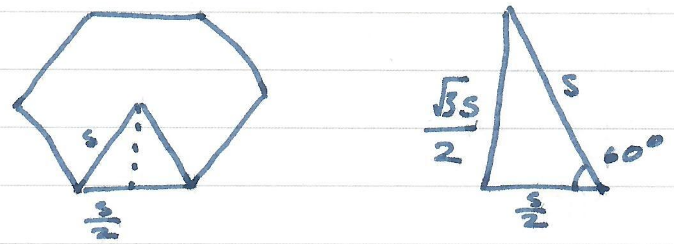
NZQA Scholarship Calculus
2008 1)a)



$$\triangle = \frac{1}{2}s\,\frac{\sqrt{3}}{2}s = \frac{\sqrt{3}}{4}s^2$$

$$\text{So } \hexagon = 6 \times \triangle = \frac{6\sqrt{3}}{4}s^2 = \underline{\underline{\frac{3}{2}\sqrt{3}\,s^2}}$$

So the area of the grid with 16 hexagons

$$= 16 \times \frac{3}{2}\sqrt{3}s^2 = \underline{\underline{24\sqrt{3}s^2}}$$

So far, so inane... but now the question gets ridiculous.

b) $$A = 6hs + \frac{3}{2}s^2\left(-\frac{\cos\theta}{\sin\theta} + \frac{\sqrt{3}}{\sin\theta}\right)$$

$$= 6hs + \frac{3}{2}s^2\left(-\cot\theta + \sqrt{3}\operatorname{cosec}\theta\right)$$

(which more useful)

Assume h and s are fixed:

$$\frac{dA}{d\theta} = \frac{3}{2}s^2\left(\operatorname{cosec}^2\theta - \sqrt{3}\operatorname{cosec}x\cot x\right)$$

$= 0$ for a turning point.

So $\operatorname{cosec} x = 0$ (can't happen)

or $\operatorname{cosec} \theta = \sqrt{3} \cot \theta$

$\sin \theta = \frac{1}{\sqrt{3}} \tan \theta$: $\cos \theta = \frac{1}{\sqrt{3}}$, $\theta = 54.74°$.

c) Now the question gets completely stupid.

Again we're told

$$A = 6hs + \frac{3}{2} s^2 \left(- \frac{\cos \theta}{\sin \theta} + \frac{\sqrt{3}}{\sin \theta} \right)$$

And also $\frac{ds}{dt} = k \sin \theta$ (I can't imagine any bee working this out.)

In particular,

$$\sqrt{2} = k \sin \frac{\pi}{4} = \frac{k}{\sqrt{2}} \text{ so } k = 2.$$

(so this was just a way of sneaking something rather trivial in.)

Now we have to differentiate A by t, assuming s is increasing - which makes no geometrical or practical sense at all.

$$\frac{dA}{dt} = 6h \underbrace{\frac{ds}{dt}}_{=2\sin\theta} + \frac{3}{2} \cdot 2s \underbrace{\frac{ds}{dt}}_{=2\sin\theta} \left(-\frac{\cos\theta}{\sin\theta} + \frac{\sqrt{3}}{\sin\theta} \right)$$

$$+ \frac{3}{2} s^2 \left(\operatorname{cosec}^2\theta - \sqrt{3} \cot\theta \operatorname{cosec}\theta \right) \frac{d\theta}{dt}$$

At this point we panic because we don't know $\frac{d\theta}{dt}$ (it's certainly not 0) but using the found value of θ gives

$$\frac{dA}{dt} = 6h \cdot 2 \frac{\sqrt{2}}{\sqrt{3}} + \frac{3}{2} \cdot 4s \frac{\sqrt{2}}{\sqrt{3}} \left(\frac{-1}{\sqrt{2}} + \frac{\sqrt{3}\sqrt{3}}{\sqrt{2}} \right)$$

$$+ \cancel{\frac{3}{2} s^2 \left(\left(\frac{\sqrt{3}}{\sqrt{2}}\right)^2 - \frac{\sqrt{3}}{\sqrt{2}} \cdot \frac{\sqrt{3}}{\sqrt{2}} \right)}$$

$= 0$, amazingly.

$$= 12h \frac{\sqrt{2}}{\sqrt{3}} + 4s \frac{\sqrt{3}}{\sqrt{2}} \left(\frac{2}{\sqrt{2}} \right)$$

$$= 4\sqrt{6}h + 4\sqrt{3}s$$

$$= \underline{\underline{4(\sqrt{6}h + \sqrt{3}s)}}$$

(this appears to be true despite what the question says).

