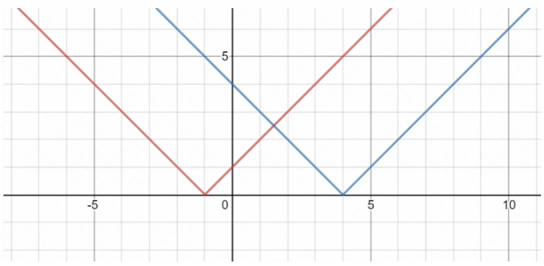
Assessment Schedule – 2019

Scholarship Calculus (93202)

Evidence Statement

Q	Solution
ONE (a)	x is real where $x \neq \pm 1$
(b)	None! Since $(x-2)^2 \geq 0$ for all real x, $-(x-2)^2 \leq 0$ and ln α is defined only for $\alpha > 0$.
(c)(i)	$6 \times \frac{5!}{2!(5-2)!} \times 1 = 60$ or $6 \times \binom{5}{4} = 60$
(ii)	$\frac{6!}{2!(6-2)!} \times \frac{4!}{2!(4-2)!} \times 1 = 90$ or $\binom{6}{2} \times \binom{4}{2} = 90$
(d)	CASE 1: $x \geq 4$, $x+1$ and $x-4$ are both positive, so $(x+1)-(x-4) \geq 1$ $5 \geq 1$ This is true for all x and $x \geq 4$ is a possible solution. CASE 2: $-1 \leq x \leq 4$, and so $x+1$ is positive, $x-4$ is negative, but $(x+1)+(x-4) \geq 1$ $2x-3 \geq 1$ $x \geq 2$ Which extends the initial solution. CASE 3: $x < 1$; $x+1$, $x-4$ both negative $-(x+1)+(x-4) \geq 1$ $-5 \geq 1$ This is false for all real values of x. So the solution to $\lvert x+1 \rvert - \lvert x-4 \rvert \geq 1$ is $x \geq 2$ Or: $y = \lvert x+1 \rvert$ $y = \lvert x-4 \rvert$  $x + 1 - (4 - x) \geq 1$ $x \geq 2$

(e)	$\sin^4 A+\cos^4 A=\frac{2}{3}$ $\sin^4 A+2\sin^2 A\cos^2 A+\cos^4 A=\frac{2}{3}+2\sin^2 A\cos^2 A$ $\left(\sin^2 A+\cos^2 A\right)^2=\frac{2}{3}+2\sin^2 A\cos^2 A$ $1-\frac{2}{3}=2\sin^2 A\cos^2 A$ $\frac{1}{3}=\frac{1}{2}\left(4\sin^2 A\cos^2 A\right)$ $\frac{2}{3}=\left(2\sin A\cos A\right)^2$ $\pm\sqrt{\frac{2}{3}}=\sin 2A$ Since $90° < A < 180°$, then $180° < 2A < 360°$. Therefore, $\sin 2A < 0$. We consider only the negative solution, i.e. $\sin 2A=-\sqrt{\frac{2}{3}}=-\frac{\sqrt{6}}{3}=-\frac{2}{\sqrt{6}}$

Q	Solution
TWO (a)	Multiplying through by $\sqrt{x}\times\sqrt{x-2}$ gives $x-2-x=\frac{k}{4}\sqrt{x}\times\sqrt{x-2}$ Squaring both sides: $4=\frac{k^2}{16}x(x-2)$ or $k^2x^2-2k^2x-64=0$ Using discriminant $\Delta=4k^4+256k^2\geq 0$ for all real k, there are no real values of k for which the equation will have imaginary roots.
(b)	$\log(x^2+y^2)=\log 130$, so $x^2+y^2=130$ A $\log\frac{x+y}{x-y}=\log 8$, so $7x-9y=0$ and $x=\frac{9y}{7}$ B Sub B into A: $\frac{81y^2}{49}+y^2=130$ and $y=\pm 7$, and $x=\pm 9$. However, for the logarithms to be valid, we require $\frac{x+y}{x-y}>0$, therefore $x=9$, $y=7$
(c)	Using symmetry, the required area equals the area of 1 quadrant multiplied by 4. In the first quadrant, $y=x\sqrt{1-x^2}$ Area $=4\int_0^1 x\sqrt{1-x^2}\,\mathrm{d}x$ $=-2\int_0^1(-2x)\sqrt{1-x^2}\,\mathrm{d}x$ $=-2\left[\left(\frac{2}{3}\right)\left(1-x^2\right)^{\frac{3}{2}}\right]_0^1$ $=\frac{-4}{3}\left(\left(0\right)^{\frac{3}{2}}-\left(1\right)^{\frac{3}{2}}\right)$ $=\frac{-4}{3}\times\left(-1\right)=\frac{4}{3}$
(d)	Consider the following areas: Let area of $ABC=X_1$, area of $BCP=X_2$, area of $CAP=X_3$, and area of $ABP=X_4$ Then $\frac{X_2}{X_1}=\frac{\frac{1}{2}\times BC\times\text{altitude from } P \text{ to } BC}{\frac{1}{2}\times BC\times\text{altitude from } A \text{ to } BC}$ $=\frac{\text{altitude from } P \text{ to } BC}{\text{altitude from } A \text{ to } BC}$ $=\frac{A'P}{A'A}$ (similar triangles) Similarly, $\frac{X_3}{X_1}=\frac{PB'}{BB'}$ and $\frac{X_4}{X_1}=\frac{PC'}{CC'}$ Adding gives: $\frac{A'P}{AA'}+\frac{PB'}{BB'}+\frac{PC'}{CC'}=\frac{X_2}{X_1}+\frac{X_3}{X_1}+\frac{X_4}{X_1}=\frac{X_2+X_3+X_4}{X_1}=1$

