Assessment Schedule – 2020

Scholarship Calculus (93202)

Evidence Statement

Q	Solution
ONE (a)	$\lim_{x\to\infty}\left(\dfrac{3x^2+2x-4}{5x^2+8x-1}\right) = \lim_{x\to\infty}\left(\dfrac{3+\frac{2}{x}-\frac{4}{x^2}}{5+\frac{8}{x}-\frac{1}{x^2}}\right)$ $=\left(\dfrac{3+0-0}{5+0-0}\right)=\dfrac{3}{5}$
(b)	Let $a^4+x^4=u$, then $\dfrac{\mathrm{d}u}{\mathrm{d}x}=4x^3$ and $\mathrm{d}x=\dfrac{\mathrm{d}u}{4x^3}$ The new limits of integration: When $x=0, u=a^4$. When $x=a, u=2a^4$ Integral is now $\displaystyle\int_{a^4}^{2a^4}\frac{x^3}{u^{\frac{1}{2}}}\frac{\mathrm{d}u}{4x^3} = \frac{1}{4}\int_{a^4}^{2a^4} u^{-\frac{1}{2}}\mathrm{d}u$ $=\dfrac{1}{4}\left[2u^{\frac{1}{2}}\right]_{a^4}^{2a^4}$ $=\dfrac{1}{2}\left(\sqrt{2}\,a^2-a^2\right)$ $=\dfrac{a^2}{2}\left(\sqrt{2}-1\right)$
(c)(i)	$\mathrm{a}x^4+\mathrm{b}x^3+\mathrm{c}x^2+\mathrm{d}x+\mathrm{e}=\mathrm{a}(x-\alpha)(x-\beta)(x-\gamma)(x-\delta)$ $=\mathrm{a}\left\{x^4-(\alpha+\beta+\gamma+\delta)x^3+(\beta\gamma+\gamma\alpha+\alpha\beta+\alpha\delta+\beta\delta+\gamma\delta)x^2-(\beta\gamma\delta+\gamma\alpha\delta+\alpha\beta\delta+\alpha\beta\gamma)x+\alpha\beta\gamma\delta\right\}$ *Equating coefficients gives*: $\alpha+\beta+\gamma+\delta=-\dfrac{\mathrm{b}}{\mathrm{a}}$ $\beta\gamma+\gamma\alpha+\alpha\beta+\alpha\delta+\beta\delta+\gamma\delta=\dfrac{\mathrm{c}}{\mathrm{a}}$ $\beta\gamma\delta+\gamma\alpha\delta+\alpha\beta\delta+\alpha\beta\gamma=-\dfrac{\mathrm{d}}{\mathrm{a}}$ $\alpha\beta\gamma\delta=\dfrac{\mathrm{e}}{\mathrm{a}}$

(ii)

Let roots be α, β, γ, and δ

$\alpha+\beta+\gamma+\delta=8$ A

$\beta\gamma+\gamma\alpha+\alpha\beta+\alpha\delta+\beta\delta+\gamma\delta=19$ B

$\beta\gamma\delta+\gamma\alpha\delta+\alpha\beta\delta+\alpha\beta\gamma=-p$ C

$\alpha\beta\gamma\delta=2$ D

Given: $\alpha+\delta=\beta+\gamma$, then from A

$\alpha+\delta=\beta+\gamma=4$ E

Rewriting B as $\beta\gamma+\alpha\delta+(\beta+\gamma)(\alpha+\delta)=19$

We have $\beta\gamma+\alpha\delta=3$

From D we can see that $\beta\gamma$ and $\alpha\delta$ are two numbers that sum to 3 and whose product is 2.

They are therefore the roots of the quadratic equation $w^2-3w+2=0=(w-1)(w-2)$

Let $\alpha\delta=1$; since $\alpha+\delta=4$, α and δ are roots of the equation x^2-4x+1, which has roots $2\pm\sqrt{3}$

Let $\beta\gamma=2$; since $\beta+\gamma=4$, β and γ are roots of the equation x^2-4x+2, which has roots $2\pm\sqrt{2}$

These are the four roots of the equation.

