_{\tan\theta}(\sin\theta) = \frac{\log\sin\theta}{\log\tan\theta} = \frac{\log\sin\theta}{\log\sin\theta-\log\cos\theta}$$

$$= \frac{1}{1-\frac{\log\cos\theta}{\log\sin\theta}} = \frac{1}{1+\frac{k}{k+1}} = \underline{\underline{\frac{k+1}{2k+1}}}$$

NZQA Scholarship Calculus
2016 3) a)

$$f(x) = x - \int_0^{\frac{\pi}{2}} f(x)\sin x \, dx$$

If (2)(c) was weird, this one's even worse. The 'obvious' things are:

a) Consider $\int f(x)\sin x\,dx = -f(x)\cos x + \int f'(x)\cos x\,dx$

But the $f'(x)$ term is a mystery, particularly since $f(x)$ itself is a mix of $x$ and a 'constant'.

b) So the published answer expects us to use ① to say $f(x) = x - a$ ... whatever a is.

(I assume it isn't really constant, but...)

So let $a = \int_0^{\frac{\pi}{2}} f(x)\sin x\,dx$

$$= \int_0^{\frac{\pi}{2}} (x-a)\sin x\,dx$$

$$= (\text{using IBP}) \quad \left[-x\cos x\right]_0^{\frac{\pi}{2}} + \int_0^{\frac{\pi}{2}} \cos x\,dx + \left[a\cos x\right]_0^{\frac{\pi}{2}}$$

$= 1 - a$ So $a = \frac{1}{2}$. So $\underline{\underline{f(x) = x - \frac{1}{2}}}$

(Another weirdly contrived question)

NZQA Scholarship Calculus
2016 3) b)

$$\frac{dy}{dx} = x - 2y$$ (Is this my favourite Tripos question?)

Actually no.

It's got 'integrating factor' written all over it - though this leads to $\int x e^{2x} dx$, which needs IBP.

All do-able, but ~~keeping the problem leads us through it~~ it's a bit of a grind - and (i) is a bit of a distraction

i) $$\frac{d(e^{2x} y)}{dx} = e^{2x}\frac{dy}{dx} + 2e^{2x}y$$

$$= e^{2x}(x - 2y) + 2e^{2x}y$$

$$= xe^{2x} + \cancel{(2y - 2y)e^{2x}} = xe^{2x}$$ as required.

ii) $$\frac{dy}{dx} + 2y = x$$

Use an IF $e^{2x}$:

$$e^{2x}\frac{dy}{dx} + 2e^{2x}y = xe^{2x}$$

$$\frac{d}{dx}(e^{2x}y) = xe^{2x}$$

From here we get

$$e^{2x}y = \int x e^{2x}\,dx \qquad \text{——— } (1)$$

Using IBP:

$$2\int x e^{2x}\,dx = x e^{2x} - \int e^{2x}\,dx$$

So $\int x e^{2x}\,dx = \frac{1}{2}\left(x e^{2x} - \frac{1}{2}e^{2x}\right)$

$$= \frac{1}{2}\left(x - \frac{1}{2}\right)e^{2x} \qquad \text{——— } (2)$$

So using (2) in (1):

$$e^{2x}y = \frac{1}{2}\left(x - \frac{1}{2}\right)e^{2x} + C$$

$$y = \frac{x}{2} - \frac{1}{4} + Ce^{-2x}$$

Subs the point $(1,1)$:

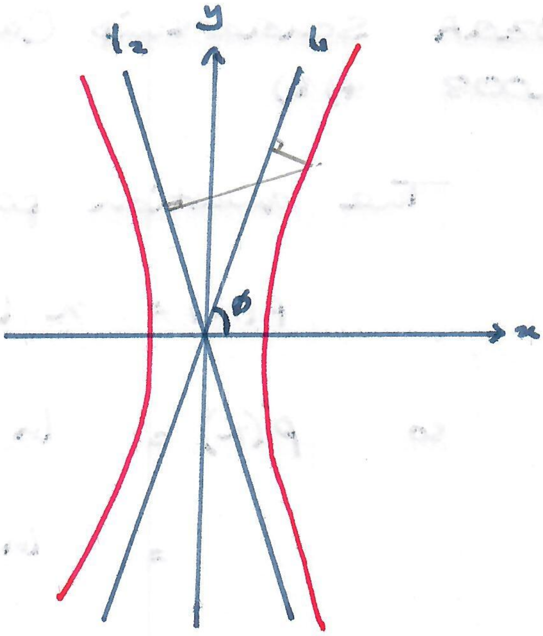
$$1 = \frac{1}{2} - \frac{1}{4} + Ce^{-2}$$