Q	Solution
THREE (a)	$f'(4) = \lim_{h\to 0}\left[\dfrac{\left((4+h)^2-4(4+h)+3\right)^2-9}{h}\right]$ $= \lim_{h\to 0}\left[\dfrac{\left(h^2+4h+3\right)^2-9}{h}\right]$ $= \lim_{h\to 0}\left[\dfrac{h^4+8h^3+22h^2+24h+9-9}{h}\right]$ Or: $\lim_{h\to 0}\left[\dfrac{\left(h^2+4h\right)\left(h^2+4h+6\right)}{h}\right]$ $= \lim_{h\to 0}\left[h^3+8h^2+22h+24\right]$ Or: $\lim_{h\to 0}(h+4)\left(h^2+4h+6\right)$ $= 24$
(b)	At time t, $\left[x(t)\right]^2+\left[y(t)\right]^2=25$. Differentiating gives $2x(t)\times\dfrac{\mathrm{d}x}{\mathrm{d}t}+2y(t)\times\dfrac{\mathrm{d}y}{\mathrm{d}t}=0$ If at $t=t_0$, $x(t_0)=3$, $y(t_0)=4$, and $y'(t_0)=-2$, then $2\times 3\times x'(t_0)+2\times 4\times(-2)=0$ and $x'(t_0)=\dfrac{8}{3}$ Or: $x^2+y^2=25$ $2x+2y\dfrac{dy}{dx}=0$ $\left.\dfrac{dy}{dx}\right\|_{(3,4)}=-\dfrac{3}{4}$ $\dfrac{dx}{dt}=\dfrac{dx}{dy}\cdot\dfrac{dy}{dt}$ $=-\dfrac{4}{3}\times -2=\dfrac{8}{3}$

(c) Since the diagonal of the lake is 2 km, and half-way between the huts, A is 1 km from the lake, and:

$AC = DB$ and $CD \parallel AB$

$AC^2 = (1+x)^2 + x^2$

$AC = \sqrt{(1+x)^2 + x^2}$

And $CD = 2 - 2x$

Let the time taken $= T = \dfrac{2 \times AC}{3} + \dfrac{CD}{2.5}$

$$T = \frac{2\sqrt{(1+x)^2 + x^2}}{3} + \frac{2-2x}{2.5}$$

$$\frac{dT}{dx} = \frac{2 \times 0.5(2+4x)}{3\sqrt{(1+x)^2 + x^2}} - \frac{4}{5}$$

When $\dfrac{dT}{dx} = 0$

$6\sqrt{(1+x)^2 + x^2} = 2.5(2+4x)$

$36(1+2x+2x^2) = 25 + 100x + 100x^2$

$28x^2 + 28x - 11 = 0$

$x = -\dfrac{1}{2} + \dfrac{3}{\sqrt{14}} = 0.3018$ km or $x = -1.3018$. Ignore the negative result.

$x = 0.3018$ km is the x coordinate for a stationary point.