Finally, from C

$-p=\beta\gamma(\alpha+\delta)+\alpha\delta(\beta+\gamma)=12$

$p=-12$

Alternative solutions

After finding $\alpha+\delta=\beta+\gamma=4$ and $\beta\gamma+\alpha\delta=3$, we may find the p value first by working on C instead of B:

$\alpha+\delta=\beta+\gamma=4$

$\beta\gamma(\delta+\alpha)+\alpha\delta(\gamma+\beta)=-p$

$4(\beta\gamma+\alpha\delta)=-p$

$4\times3=-p$

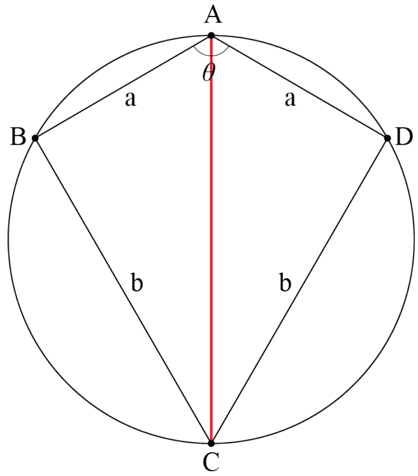
$p=-12$

Q	Solution
TWO (a)	$\left(x^4+\frac{1}{x^4}\right)^2 = x^8+2+\frac{1}{x^8}=49$ $x^8+\frac{1}{x^8}=47$
(b)(i)	$f'(x)=\frac{(2+\sin x)(-\sin x)-\cos x(\cos x)}{(2+\sin x)^2}=-\frac{2\sin x+1}{(2+\sin x)^2}$ The denominator is always positive. $f'(x)=0$ when $2\sin x+1=0$ on $[0,2\pi]$. i.e. $\sin x=-\frac{1}{2}$, giving $x=\frac{7\pi}{6}$ or $x=\frac{11\pi}{6}$ Turning points of $f(x)$ are $\left(\frac{7\pi}{6},-\frac{1}{\sqrt{3}}\right)$ and $\left(\frac{11\pi}{6},\frac{1}{\sqrt{3}}\right)$. (NB! The question does not ask for identification of max / min.)
(ii)	$f''(x)=\frac{-2\cos(2+\sin x)^2+(2\sin x+1)\cdot 2(2+\sin x)\cos x}{(2+\sin x)^4}$ $f''(x)=-\frac{2\cos x(1-\sin x)}{(2+\sin x)^3}$ The inflection points are: $\left(\frac{\pi}{2},0\right)$ and $\left(\frac{3\pi}{2},0\right)$. $(2+\sin x)^3>0$ on the domain and $(1-\sin x)\geq 0$ Hence, $f''(x)>0$ when $\cos x<0$, which is when $\frac{\pi}{2}<x<\frac{3\pi}{2}$; $f''(x)<0$ when $\cos x>0$, which is when $0<x<\frac{\pi}{2}$ and $\frac{3\pi}{2}<x<2\pi$. $f(x)$ is concave-up on $\frac{\pi}{2}<x<\frac{3\pi}{2}$ and concave-down on $0<x<\frac{\pi}{2}$ and $\frac{3\pi}{2}<x<2\pi$.

(c)(i)

Join AC, then $\Delta ABC \equiv \Delta ADC$ since all corresponding sides are the same length.