NZQA Scholarship Calculus
2008 2) a)

$(z+1)^3 = 8 \quad = 2^3 \operatorname{cis}(0)$

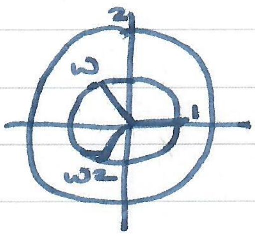
$(z+1) \quad = 2 \operatorname{cis}(0),$
$2 \operatorname{cis}\left(\frac{2\pi}{3}\right)$
$2 \operatorname{cis}\left(\frac{4\pi}{3}\right)$ or $2 \operatorname{cis}\left(-\frac{2\pi}{3}\right)$

$= \quad 2, \; 2\omega, \; 2\omega^2 \; (2\bar{\omega}) \quad$ where $\omega = \operatorname{cis}\left(\frac{2\pi}{3}\right)$ or $\operatorname{cis}\left(-\frac{2\pi}{3}\right)$

So $z = 1, \; 2\omega - 1$ or $2\omega^2 - 1 \; (2\bar{\omega} - 1)$.

where $\omega = -\frac{1}{2} + \frac{\sqrt{3}}{2} i$

Sum of the roots is



$1 + 2\omega - 1 + 2\bar{\omega} - 1$
$= 1 - 1 - 1 - 1 - 1 = \underline{-3}$

(or) in $(z+1)^3 = 8$

$z^3 + \underset{\alpha}{3}z^2 + 3z - 7 = 0$

so the sum of the roots is $-\alpha = \underline{-3}$.

NZQA Scholarship Calculus
2008 2) b)

$$(z+1)^3 = 8(z-1)^3$$

You can do this by division and reference back to (a), but in fact it's easier just to use polynomials ... partic. since it looks like there might be a $\mathbb{R}$ solution.

$$z^3 + 3z^2 + 3z + 1 = 8z^3 - 24z^2 + 24z - 8$$

so $$7z^3 - 27z^2 + 21z - 9 = 0$$

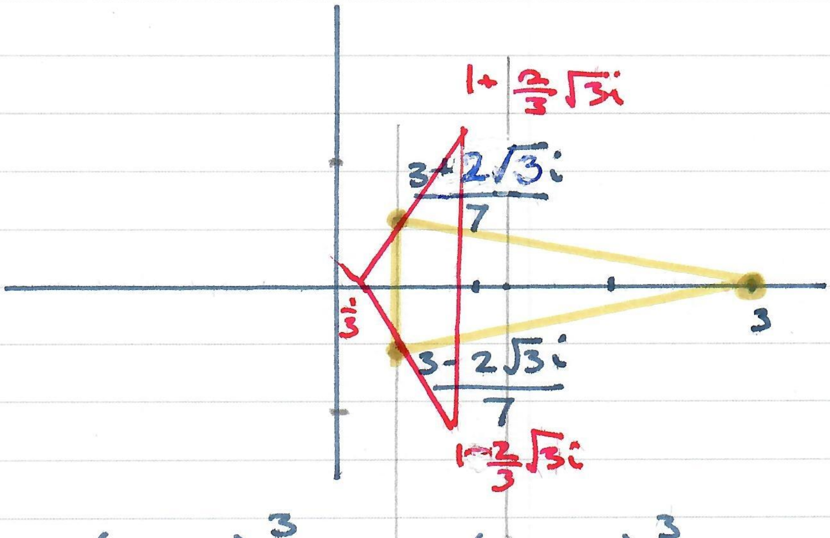
Observe $z = 3$ is a root, so this is (it's horrid to work out):

$$(z-3)(7z^2 - 6z + 3) = 0$$

So $z = 3$ or $\dfrac{6 \pm \sqrt{36-84}}{2 \times 7}$

ie 3 or $\dfrac{3 \pm 2\sqrt{3}\, i}{7}$

NZQA Scholarship Calculus
2008 2)c).



The equation $\left(\frac{1}{z}+1\right)^3 = 8\left(\frac{1}{z}-1\right)^3$

has root = the reciprocals of the roots already found.

So $\underline{\frac{1}{3}} = \frac{1}{3}+0i$

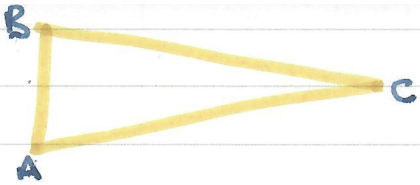
$$\frac{1}{\frac{3+2\sqrt{3}i}{7}} = \frac{7}{3+2\sqrt{3}i} = \frac{7(3-2\sqrt{3}i)}{(3+2\sqrt{3}i)(3-2\sqrt{3}i)}$$

$$= \frac{21-14\sqrt{3}i}{9+4.3} = \frac{21-14\sqrt{3}i}{21}$$

$$= \underline{\underline{1-\frac{2}{3}\sqrt{3}i}}$$

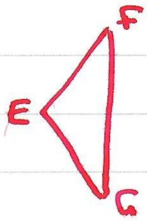
And the other root is the conjugate

$$\underline{\underline{1+\frac{2}{3}\sqrt{3}i}}$$

Area of 

$$= \left(3 - \frac{3}{7}\right) \times \frac{2}{7}\sqrt{3} \times 2 \times \frac{1}{2}$$

$$= \frac{18}{7} \times \frac{2}{7}\sqrt{3} = \frac{36}{49}\sqrt{3}$$

Area of 

$$= \left(1 - \frac{1}{3}\right) \times \frac{2}{3}\sqrt{3} \times 2 \times \frac{1}{2}$$

$$= \frac{4}{9}\sqrt{3}$$

So the ratio $\dfrac{\Delta ABC}{\Delta EFG} = \dfrac{36}{49}\sqrt{3} \times \dfrac{9}{4\sqrt{3}}$

$$= \frac{9 \times 9}{49} = \underline{\underline{\frac{81}{49}}}$$ as required.