$$\frac{3}{4} = Ce^{-2} \qquad C = \frac{3}{4}e^{2}$$

$$y = \frac{x}{2} - \frac{1}{4} + \frac{3}{4}e^{2-2x}$$

$$= \frac{1}{4}\left(2x - 1 + 3e^{2-2x}\right)$$

NZQA Scholarship Calculus
2016 4)a)




$$\frac{x^2}{4} + \frac{y^2}{36} = 1$$

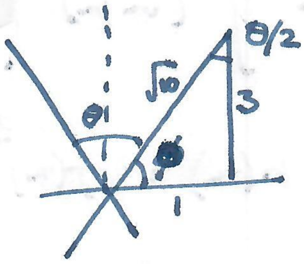
Standard result gives the assymptotes are

$$\frac{x}{2} = \pm \frac{y}{6}$$

$$y = \pm 3x$$

So $\phi = \tan^{-1} 3$

Draw a picture:



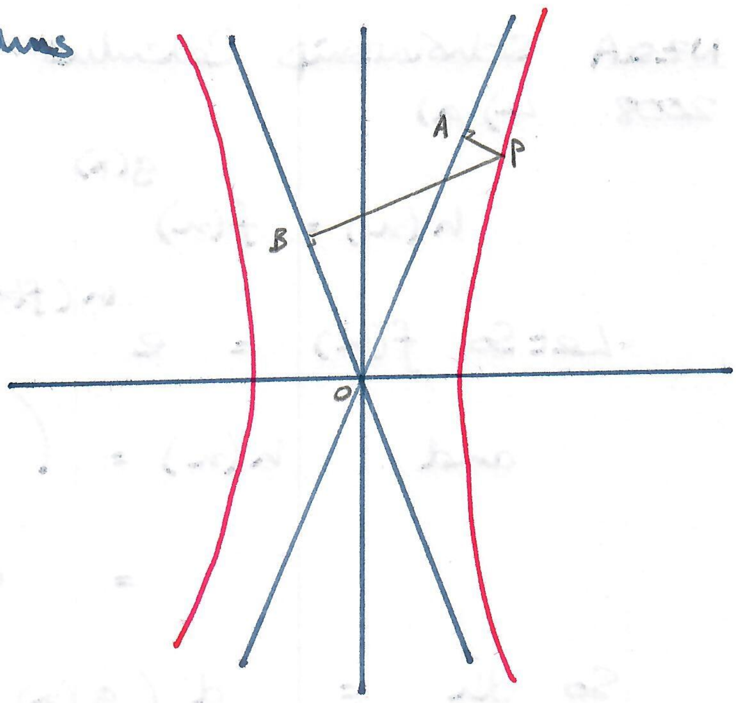

$\sin\phi = \frac{3}{\sqrt{10}}$ $\quad \cos\phi = \frac{1}{\sqrt{10}}$

$$\sin\theta = 2\sin\frac{\theta}{2}\cos\frac{\theta}{2} = 2.\frac{3}{\sqrt{10}}.\frac{1}{\sqrt{10}} = \frac{6}{10} = \underline{\underline{0.6}}$$

as required.

NZQA Scholarship Calculus

2016 4) b)




This is a nice general result which applies to all hyperbolae ... the mechanics of finding A and B is a little tedious but it works ... but there's a quick way using standard 2D/3D coordinate analysis. I'm not sure it's taught for 2D (it is for 3D, though prob. only for Further Maths) but since it basically just helps with the book-keeping I'll do it here.

Essentially if we have a line

$$ax + by = c$$

and a point $(h, k)$

then the perp. distance from $(h, k)$ to the line is

$$\frac{ah + by - c}{\sqrt{a^2+b^2}}$$

And we have two lines:

$$3x - y = 0 \quad (A)$$
$$3x + y = 0 \quad (B)$$

So $|AP| = \dfrac{3h-k}{\sqrt{10}}$

$|BP| = \dfrac{3h+k}{\sqrt{10}}$

$|AP||BP| = \dfrac{(3h-k)(3h+k)}{10}$

$= \dfrac{9h^2-k^2}{10}$

But $9h^2-k^2 = 36$ (eq$^n$ of the hyperbola)

So $|AP||BP| = \dfrac{36}{10} = \underline{\underline{3.6}}$

## NZ NCEA Scholarship Calculus 2016

(ChatGPT answers because I ran out of steam with tedious algebra – though I got most of the way.)