$\angle B = \angle D = 90°$ since ABCD is cyclic.



$$\cot\left(\frac{\theta}{2}\right) = \frac{a}{b}$$

$$\frac{1+\cos\theta}{\sin\theta} = \frac{1+\cos\left(2\frac{\theta}{2}\right)}{\sin\left(2\frac{\theta}{2}\right)}$$

$$= \frac{1+2\cos^2\left(\frac{\theta}{2}\right)-1}{2\sin\left(\frac{\theta}{2}\right)\cos\left(\frac{\theta}{2}\right)}$$

$$= \cot\left(\frac{\theta}{2}\right)$$

$$\therefore \frac{a}{b} = \frac{1+\cos\theta}{\sin\theta}$$

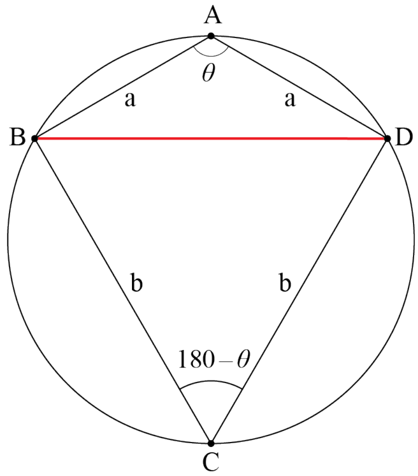
Alternative solution

Using the cosine rule:

$$(\mathrm{BD})^2 = 2a^2 - 2a^2\cos\theta$$

$$\text{or } (\mathrm{BD})^2 = 2b^2 - 2b^2\cos(180-\theta)$$

$$\Rightarrow a^2 - a^2\cos\theta = b^2(1+\cos\theta)$$



$$\frac{a^2}{b^2} = \frac{1+\cos\theta}{1-\cos\theta}$$

$$\frac{a}{b} = \sqrt{\frac{1+\cos\theta}{1-\cos\theta}}$$

$$\frac{a}{b} = \sqrt{\frac{(1+\cos\theta)^2}{1-\cos^2\theta}}$$

$$\frac{a}{b} = \sqrt{\frac{(1+\cos\theta)^2}{\sin^2\theta}}$$ $\sin\theta$ and $1+\cos\theta$ are positive for $0 < \theta < 180°$

$$\therefore \frac{a}{b} = \frac{1+\cos\theta}{\sin\theta}$$

(ii)

$$\cot\frac{\theta}{2} = \frac{\mathrm{a}}{\mathrm{b}} = \frac{\mathrm{a}}{\mathrm{a}/\sqrt{3}} = \sqrt{3}$$

So $\frac{\theta}{2} = \frac{\pi}{6}$ or 30° and $\theta = \frac{\pi}{3}$ or 60°.

