NZQA Scholarship Calculus
2009 1) a)

This would be more interesting if it said 'find...' but there we go.

$$\frac{d^2y}{dx^2} = -k^2y.$$

Try $y = a\sin kx + b\cos kx$ —— ①

$$\frac{dy}{dx} = ak\cos kx - bk\sin kx$$

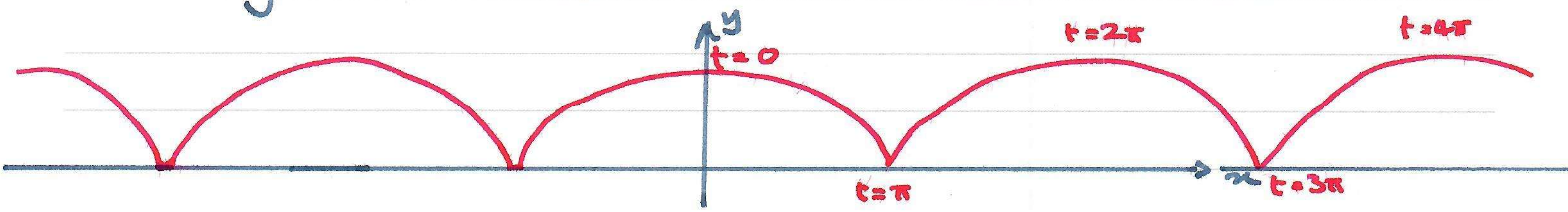
$$\frac{d^2y}{dx^2} = -ak^2\sin kx - bk^2\cos kx \quad \text{—— ②}$$

So clearly taking ② = $-k^2$① we get the result required:

$$\underline{\frac{d^2y}{dx^2} = -k^2y.}$$

NZQA Scholarship Calculus
2009 1) b)

$$x = t + \sin t$$
$$y = 1 + \cos t$$



(The question assumes the parameter t is the same as time)

speed s (unusual usage):

$$s(t) = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}$$

$$\frac{dx}{dt} = 1 + \cos t$$

$$\frac{dy}{dt} = -\sin t$$

So $$\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 = 1 + 2\cos t + \underline{\cos^2 t + \sin^2 t}$$
$$= 2(1 + \cos t)$$
$$= 2\left(2\cos^2 \frac{t}{2}\right)$$

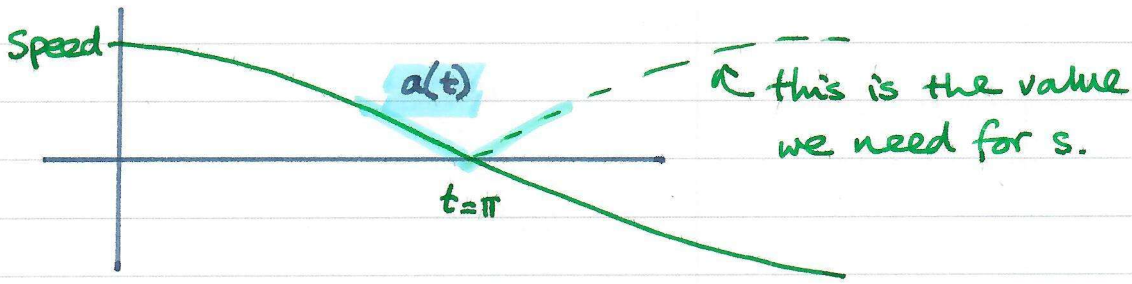
So $s(t) = 2\cos\frac{t}{2}$

$$a(t) = -2 \cdot \frac{1}{2} \sin\frac{t}{2} = -\sin\frac{t}{2}$$

This appears to be all continuous. but the puzzle is in the | | of speed and acceleration, since neither can be negative. So we need to check signs.

So as $t \to \pi-$, $a(t) = -1$

And as $t \to \pi+$, $a(t) = 1$.



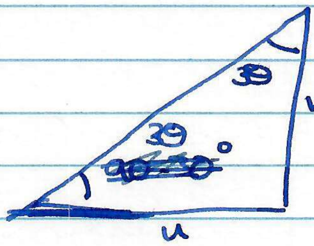
From this it's clear $a(t)$ is not continuous through the turning point.