Worked Solution: Question 4(c)

## Question 4(c)

The asymptotes of

$$\frac{x^2}{4} - \frac{y^2}{36} = 1$$

are

$$l_1\colon\ y = 3x, \qquad l_2\colon\ y = -3x.$$

Let

$$P = (x, y),$$

where $P$ is in the first quadrant. Write the points on the two asymptotes as

$$C = (u, 3u), \qquad D = (v, -3v).$$

Since

$$CP : PD = 1 : \lambda,$$

the internal section formula gives

$$P = \frac{\lambda C + D}{\lambda + 1}.$$

Therefore,

$$(x, y) = \left(\frac{\lambda u + v}{\lambda + 1}, \frac{3\lambda u - 3v}{\lambda + 1}\right).$$

Thus

$$\lambda u + v = (\lambda + 1)x$$

and

$$\lambda u - v = \frac{\lambda + 1}{3}y.$$

Adding these equations,

$$2\lambda u = (\lambda + 1)\left(x + \frac{y}{3}\right),$$

so

$$u = \frac{\lambda + 1}{2\lambda}\left(x + \frac{y}{3}\right).$$

Subtracting the second equation from the first,

$$2v = (\lambda + 1)\left(x - \frac{y}{3}\right),$$

so

$$v = \frac{\lambda + 1}{2}\left(x - \frac{y}{3}\right).$$

### Area of triangle $COD$

Using $O = (0,0)$, $C = (u, 3u)$, and $D = (v, -3v)$,

$$A = \frac{1}{2}|u(-3v) - (3u)v| = \frac{1}{2}|-6uv| = 3uv.$$

Substituting the expressions for $u$ and $v$,

$$A = 3\left[\frac{\lambda + 1}{2\lambda}\left(x + \frac{y}{3}\right)\right]\left[\frac{\lambda + 1}{2}\left(x - \frac{y}{3}\right)\right].$$

Therefore,

$$A = \frac{3(\lambda + 1)^2}{4\lambda}\left(x^2 - \frac{y^2}{9}\right).$$

But $P$ lies on the hyperbola, so

$$\frac{x^2}{4} - \frac{y^2}{36} = 1.$$

Multiplying by 4,

$$x^2 - \frac{y^2}{9} = 4.$$

Hence

$$A = \frac{3(\lambda + 1)^2}{\lambda} = 3\left(\lambda + 2 + \frac{1}{\lambda}\right).$$

Differentiate:

$$\frac{dA}{d\lambda} = 3\left(1 - \frac{1}{\lambda^2}\right).$$

For a stationary value,

$$1 - \frac{1}{\lambda^2} = 0.$$

Since $\lambda > 0$,

$$\lambda = 1.$$

Also,

$$\frac{d^2A}{d\lambda^2} = \frac{6}{\lambda^3} > 0,$$

so this stationary value is a minimum.

Finally,

$$A_{\min} = 3(1 + 2 + 1) = 12.$$

**Answer:**

$$\boxed{\lambda = 1, \qquad A_{\min} = 12 \text{ square units.}}$$

Thus the minimum occurs when $P$ is the midpoint of $CD$.

NZQA Scholarship Calculus

2016 5) a)

$z^{11} = 1$ so $z = \text{cis}\left(\frac{2\pi n}{11}\right)$ for $n = 0, \dots 10$

and $z^{11} - 1 = 0$. ①

Factorising ① and

$(z-1)(z^{10} + z^9 + z^8 + \text{———} + z + 1) = 0$

So the roots $\text{cis}\left(\frac{2\pi n}{11}\right)$ for $n = 1, \text{———} 10$

are the roots of $z^{10} + \text{———} + 1 = 0$.

The sum of these roots is - the coeff (check - ve. since 10 roots) of $z^9$ - i.e. -1.

So if we take the $\mathbb{R}$ component :-

$\cos\left(\frac{2\pi}{11}\right) + \cos\left(\frac{4\pi}{11}\right) + \cos\left(\frac{6\pi}{11}\right) + \cos\left(\frac{8\pi}{11}\right) + \cos\left(\frac{10\pi}{11}\right)$

$= \quad = \quad = \quad = \quad =$

$+ \cos\left(\frac{20\pi}{2}\right) + \cos\left(\frac{18\pi}{11}\right) + \cos\left(\frac{16\pi}{11}\right) + \cos\left(\frac{14\pi}{11}\right) + \cos\left(\frac{12\pi}{11}\right)$

$= -1$

Since these equate in pairs,

$\cos\left(\frac{2\pi}{11}\right) + \cos\left(\frac{4\pi}{11}\right) + \cos\left(\frac{6\pi}{11}\right) + \cos\left(\frac{8\pi}{11}\right) + \cos\left(\frac{10\pi}{11}\right) = -\frac{1}{2}$

as required.