Alternative solution

$$\frac{a}{b} = \frac{1+\cos\theta}{\sin\theta} = \sqrt{3}$$

$$1+\cos\theta = \sqrt{3}\sin\theta$$

$$1+\cos^2\theta + 2\cos\theta = 3\sin^2\theta = 3(1-\cos^2\theta)$$

$$4\cos^2\theta + 2\cos\theta - 2 = 0$$

$$(4\cos\theta - 2)(\cos\theta + 1) = 0 \Rightarrow \cos\theta = \frac{1}{2}$$

$\theta = 60°$ only, as $a > b$

or

$$\sqrt{3}\sin\theta - \cos\theta = 1$$

$$2\sin(\theta - 30^0) = 1$$

$$\theta - 30^0 = 30^0$$

Q	Solution
THREE (a)	Since $f''(0)$ exists, $f'(0)$ must exist. $f(0)=b, \lim_{x\to 0^+} f(x) = e^0 + \sin 0 \to b=1$ $f'(x) = \begin{cases} 2x+a & x\le 0 \\ e^x+\cos x & x>0 \end{cases}$ Since $f''(0)$ exists, $\lim_{x\to 0^+} f'(x) = \lim_{x\to 0^-} f'(x)$ $\to a = 1+1=2$
(b)	The gradient of the normal line at any point (x_0,y_0) *is*: $\left[-\dfrac{dx}{dy}\right]_{(x_0,y_0)}$ The equation of the normal line at (x_0,y_0) is: $y-y_0 = \left[-\dfrac{dx}{dy}\right]_{(x_0,y_0)}(x-x_0)$ Since (0,0) is on the line: $-y_0 = \left[-\dfrac{dx}{dy}\right]_{(x_0,y_0)}(-x_0)$, i.e. $y = -\dfrac{dx}{dy}\cdot x$ $\int y\,dy = \int -x\,dx \to \frac{1}{2}x^2 + \frac{1}{2}y^2 = C$ Sub $\left(\sqrt{2},-\sqrt{2}\right)$ in: $C=2$ $x^2+y^2=4$
(c)	$s^2 = l^2 + w^2$ so, at given instant $s^2 = 9^2+12^2$ and $s=15$ Differentiating implicitly $2s\dfrac{ds}{dt} = 2l\dfrac{dl}{dt} + 2w\dfrac{dw}{dt}$ $15\dfrac{ds}{dt} = 12\times 2 + 9\times 3$ $\dfrac{ds}{dt} = 3.4 \text{ cm s}^{-1}$ **Alternative solutions** $x = 12+2t \quad (x = a+2t)$ $y = 9+3t \quad (y=b+3t)$ $s = \sqrt{(12+2t)^2+(9+3t)^2}$ $\dfrac{ds}{dt} = \dfrac{4(12+24)+6(9+3t)}{2\sqrt{(12+2t)^2+(9+3t)^2}}$ $t=0: \dfrac{ds}{dt} = 3.4$ OR $\dfrac{dy}{dx} = \dfrac{3}{2}$ $s = \sqrt{x^2+y^2}$ $\dfrac{ds}{dx} = \dfrac{2x+2y\dfrac{dy}{dx}}{2\sqrt{x^2+y^2}}, \quad x=12, y=9: \dfrac{ds}{dx} = 1.7$ $\dfrac{ds}{dt} = \dfrac{ds}{dx}\cdot\dfrac{dx}{dt} = 1.7\times 2 = 3.4$

(d)	The distance from $P\,(1,0)$ to an arbitrary point $Q\,(x,y)$ on the parabola is given by $l = \sqrt{(x-1)^2 + y^2}$ $= \sqrt{(x-1)^2 + x}$ $= \sqrt{x^2 - x + 1}$ Minimising $\sqrt{x^2 - x + 1}$ on the domain $\left[0,+\infty\right]$ is equivalent to minimising $F(x) = x^2 - x + 1$ $F'(x) = 2x - 1$ $F''(x) = 2$ The only possible turning point of $F(x)$ is at $x = \frac{1}{2}$ Since $F''(x) > 0$, $F(x)$ has a relative min at $x = \frac{1}{2}$. Since $F(x)$ is continuous on the interval $\left[0,+\infty\right]$ and there is only one relative min, this must be an absolute min. So: $y^2 = x = \frac{1}{2}$ and $y = \pm\frac{1}{\sqrt{2}} = \pm\frac{\sqrt{2}}{2}$ The points on the parabola closest to (1,0) are $\left(\frac{1}{2},\frac{\sqrt{2}}{2}\right)$ and $\left(\frac{1}{2},\frac{-\sqrt{2}}{2}\right)$. **Alternative solutions** $l^2 = (x-1)^2 + (y-0)^2 = (y^2-1)^2 + y^2$ $= \left(y^2 - \frac{1}{2}\right)^2 + \frac{3}{4}$ $\min l^2 = \frac{3}{4}$, when $y^2 = \frac{1}{2}$, *i.e.* $l = \frac{\sqrt{3}}{2}$ when $\left(\frac{1}{2},\pm\frac{\sqrt{2}}{2}\right)$