1) NZQA 2009

$$(1+\operatorname{cis} 2\theta)(1+\operatorname{cis} 4\theta) = u+iv.$$

Show $\frac{u}{v} = \cot 3\theta$ $\qquad \frac{v}{u} = \tan 3\theta.$

This is essentially saying show $\arg(x) = \cancel{90°}\ 3\theta.$



$(1+\operatorname{cis} 2\theta)(1+\operatorname{cis} 4\theta)$

$= (1+\cos 2\theta + i\sin 2\theta)(1+\cos 4\theta + i\sin 4\theta)$

$= (1+\cos 2\theta)(1+\cos 4\theta) - \sin 2\theta \sin 4\theta$ $\qquad \mathbb{R}$

$\quad + i\left(\sin 2\theta(1+\cos 4\theta) + \sin 4\theta(1+\cos 2\theta)\right)$ $\qquad \mathbb{I}$

$= 1 + \cos 2\theta + \cos 4\theta + \cos 2\theta\cos 4\theta - \sin 2\theta \sin 4\theta$

$\quad + i\left(\sin 2\theta + \sin 4\theta + \sin 2\theta\cos 4\theta + \sin 4\theta \cos 2\theta\right)$

Using the trig. identities:

$= 1 + 2\cos 3\theta\cos\theta + \cos 6\theta$

$\quad + i\left(2\sin 3\theta \cos\theta + \sin 6\theta\right)$

$$= 1 + 2\cos 3\theta \cos\theta + \cos^2 3\theta - \sin^2 3\theta$$
$$+ i(2\sin 3\theta\cos\theta + 2\cos 3\theta \sin 3\theta)$$
$$= (1-\sin^2 3\theta) + \cos 3\theta(2\cos\theta + \cos 3\theta)$$
$$+ 2i\sin 3\theta\ (\cos\theta + \cos 3\theta)$$
$$= \cancel{\cos^2 3\theta} + \cos 3\theta(2\cos\theta + \cancel{\cos 3\theta}) = \frac{2\cos^2 3\theta + 2\cos\theta}{\cos 3\theta}$$
$$+ 2i\sin 3\theta(\cos\theta + \cos 3\theta)$$

which is close...

II: $2i\sin 3\theta\,(2\cos 2\theta\cos\theta)$

So $\tan(\arg u+iv) = \dfrac{4\sin 3\theta\cos 2\theta\cos\theta}{2\cos 3\theta\cos\theta}$

$$= 2\ \tan 3\theta\cos 2\theta.$$

(we could expand $2\cos 2\theta$, but it's not equal to 1).

So $\tan(\arg(u+iv))$

$$= \frac{4\sin 3\theta\cos 2\theta\cos\theta}{2\cos^2 3\theta + 2\cos\theta\cos 3\theta}$$

$$= \frac{2\sin 3\theta}{\cos 3\theta}\left(\frac{\cos 2\theta\cos\theta}{\boxed{\cos 3\theta + \cos\theta}}\right)$$

$$= 2\cos 2\theta\cos\theta.$$

So everything in the bracket cancels, and the 2s.

$\tan(\arg u+iv) = \dfrac{\sin 3\theta}{\cos 3\theta}$ So $\dfrac{u}{v} = \cot 3\theta.$

NZQA Scholarship Calculus
2009 2)a)

It's tempting to use standard identities - but perhaps we're not allowed them.

$$\cosh 3x = \underline{\tfrac{1}{2}\left(e^{3x} + e^{-3x}\right)}$$

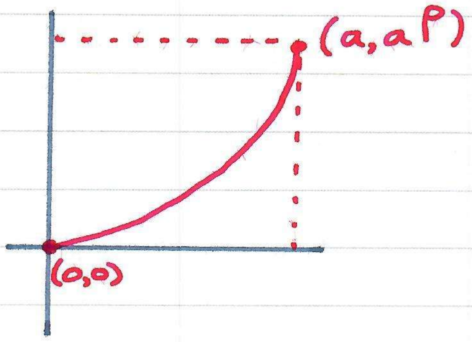
$$4(\cosh x)^3 - 3\cosh x$$

$$= 4 \cdot \tfrac{1}{8}\left(e^{3x} + \cancel{3e^{x}} + \cancel{3e^{-x}} + e^{-3x}\right) - \tfrac{3}{2}\left(\cancel{e^{x}} + \cancel{e^{-x}}\right)$$

$$= \underline{\tfrac{1}{2}\left(e^{3x} + e^{-3x}\right)}$$

So the identity is proved as required.