NZQA Scholarship Calculus
2016 5) b) 1

Constraints are around the number of questions and the time allowed

| | Time | Marks | Marks/min |
|---|---|---|---|
| Group 1 | 2 | 4 | 2 |
| Group 2 | 3 | 5 | $1\frac{2}{3}$ |
| Group 3 | 4 | 6 | $1\frac{1}{2}$ |

So $x_1 + x_2 + x_3 \leq 100$ as given

and $2x_1 + 3x_2 + 4x_3 \leq 210$

$2x_1 + 3x_2 \leq 150$

So $4x_3 \leq 60$ ∴ $x_3 \leq 15$.

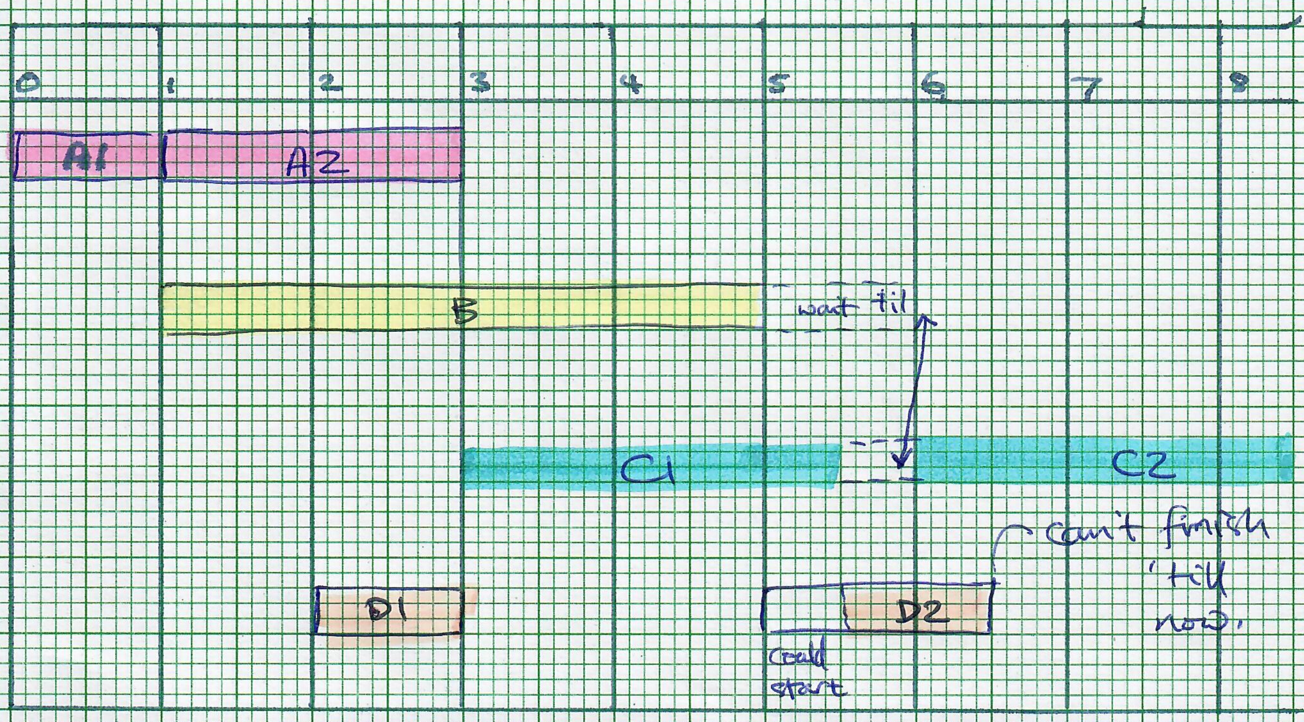
The student wants to maximise the marks

i.e. maximise $4x_1 + 5x_2 + 6x_3$

Consider the marks/min. Clearly the best earner is Group 1, and since there's apparently no limit on them the strategy should be: $2\frac{1}{2}$ hrs answer 75 Group 1 for 300 marks
1 hr ——— 15 Group 3 — 90 marks
giving a total of 390 marks for (75, 0, 15).

NZQA Scholarship Calculus
2016 5) b) ②

This is basically an exercise in drawing a GANTT Chart with every kind of wrinkle...




Critical Path: A, B, wait, C2 = 8.5 hours.