Q	Solution
FOUR (a)	$\frac{\mathrm{d}}{\mathrm{d}x}\left(f(x).g(x)\right) = \lim_{h\to 0}\left[\frac{f(x+h)g(x+h)-f(x)g(x)}{h}\right]$ $= \lim_{h\to 0}\left[\frac{f(x+h)g(x+h)-f(x+h).g(x)+f(x+h).g(x)-f(x)g(x)}{h}\right]$ $= \lim_{h\to 0}\left(\frac{f(x+h).\left(g(x+h)-g(x)\right)}{h}\right)+\lim_{h\to 0}\left(\frac{g(x).\left(f(x+h)-f(x)\right)}{h}\right)$ $= \lim_{h\to 0} f(x+h).\lim_{h\to 0}\left(\frac{g(x+h)-g(x)}{h}\right)+g(x).\lim_{h\to 0}\left(\frac{f(x+h)-f(x)}{h}\right)$ $= f(x).\frac{\mathrm{d}g(x)}{\mathrm{d}x}+\frac{\mathrm{d}f(x)}{\mathrm{d}x}.g(x)$
(b)(i)	$\int \mathrm{e}^{-x}\cos x\,\mathrm{d}x = -\mathrm{e}^{-x}\cos x-\int \mathrm{e}^{-x}\sin x\,\mathrm{d}x+c$ $= -\mathrm{e}^{-x}\cos x-\left[-\mathrm{e}^{-x}\sin x-\int -\mathrm{e}^{-x}\cos x\mathrm{d}x\right]+C$ $= -\mathrm{e}^{-x}\cos x+\mathrm{e}^{-x}\sin x-\int \mathrm{e}^{-x}\cos x\,\mathrm{d}x+C$ so $2\int \mathrm{e}^{-x}\cos x\,\mathrm{d}x = -\mathrm{e}^{-x}\cos x+\mathrm{e}^{-x}\sin x+C$ $\int \mathrm{e}^{-x}\cos x\,\mathrm{d}x = \frac{1}{2}\left(\mathrm{e}^{-x}\sin x-\mathrm{e}^{-x}\cos x\right)+C$ OR $\int \mathrm{e}^{-x}\cos x\,\mathrm{d}x = \mathrm{e}^{-x}\sin x-\int -\mathrm{e}^{-x}\sin x\,\mathrm{d}x+c$ $= \mathrm{e}^{-x}\sin x+\left[-\mathrm{e}^{-x}\cos x-\int \mathrm{e}^{-x}\cos x\mathrm{d}x\right]+C$ $= \mathrm{e}^{-x}\sin x-\mathrm{e}^{-x}\cos x-\int \mathrm{e}^{-x}\cos x\,\mathrm{d}x+C$ so $2\int \mathrm{e}^{-x}\cos x\,\mathrm{d}x = \mathrm{e}^{-x}\sin x-\mathrm{e}^{-x}\cos x+C$ $\int \mathrm{e}^{-x}\cos x\,\mathrm{d}x = \frac{1}{2}\left(\mathrm{e}^{-x}\sin x-\mathrm{e}^{-x}\cos x\right)+C'$
(ii)	Integration intervals for area are $\left[0,\frac{\pi}{2}\right],\left[\frac{\pi}{2},\frac{3\pi}{2}\right]$ and $\left[\frac{3\pi}{2},2\pi\right]$ Area $=\left\|\int_0^{\frac{\pi}{2}} \mathrm{e}^{-x}\cos x\,\mathrm{d}x\right\|+\left\|\int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} \mathrm{e}^{-x}\cos x\,\mathrm{d}x\right\|+\left\|\int_{\frac{3\pi}{2}}^{2\pi} \mathrm{e}^{-x}\cos x\,\mathrm{d}x\right\|$ $=\left\|\frac{1}{2}\mathrm{e}^{-x}(\sin x-\cos x)\Big\|_0^{\frac{\pi}{2}}\right\|+\left\|\frac{1}{2}\mathrm{e}^{-x}(\sin x-\cos x)\Big\|_{\frac{\pi}{2}}^{\frac{3\pi}{2}}\right\|+\left\|\frac{1}{2}\mathrm{e}^{-x}(\sin x-\cos x)\Big\|_{\frac{3\pi}{2}}^{2\pi}\right\|$ $=\frac{1}{2}\left[\left\|\left(\mathrm{e}^{-\frac{\pi}{2}}(1-0)\right)-\left(\mathrm{e}^{0}(0-1)\right)\right\|+\left\|\left(\mathrm{e}^{-\frac{3\pi}{2}}(-1-0)\right)-\left(\mathrm{e}^{-\frac{\pi}{2}}(1-0)\right)\right\|+\left\|\left(\mathrm{e}^{-2\pi}(0-1)\right)-\left(\mathrm{e}^{-\frac{3\pi}{2}}(-1-0)\right)\right\|\right]$ $=\frac{1}{2}\left[\left(\mathrm{e}^{-\frac{\pi}{2}}+1\right)+\left(\mathrm{e}^{-\frac{3\pi}{2}}+\mathrm{e}^{-\frac{\pi}{2}}\right)+\left(-\mathrm{e}^{-2\pi}+\mathrm{e}^{-\frac{3\pi}{2}}\right)\right]$ $=\frac{1}{2}\left[1+2\mathrm{e}^{-\frac{\pi}{2}}+2\mathrm{e}^{-\frac{3\pi}{2}}-\mathrm{e}^{-2\pi}\right]$