NZQA Scholarship Calculus
2009 . 2) b)



we want to know $V_x = \int_0^a \pi y^2 \, dx$

where $y = x^p$

and $V_y = \int_0^{a^p} \pi x^2 \, dy$

$$V_x = \pi \int_0^a x^{2p} \, dx = \pi \left[\frac{1}{2p+1} x^{2p+1} \right]_0^a$$

$$= \underline{\frac{\pi}{2p+1} a^{2p+1}}$$

$$V_y = \pi \int_0^{a^p} \left(y^{\frac{1}{p}}\right)^2 dy = \pi \int_0^{a^p} y^{\frac{2}{p}} \, dy$$

$$= \pi \left[\frac{1}{\frac{2}{p}+1} y^{\frac{2}{p}+1} \right]_0^{a^p} = \underline{\frac{\pi p}{2+p} a^{2+p}}$$

So for the required equality $V_x = V_y$:

$$\frac{\pi}{2p+1} a^{2p+1} = \frac{\pi p}{2+p} a^{2+p}$$

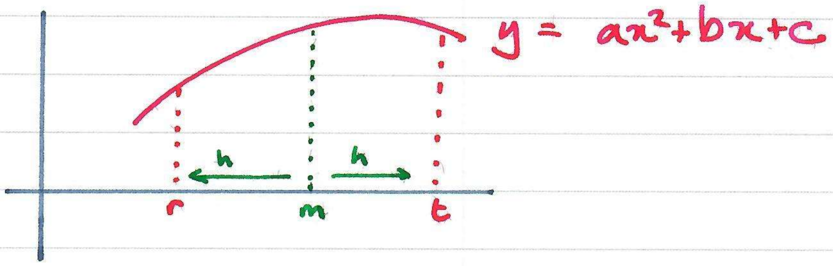
$$a^{p-1} = \frac{(2p+1)p}{2+p}.$$

So when $p \neq 1$, $a = \left[\frac{(2p+1)p}{2+p}\right]^{\frac{1}{p-1}}$

and when $p = 1$, the curve is simply $y = x$

and $V_x = V_y = \frac{\pi a^3}{3}$ for any +ve a.

NZQA Scholarship Calculus
2009 2)c)



This feels like a very standard numerical analysis approach for a quadratic approximation - but the way it's set up seems a lot of work. Perhaps I should do it using splines... though in fact yes, this is precisely what Simpson's rule is.

① The exact integral:

$$I = \int_r^t (ax^2+bx+c)\,dx = \left[\frac{a}{3}x^3 + \frac{b}{2}x^2 + cx\right]_r^t$$

$$= \frac{a}{3}(t^3-r^3) + \frac{b}{2}(t^2-r^2) + c(t-r).$$

② Simpson's Rule:

$$I = \frac{t-r}{6}\left[f(r) + 4f\left(\frac{r+t}{2}\right) + f(t)\right]$$

(I'm not sure where students are supposed to get Simpson's rule from - pres. a formula sheet, and pres. in some formulation like this)

So this is just a matter of multiplying it out...

$$= \frac{t-r}{6}\left[ar^2 + br + c + 4\left(a\left(\frac{r+t}{2}\right)^2 + b\left(\frac{r+t}{2}\right) + c\right) + at^2 + bt + c \right]$$

$$= \frac{t-r}{6}\left[a(r^2+t^2) + b(r+t) + 2c + a(r^2 + 2rt + t^2) + 2b(r+t) + 4c \right]$$

$$= \frac{t-r}{6}\left[2a(r^2 + rt + t^2) + 3b(r+t) + 6c \right]$$

$$= \frac{1}{6}\left[2a(\cancel{r^2t} + \cancel{rt^2} + t^3 - r^3 - \cancel{r^2t} - \cancel{rt^2}) + 3b(-r^2 + t^2) + 6c(t-r) \right]$$

$$= \frac{a}{3}(t^3 - r^3) + \frac{b}{2}(t^2 - r^2) + c(t-r)$$

which equals the exact integral and proves the required result.