To show a local minimum: check the second derivative or use the first derivative and interval test.

$$T'' = \frac{2(2x^2+2x+1) - (2x+1)^2}{3(2x^2+2x+1)\sqrt{2x^2+2x+1}}$$

$T''(x = 0.3018) = 0.2794 > 0$

Or

x	0.2	0.3018	0.4
$\dfrac{dT}{dx} = \dfrac{2 \times 0.5(2+4x)}{3\sqrt{(1+x)^2 + x^2}} - \dfrac{4}{5}$	–0.0328	0	0.0242
Gradient	< 0	$= 0$	> 0

Check the end points to show it's the actual minimum over the domain:

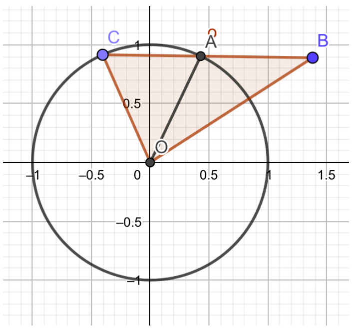
$x = 0$: $T = 1.467$

$x = 1$: $T = 1.491$

$x = 0.3018$: $T = 1.449$

Q	Solution
FOUR (a)	$\left[p(m)\right]^2 = a^2 - \frac{2a(a-b)}{t_p}t + \frac{(a-b)^2}{t_p^{\ 2}}t^2$ $p(f) = 1 - p(m) = 1 - \left(a - \frac{a-b}{t_p}t\right) = (1-a) + \frac{a-b}{t_p}t$ $\left[p(f)\right]^2 = 1 - 2a + a^2 + \frac{2(a-b)}{t_p}t - \frac{2a(a-b)}{t_p}t + \frac{(a-b)^2}{t_p^{\ 2}}t^2$ $T = \frac{1}{t_p}\int_0^{t_p}\left\{1 - 2a + 2a^2 + \frac{2(a-b)}{t_p}t - \frac{4a(a-b)}{t_p}t + \frac{2(a-b)^2}{t_p^{\ 2}}t^2\right\}\mathrm{d}t$ $= \frac{1}{t_p}\left[t - 2at + 2a^2t + \frac{2(a-b)}{2t_p}t^2 - \frac{4a(a-b)}{2t_p}t^2 + \frac{2(a-b)^2}{3t_p^{\ 2}}t^3\right]_0^{t_p}$ $= 1 - 2a + 2a^2 + (a-b) - 2a(a-b) + \frac{2}{3}(a-b)^2$ $= 1 - a + b(2a-1) + \frac{2}{3}(a-b)^2$ Or: Use reversed chain rule or substitution $T = \frac{1}{t_p}\int_0^{t_p}\left[a - \frac{(a-b)}{t_p}t\right]^2 + \left[1 - a + \frac{(a-b)}{t_p}t\right]^2 \mathrm{d}t$ $= \frac{1}{t_p}\times\frac{1}{3}\left\{\left[a - \frac{(a-b)}{t_p}t\right]^3 \times \frac{-t_p}{a-b} + \left[1 - a + \frac{(a-b)}{t_p}t\right]^3 \times \frac{t_p}{a-b}\right\}_0^{t_p}$ $= \frac{1}{3(a-b)}\left[\left(-b^3 + (1-b)^3\right) - \left(-a^3 + (1-a)^3\right)\right]$ $= 1 - a + b(2a-1) + \frac{2}{3}(a-b)^2$

(b)	$y = ux$ $\dfrac{\mathrm{d}y}{\mathrm{d}x} = \dfrac{\mathrm{d}u}{\mathrm{d}x}x + u$ $4x^2\left(\dfrac{\mathrm{d}u}{\mathrm{d}x}x + u\right) = (ux)^2 - 2x(ux)$ $4x^3\dfrac{\mathrm{d}u}{\mathrm{d}x} + 4x^2u = u^2x^2 - 2ux^2$ $4x\dfrac{\mathrm{d}u}{\mathrm{d}x} = u^2 - 6u$ $\int \dfrac{4\mathrm{d}u}{u(u-6)} = \int \dfrac{\mathrm{d}x}{x}$ Now $\dfrac{4}{u(u-6)} = \dfrac{A}{u} + \dfrac{B}{u-6}$ $4 = A(u-6) + Bu$ Hence $-6A = 4 \Rightarrow A = -\dfrac{2}{3}$ and $Au + Bu = 0 \Rightarrow B = \dfrac{2}{3}$ Substituting into the separated DE $\int \dfrac{-2}{3u}\mathrm{d}u + \int \dfrac{2}{3(u-6)}\mathrm{d}u = \int \dfrac{\mathrm{d}x}{x}$ $\dfrac{-2}{3}\ln\lvert u\rvert + \dfrac{2}{3}\ln\lvert u-6\rvert = \ln\lvert x\rvert + c$ But $u = \dfrac{y}{x}$. A general solution is: $\dfrac{-2}{3}\ln\left\lvert \dfrac{y}{x}\right\rvert + \dfrac{2}{3}\ln\left\lvert \dfrac{y}{x} - 6\right\rvert = \ln\lvert x\rvert + c$ (or equivalent) $\ln\left\lvert \dfrac{\frac{y}{x} - 6}{\frac{y}{x}} \right\rvert^{\frac{2}{3}} = \ln\lvert kx\rvert$ $\left\lvert \dfrac{\frac{y}{x} - 6}{\frac{y}{x}} \right\rvert^{\frac{2}{3}} = kx$ $\dfrac{\frac{y}{x} - 6}{\frac{y}{x}} = Ax^{\frac{3}{2}}$ $y = \dfrac{6x}{1 - Ax^{\frac{3}{2}}}$ Given $f(1) = -6$ $-6 = \dfrac{6}{1-A}$ so $A = 2$, from which $f(4) = -1.6$