(c)	$xy+e^y=2x+1$ $x\dfrac{dy}{dx}+y+e^y\dfrac{dy}{dx}=2$ $\dfrac{dy}{dx}=\dfrac{2-y}{x+e^y}$ $\dfrac{d^2y}{dx^2}=\dfrac{-(x+e^y)\dfrac{dy}{dx}-(2-y)\left(1+e^y\dfrac{dy}{dx}\right)}{\left(x+e^y\right)^2}$ When $x=0$, $e^y=1\Rightarrow y=0$ and $\dfrac{dy}{dx}=\dfrac{2-0}{0+1}=2$ Hence $\dfrac{d^2y}{dx^2}=\dfrac{-(x+e^y)\dfrac{dy}{dx}-(2-y)\left(1+e^y\dfrac{dy}{dx}\right)}{\left(x+e^y\right)^2}$ $=\dfrac{-(0+1)2-(2-0)(1+1\times 2)}{(0+1)^2}$ $=-8$ **Alternative solutions** $\left(y+xy'\right)+e^y y'=2$ $y'+\left(y'+xy''\right)+\left(e^y\left(y'\right)^2+e^y y''\right)=0$ sub $x=0$ into original and the first derivative: $y=0,\ y'=2$ $4+4+y''=0$ $y''=-8$