NZQA Scholarship Calculus
2009 3)a)

The correct approach would be to say

$$\frac{1}{a+ib} = \frac{a-ib}{a^2+b^2}$$

whereas $\frac{1}{a} + \frac{1}{ib} = \frac{1}{a} - i\frac{1}{b}$

Equating real parts:

$$\frac{a}{a^2-b^2} = \frac{1}{a} \quad \text{so} \quad a^2 = a^2 - b^2$$

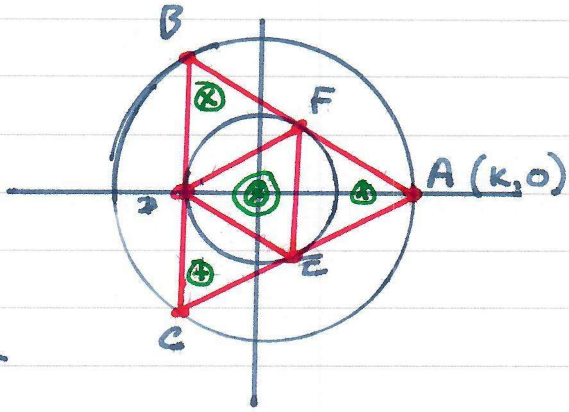
So $b=0$ – which contradicts the assumption in the question* that $b \neq 0$

* actually it's not quite clear: we're told $a+ib \neq 0$, but it's also assumed that $b \neq 0$ and indeed $a \neq 0$.

NZQA Scholarship Calculus
2009 3) b)

Complex Numbers are fairly robust things, so you can move quite fast on this one.

We're not told much about the roots (points) but we know there's one +ve real root - which must be A.



So the other roots on that circle are

$$\underset{B}{k \operatorname{cis}\left(\frac{2\pi}{3}\right)} \text{ and } \underset{C}{k \operatorname{cis}\left(\frac{-2\pi}{3}\right)}$$

Using simple geometry the inner ⊙ and Δ are half the size of ABC, so those roots are

$$\underset{D}{-\frac{k}{2}}, \quad \underset{E}{\frac{k}{2} \operatorname{cis}\left(-\frac{\pi}{3}\right)}, \quad \underset{F}{\frac{k}{2} \operatorname{cis}\left(\frac{\pi}{3}\right)}.$$

This lists the exact roots, but in coord. form they are:

$$k, \ -\frac{k}{2} + \frac{\sqrt{3}k}{2}i, \ -\frac{k}{2} - \frac{\sqrt{3}}{2}i, \ -\frac{k}{2}, \ \frac{k}{4} - \frac{\sqrt{3}}{4}i, \ \frac{k}{4} + \frac{\sqrt{3}}{4}i$$

We form polynomials from these by using the conjugate pairs:

$$\left(x - \left(-\frac{k}{2} + \frac{\sqrt{3}}{2}ki\right)\right)\left(x - \left(-\frac{k}{2} - \frac{\sqrt{3}}{2}ki\right)\right)$$

or by observing we want the quadratic formula to produce roots

$$\frac{-k}{2} \pm \frac{\sqrt{-3}k}{2} = \frac{-k \pm \sqrt{-3}k}{2}$$

and this matches the polynomial

$$x^2 + kx + k^2.$$

Similarly the quadratic required for the inner roots is

$$x^2 - \frac{k}{2}x + \frac{k^2}{4}$$

So $p(x)$ is

$$(x-k)(x^2+kx+k^2)\left(x+\frac{k}{2}\right)\left(x^2-\frac{k}{2}x+\frac{k^2}{4}\right)$$

which is also

$$(x^3-k^3)\left(x^3+\frac{k^3}{8}\right)$$

ii) (This is a strange twist) $p'(x) = 3x^2\left(x^3+\frac{k^3}{8}\right)+3x^2(x^3-k^3)$

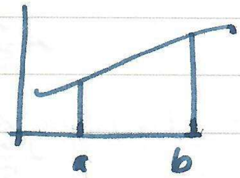
$= 3x^2\left(2x^3 - \frac{7}{8}k^3\right)$ So $x = 0$ (twice) or

$x = \sqrt[3]{\frac{7}{16}}k, \sqrt[3]{\frac{7}{16}}k\operatorname{cis}\left(\frac{2\pi}{3}\right), \sqrt[3]{\frac{7}{16}}k\operatorname{cis}\left(\frac{-2\pi}{3}\right)$