Q	Solution
FIVE (a)	$\frac{w-1}{w+1}=\frac{\cos\theta+\mathrm{i}\sin\theta-1}{\cos\theta+\mathrm{i}\sin\theta+1}$ $=\frac{\cos\theta-1+\mathrm{i}\sin\theta}{\cos\theta+1+\mathrm{i}\sin\theta}$ $=\frac{-2\sin^2\frac{\theta}{2}+\mathrm{i}2\sin\frac{\theta}{2}\cos\frac{\theta}{2}}{2\cos^2\frac{\theta}{2}+\mathrm{i}2\sin\frac{\theta}{2}\cos\frac{\theta}{2}}$ $=\frac{2\sin\frac{\theta}{2}}{2\cos\frac{\theta}{2}}\left(\frac{-\sin\frac{\theta}{2}+\mathrm{i}\cos\frac{\theta}{2}}{\cos\frac{\theta}{2}+\mathrm{i}\sin\frac{\theta}{2}}\right)$ $=\tan\frac{\theta}{2}\left(\frac{\left(-\sin\frac{\theta}{2}+\mathrm{i}\cos\frac{\theta}{2}\right)\left(\cos\frac{\theta}{2}-\mathrm{i}\sin\frac{\theta}{2}\right)}{\left(\cos\frac{\theta}{2}+\mathrm{i}\sin\frac{\theta}{2}\right)\left(\cos\frac{\theta}{2}-\mathrm{i}\sin\frac{\theta}{2}\right)}\right)$ $=\tan\frac{\theta}{2}\times\mathrm{i}\left(\frac{\cos^2\frac{\theta}{2}+\sin^2\frac{\theta}{2}}{\cos^2\frac{\theta}{2}+\sin^2\frac{\theta}{2}}\right)$ $=\mathrm{i}\tan\frac{\theta}{2}$ Or:  $\angle BOC=90°$ $\angle CBO=\frac{1}{2}\theta$
(b)(i)	$\tan\phi=\frac{b\sin\theta}{a\cos\theta}=\frac{b}{a}\tan\theta$

(ii)	Let $f(\theta)=\tan(\theta-\phi)$. Since $f(\theta)$ increases as $\theta-\phi$ increases, we shall now maximise $f(\theta)$. $f(\theta)=\dfrac{\tan\theta-\tan\phi}{1+\tan\theta\tan\phi}=\dfrac{\tan\theta-\frac{b}{a}\tan\theta}{1+\tan\theta\frac{b}{a}\tan\theta}=\dfrac{\tan\theta(a-b)}{a+b\tan^2\theta}$ $f'(\theta)=\dfrac{(a-b)\sec^2\theta(a+b\tan^2\theta)-(a-b)\tan\theta 2b\tan\theta\sec^2\theta}{(a+b\tan^2\theta)^2}$ $f'(0)=0$ when the numerator $=0$ $(a-b)\sec^2\theta(a+b\tan^2\theta)-(a-b)\tan\theta\ 2b\tan\theta\sec^2\theta=0$ $(a-b)\sec^2\theta\left[a+b\tan^2\theta-2b\tan^2\theta\right]=0$ $\sec^2\theta\neq 0$ for any θ and $a\neq b$; hence we require a value(s) for θ where $a-b\tan^2\theta=0$ $\tan\theta=\pm\sqrt{\dfrac{a}{b}}$ Since $\theta-\phi=0$ at the x and y intercepts, $\theta=\tan^{-1}\left(\pm\sqrt{\dfrac{a}{b}}\right)$ is where $\lvert\theta-\phi\rvert$ is max $\phi=\tan^{-1}\pm\sqrt{\dfrac{b}{a}}$

Sufficiency Statement

Score 1–4, no award	Score 5–6, Scholarship	Score 7–8, Oustanding Scholarship
Shows understanding of relevant mathematical concepts, and some progress towards solution to problems.	Application of high-level mathematical knowledge and skills, leading to partial solutions to complex problems.	Application of high-level mathematical knowledge and skills, perception, and insight / convincing communication shown in finding correct solutions to complex problems.

Cut Scores

Scholarship	Outstanding Scholarship
21 – 33	34 – 40