Q	Solution
FIVE (a)(i)	$e^{i\theta}+e^{-i\theta}=\cos\theta+i\sin\theta+\cos\theta-i\sin\theta$ $e^{i\theta}+e^{-i\theta}=2\cos\theta$ $\cos\theta=\dfrac{e^{i\theta}+e^{-i\theta}}{2}$
(ii)	$e^{i\theta}-e^{-i\theta}=\cos\theta+i\sin\theta-(\cos\theta-i\sin\theta)$ $e^{i\theta}-e^{-i\theta}=2i\sin\theta$ $\sin\theta=\dfrac{e^{i\theta}-e^{-i\theta}}{2i}$ $\sin^3\theta=\left(\dfrac{e^{i\theta}-e^{-i\theta}}{2i}\right)^3$ $=\dfrac{\left(e^{i\theta}\right)^3-3\left(e^{i\theta}\right)^2\left(e^{-i\theta}\right)+3\left(e^{i\theta}\right)\left(e^{-i\theta}\right)^2-\left(e^{-i\theta}\right)^3}{8i^3}$ $=\dfrac{-1}{8i}\left(e^{3i\theta}-3e^{i\theta}+3e^{-i\theta}-e^{-3i\theta}\right)$ $=\dfrac{-1}{8i}\left(-3e^{i\theta}+3e^{-i\theta}\right)+\dfrac{-1}{8i}\left(e^{3i\theta}-e^{-3i\theta}\right)$ $=\dfrac{3}{4}\left(\dfrac{e^{i\theta}-e^{-i\theta}}{2i}\right)-\dfrac{1}{4}\left(\dfrac{e^{3i\theta}-e^{-3i\theta}}{2i}\right)$ $=\dfrac{3}{4}\sin\theta-\dfrac{1}{4}\sin(3\theta)$ **Alternative solutions** $\sin(3\theta)=\sin(\theta+2\theta)$ $=\sin\theta\cos(2\theta)+\cos\theta\sin(2\theta)$ $=\sin\theta(1-2\sin^2\theta)+2\sin\theta(1-\sin^2\theta)$ $\therefore\sin^3\theta=\dfrac{3}{4}\sin\theta-\dfrac{1}{4}\sin(3\theta)$ OR $\sin 3\theta=\mathrm{Im}(e^{i3\theta})=\mathrm{Im}\left((e^{i\theta})^3\right)=\mathrm{Im}(\cos\theta+i\sin\theta)^3$ $(\cos\theta+i\sin\theta)^3=(\cos^3\theta-3\cos\theta\sin^2\theta)+i(3\cos^2\theta\sin\theta-\sin^3\theta)$ $\therefore\sin 3\theta=3(1-\sin^2\theta)\sin\theta-\sin^3\theta$ $\therefore\sin^3\theta=\dfrac{3}{4}\sin\theta-\dfrac{1}{4}\sin 3\theta$

(b)	$\int\left(e^{ix}\cdot e^{x}\right)dx=\int\left(\cos x\cdot e^{x}+i\sin x\cdot e^{x}\right)dx$ Hence, $\int\cos x\cdot e^{x}\,dx=\mathrm{Re}\left(\frac{1}{i+1}e^{(i+1)x}\right)$ $\mathrm{Re}\left(\frac{1}{i+1}e^{(i+1)x}\right)=\mathrm{Re}\left(\frac{e^{x}(\cos x+i\sin x)}{i+1}\right)$ $=e^{x}\mathrm{Re}\left(\frac{(\cos x+i\sin x)(1-i)}{(1+i)(1-i)}\right)$ $=\frac{1}{2}e^{x}\mathrm{Re}\left((\cos x+\sin x)+i(\sin x-\cos x)\right)$ $=\frac{1}{2}e^{x}(\cos x+\sin x)$ **Alternative solution** $\cos x=\frac{e^{ix}-e^{-ix}}{2}$ So $\int e^{x}\cos x\,dx=\frac{1}{2}\int e^{(i+1)x}+e^{(1-i)x}\,dx$ $=\frac{1}{2}\left[\frac{e^{(i+1)x}}{i+1}+\frac{e^{(1-i)x}}{1-i}\right]$ $=\frac{1}{2}\left[\frac{(1-i)e^{ix}e^{x}+(i+1)e^{-ix}e^{x}}{(i+1)(1-i)}\right]$ $=\frac{e^{x}}{2}\left[\frac{(e^{ix}-ie^{-ix}+ie^{-ix}+e^{-ix}}{2}\right]$ $=\frac{e^{x}}{2}\left[\frac{e^{ix}+e^{-ix}}{2}+\frac{ie^{-ix}-ie^{ix}}{2}\right]$ Note $\frac{e^{ix}-ie^{-ix}}{2i}=\sin x$ so $\frac{ie^{-ix}-ie^{-ix}}{2}=\sin x$ after multiplication by $\frac{-i}{-i}$ so $I=\frac{e^{x}}{2}\left[\cos x+\sin x\right]+c$