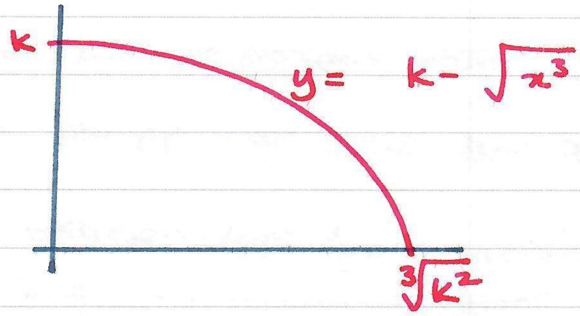
(see diagram)

NZ Scholarship Calculus
2009 4)a)

$$L = \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx.$$



In this case, the curve does something like:



with the x-intercept being

$$0 = k - \sqrt{x^3}$$
$$x^3 = k^2$$
$$x = \sqrt[3]{k^2}$$

we know $y = k - \sqrt{x^3} = k - x^{\frac{3}{2}}$

$$\frac{dy}{dx} = -\frac{3}{2} x^{\frac{1}{2}}$$

$$1 + \left(\frac{dy}{dx}\right)^2 = 1 + \frac{9}{4}x$$

and the limits for L are 0 and $\sqrt[3]{k^2}$

So $$L = \int_0^{\sqrt[3]{k^2}} \sqrt{1 + \frac{9}{4}x}\, dx$$

Let $1+\frac{9}{4}x = u$ $\quad \frac{9}{4}dx = du$

$dx = \frac{4}{9}du$

And the limits are: $\quad x = 0 \quad u = 1$

$x = \sqrt[3]{k^2} \quad u = 1 + \frac{9}{4}\sqrt[3]{k^2}$

$$\text{So } L = \frac{4}{9}\int_1^{1+\frac{9}{4}\sqrt[3]{k^2}} u^{\frac{1}{2}}\,du$$

$$= \frac{4}{9}\left[\frac{2}{3}u^{\frac{3}{2}}\right]_1^{1+\frac{9}{4}\sqrt[3]{k^2}}$$

$$= \frac{8}{27}\left(\left(1+\frac{9}{4}\sqrt[3]{k^2}\right)^{3/2} - 1\right)$$

Hmm... we want this to equal $\frac{56}{27}$. Which means

$$\left(1+\frac{9}{4}\sqrt[3]{k^2}\right)^{3/2} - 1 = 7$$

$$\left(1+\frac{9}{4}\sqrt[3]{k^2}\right)^{3/2} = 8$$

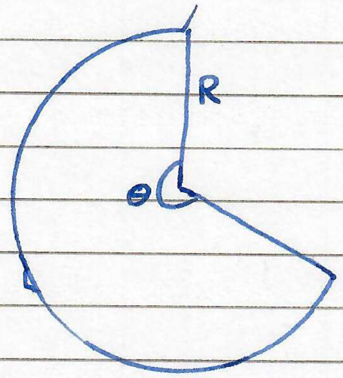
$$1+\frac{9}{4}\sqrt[3]{k^2} = 4$$

$$\frac{9}{4}\sqrt[3]{k^2} = 3$$

$$\sqrt[3]{k^2} = \frac{12}{9} = \frac{4}{3}$$

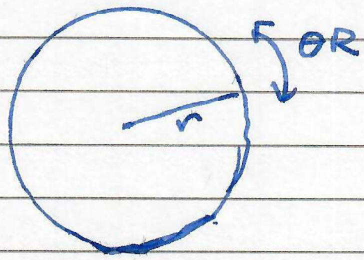
$$k^2 = \frac{64}{27} \qquad \underline{\underline{k = \frac{8}{3\sqrt{3}}}} = \underline{\underline{\frac{8\sqrt{3}}{9}}}$$

4b) Curved edge $= \theta R$



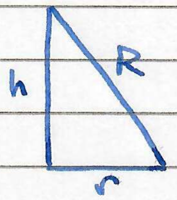
Circ. of cone $= 2\pi r = \theta R$

So $r = \dfrac{\theta R}{2\pi}$



Area $= \pi r^2 = \pi \left(\dfrac{\theta R}{2\pi}\right)^2$

$h = \sqrt{R^2 - r^2}$



$= \sqrt{R^2 - R^2\left(\dfrac{\theta}{2\pi}\right)^2}$

$= R\sqrt{1 - \left(\dfrac{\theta}{2\pi}\right)^2}$

So Vol. of cone $= \dfrac{1}{3} \times h \times \left(\dfrac{\theta R}{2\pi}\right)^2$

$= \dfrac{R}{3}\sqrt{1 - \left(\dfrac{\theta}{2\pi}\right)^2}\left(\dfrac{\theta R}{2\pi}\right)^2$

$= \dfrac{R^3}{12\pi}\sqrt{1 - \left(\dfrac{\theta}{2\pi}\right)^2}\left(\theta\right)^2 = \dfrac{R^3}{12\pi}\sqrt{1 - \left(\dfrac{\theta}{2\pi}\right)^2}\,\theta^2$

For max:

$0 = \frac{dV}{d\theta} =$ (ignore R^3 factor)

$$\frac{2\theta\sqrt{1-\left(\frac{\theta}{2\pi}\right)^2} - \frac{1}{2}\theta^2 \cdot \frac{2\theta}{4\pi^2}}{\sqrt{1-\left(\frac{\theta}{2\pi}\right)^2}}$$

$$= 2\theta\left(1-\left(\frac{\theta}{2\pi}\right)^2\right) - \frac{\theta^3}{2 \cdot 2\pi^2}$$ (again, ignore denom. which is +ve)

$$= 2\theta - \frac{\theta^3}{2\pi^2} - \frac{\theta^3}{4\pi^2}$$

$$= 2\theta - \frac{3}{4}\frac{\theta^2}{\pi^2}$$

$$= \theta\left(2 - \frac{3\theta^2}{4\pi^2}\right)$$

$$= 0$$

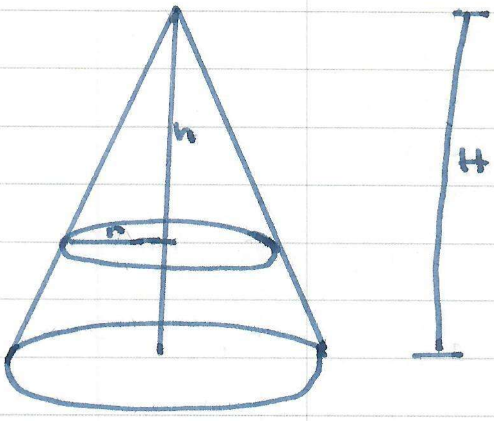
So $\theta = 0$ (which would be a min.)

or $2 = \frac{3\theta^2}{4\pi^2}$

So $\theta^2 = \frac{8\pi^2}{3}$ $\quad \theta = 2\pi\sqrt{\frac{2}{3}}$.

NZQA Scholarship Calculus
2009 4) c)

The hard thing here is working out what to include and what to leave out - since we're given very little and it looks like a simple $\frac{1}{3}$ question.



Start by measuring from the top downwards, and let the ~~initial~~ (full) volume be V_{full}.

Then $V(\text{remaining}) = V_{full} - \frac{\pi r^2 h}{3}$

We're not told how r relates to h, but it's a constant ratio - and since the question's phrased in terms of h let's say $r = kh$.

So $V(\text{remaining}) = V_{full} - \frac{\pi k^2 h^3}{3}$ —— ①

But we also know $\frac{dV}{dt} = \alpha \pi r^2 = \alpha \pi k^2 h^2$ —— ②
for some α.

Differentiating ①:

$$\frac{dV}{dt} = -\frac{\cancel{3}\pi k^2 h^2}{\cancel{3}} \frac{dh}{dt} \quad \text{——} \; ③$$

Equating ② and ③:

$$-\pi k^2 h^2 \frac{dh}{dt} = \alpha \pi k^2 h^2$$

So $\frac{dh}{dt} = \frac{\alpha}{k^2} = A$ (say) – ie it's constant.

So $h = At + C$.

when $t=0$ $h = \frac{H}{8}$: $\frac{H}{8} = 0 + C$ so $C = \frac{H}{8}$

$$h = At + \frac{H}{8}$$

When $t=10$ $h = H$: $H = 10A + \frac{H}{8}$ so $A = \frac{7}{80}H$

$$h = \frac{7H}{80}t + \frac{H}{8}$$

so when $h = \frac{H}{2}$:

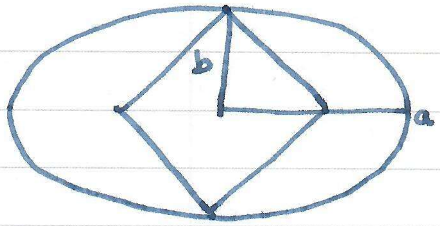
$$\frac{H}{2} = \frac{7H}{80}t + \frac{H}{8}$$

$$40H = 7Ht + 10H$$

$$30H = 7t$$

$$t = \frac{30}{7} = \underline{4\tfrac{2}{7} \text{ days.}}$$

NZQA Scholarship Calculus
2009 5) a)



Area of the ellipse = $A = \pi ab$

Area of the square $= 2b^2 = \frac{\pi}{2}ab$ (given) $\longrightarrow$ ①

Eccentricity of the ellipse is defined as

$$e = \sqrt{1 - \frac{b^2}{a^2}} \longrightarrow ②$$

So from ①: $\frac{4b}{\pi} = a$

So in ② $e = \sqrt{1 - \frac{b^2\pi^2}{16b^2}} = \underline{\underline{\sqrt{1 - \frac{\pi^2}{16}}}}$

NZQA Scholarship Calculus
2009 5) b)

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

This is a basic result about hyperbolas ... because a tangent is alway contained 'within' the assymptotes.

Proving it with these coordinates, though, is a pain - and we can usefully scale it by saying

$$X = \frac{x}{a} \quad Y = \frac{y}{b} \quad X_0 = \frac{x_0}{a} \quad Y_0 = \frac{y_0}{b}.$$

Then $X^2 - Y^2 = 1$ ⓪ and $XX_0 - YY_0 = 1.$

and the tangent given is $XX_0 - YY_0 = 1$ (the same) —— ①

But also, because (X_0, Y_0) is on the hyperbola:

$$X_0^2 - Y_0^2 = 1. \quad \text{—— ②}$$

① gives $XX_0 = 1 + YY_0$

so $X^2X_0^2 = X_0^2 + YY_0X_0^2$ —— ③

and ⓪ gives $X^2 = 1 + Y^2$

so $X^2X_0^2 = X_0^2 + X_0^2Y^2$ —— ④

3 and 4 together give

$(X^2 X_0^2 =) \quad X_0^2 + Y Y_0 X_0^2 = X_0^2 + X_0^2 Y^2$

$\therefore \quad X_0^2 \, Y(Y - Y_0) = 0.$

Since $X_0 = 0$ and $Y_0 = 0$ (they're both on the curve)

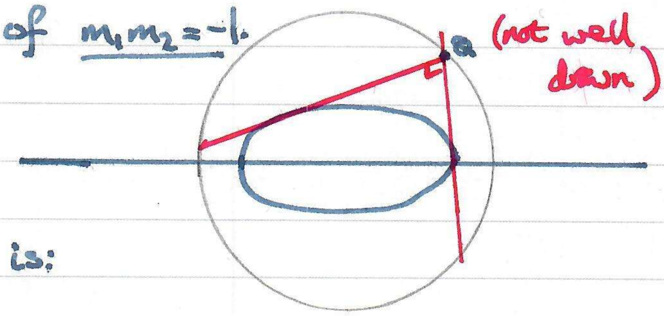
this means $Y = Y_0$ (the original point)

or $Y = 0$ which can't happen.

(I suspect another proof would consider the actual gradients - but this feels like enough when combined with knowledge of a hyperbola.)

NZQA Scholarship Calculus
2009 5) c)

The key here is to make good use of $\underline{m_1 m_2 = -1}$.



Let Q be the point (h, k)

A tangent with gradient m is:

$$y = mx \pm \sqrt{a^2m^2 + b^2}$$

And since this passes through (h, k):

$$k = mh \pm \sqrt{a^2m^2 + b^2}$$

$$(k - mh)^2 = a^2m^2 + b^2$$

i.e. $m^2(h^2 - a^2) - 2hkm + (k^2 - b^2) = 0$

This is a quadratic in m whose roots are m_1 and m_2: and because the tangents are perpendicular, $m_1 m_2 = -1$.

So using the usual product of roots:

$$\frac{k^2 - b^2}{h^2 - a^2} = -1 \quad \therefore \quad k^2 - b^2 = a^2 - h^2$$

$$\text{or} \quad h^2 + k^2 = a^2 + b^2$$

So the locus of Q is $x^2 + y^2 = a^2 + b^2$, a ⊙ centred at the origin with radius $\sqrt{a^2 + b^2}$.