(c)	$w=\frac{\mathrm{i}+z}{\mathrm{i}-z}$ $w(\mathrm{i}-z)=\mathrm{i}+z$ $\mathrm{i}w-\mathrm{i}=z(\mathrm{i}+w)$ $z=\frac{\mathrm{i}w-\mathrm{i}}{1+w}$ $\|z\|=1\to\|\mathrm{i}w-\mathrm{i}\|=\|1+w\|$ $\|\mathrm{i}w-\mathrm{i}\|=\|w-1\|$ $\|w-1\|=\|w+1\|$ Therefore, the image is the perpendicular bisector of the lines $x=1$ and $x=-1$, i.e. the y-axis. **Alternative solutions** Let $z=x+\mathrm{i}y$ $w=\frac{z+\mathrm{i}}{\mathrm{i}-z}=\frac{x+\mathrm{i}y+\mathrm{i}}{\mathrm{i}-(x+\mathrm{i}y)}$ $=\frac{x+(y+1)\mathrm{i}}{-x+(1-y)\mathrm{i}}\times\frac{-x-1-y\mathrm{i}}{-x-(1-y)\mathrm{i}}$ $=\frac{-x^2-(y+1)x\mathrm{i}-(1-y)x\mathrm{i}+(1-y)(y+1)}{x^2+(1-y)^2}$ $=\frac{-x^2-y^2+1-2x\mathrm{i}}{x^2+y^2+1-2y}$ Now as $\|z\|=1$, we have $x^2+y^2=1$ *so* $w=\frac{-2x\mathrm{i}}{2-2y}$ $=\left(\frac{x}{y-1}\right)\mathrm{i}$ $=k\mathrm{i}$ for some k As $-1\le x\le 1$ and $-1\le y\le 1$, we can get all possible real values of k so the locus of w is the y-axis.

(d)	Subtracting (1) from (2), we get: $y^2 - x^2 + yz - zx = (y-x)(x+y+z) = 1$ Subtracting (2) from (3), we get: $z^2 - y^2 + zx - xy = (z-y)(x+y+z) = 1$ So $y - x = \frac{1}{x+y+z} = z - y$ x, y, and z form an arithmetic progression, common difference $d = \frac{1}{x+y+z}$ Using the second given equation: $y^2 - (y-d)(y+d) = y^2 - \left(y^2 - d^2\right) = d^2 = 2$ So $d = \pm\sqrt{2}$ Since x, y, and z are in arithmetic progression, $y = \frac{x+y+z}{3}$ But $d = \frac{1}{x+y+z}$, so $y = \frac{1}{3d} = \frac{\pm 1}{3\sqrt{2}}$ Which gives $x = \frac{1}{3\sqrt{2}} - \sqrt{2} = \frac{-5}{3\sqrt{2}}$, $y = \frac{1}{3\sqrt{2}}$, and $z = \frac{1}{3\sqrt{2}} + \sqrt{2} = \frac{7}{3\sqrt{2}}$ Or $x = \frac{-1}{3\sqrt{2}} + \sqrt{2} = \frac{5}{3\sqrt{2}}$, $y = \frac{-1}{3\sqrt{2}}$, and $z = \frac{-1}{3\sqrt{2}} - \sqrt{2} = \frac{-7}{3\sqrt{2}}$ **Alternative solutions** $x + z = 2y$: Let $x = y + d,\ y,\ z = y - d$ (I) Sub (I) into (2): $y^2 - (y+d)(y-d) = 2 \rightarrow d = \pm\sqrt{2}$ Sub (I) into (1): $3yd = 1 \rightarrow y = \pm\frac{1}{3\sqrt{2}}$

Sufficiency Statement

Score 1–4, no award	Score 5–6, Scholarship	Score 7–8, Oustanding Scholarship
Shows understanding of relevant mathematical concepts, and some progress towards solution to problems.	Application of high-level mathematical knowledge and skills, leading to partial solutions to complex problems.	Application of high-level mathematical knowledge and skills, perception, and insight / convincing communication shown in finding correct solutions to complex problems.

Cut Scores

Scholarship	Outstanding Scholarship
22 – 34	35 – 40