NZQA Scholarship Calculus
2008 3) a)

$$\frac{A}{x} + \frac{B}{P-x} = \frac{1}{x(P-x)}$$

$$= \frac{A(P-x) + Bx}{x(P-x)}$$

Equating coefficients:

x: $\quad -A + B = 0$

1: $\quad AP = 1$

So $A = \frac{1}{P}$ and $B = \frac{1}{P}$.

and the expression given is

$$\frac{1}{Px} + \frac{1}{P(P-x)} = \frac{1}{x(P-x)}$$

NZQA Scholarship Calculus

2008 3) b)

we are told: $\frac{dN}{dt} = \alpha N(P-N)$ for some α

$$\frac{dN}{N(P-N)} = \alpha\, dt$$

using part (a):

$$\int \left(\frac{1}{PN} + \frac{1}{P(P-N)} \right) dN = \int \alpha\, dt$$

$$\frac{1}{P} \left(\ln|N| - \ln|P-N| \right) = \alpha t + c$$

$$\frac{1}{P} \ln\left|\frac{N}{P-N}\right| = \alpha t + c$$

$$\ln\left|\frac{N}{P-N}\right| = P(\alpha t + c)$$

$$\frac{N}{P-N} = e^{P(\alpha t + c)} = Ae^{P\alpha t} \text{ for some } A.$$

So $N = (P-N)Ae^{P\alpha t}$

$$N + NAe^{P\alpha t} = PAe^{P\alpha t}$$

$$N = \frac{PAe^{P\alpha t}}{1 + Ae^{P\alpha t}}$$

NZQA Scholarship Calculus
2008 3)c).

when $t = 0$, $N = 0.005P$

So $0.005\cancel{P} = \dfrac{PAe^0}{1 + Ae^0} = \dfrac{\cancel{P}A}{1+A}$

$0.005 + 0.005A = A$

$A = \dfrac{0.005}{0.995} = \dfrac{1}{199}$

So $N = \dfrac{\frac{P}{199} e^{P\alpha t}}{1 + \frac{e^{P\alpha t}}{199}}$

And $\dfrac{N}{P-N} = \dfrac{e^{P\alpha t}}{199}$ ——— ①

And now we know:

$\dfrac{dN}{dt} = 0.08P = \alpha . \dfrac{P}{5} . \dfrac{4P}{5} = \dfrac{4\alpha P^2}{25}$

(the use of %s of P seems confusing)

$\alpha = \dfrac{25}{4} \times \dfrac{0.08}{P} = \dfrac{1}{2P}.$

So when $N = \frac{P}{2}$,

$$\frac{\frac{P}{2}}{P - \frac{P}{2}} = \frac{e^{\frac{P}{2P}t}}{199}$$

$$\frac{P/2}{P/2} = 1 = \frac{e^{\frac{t}{2}}}{199}$$

So $e^{\frac{t}{2}} = 199$

$$\frac{t}{2} = \ln(199) = 5.293$$

So $\underline{\underline{t = 10.597}}$ (days)

So the principal must act within 11 days.

NZQA Scholarship Calculus
2008 4) a)

$$h(x) = f(x)^{g(x)}$$

So $f(x) = e^{\ln(f(x))}$

and $h(x) = \left(e^{\ln(f(x))}\right)^{g(x)}$

$$= e^{g(x)\ln(f(x))}$$

So $\dfrac{dh}{dx} = \dfrac{d}{dx}\left(g(x)\ln(f(x))\right)e^{g(x)\ln(f(x))}$

$$= \left(g'(x)\ln(f(x)) + g(x)\frac{f'(x)}{f(x)}\right) \times \left(f(x)\right)^{g(x)}$$

$$= \left(f(x)\right)^{g(x)}\left(g'(x)\ln(f(x)) + g(x)\frac{f'(x)}{f(x)}\right)$$

as required.

NZQA Scholarship Calculus
2008 4) b)

The question paper's squiffy, so let's try:

$$p(x) = x\ln(\ln x)$$

so $$p'(x) = \ln(\ln x) + x\frac{1}{\ln x}\cdot\frac{1}{x}$$
$$= \ln(\ln x) + \frac{1}{\ln x}$$

It seems what's wanted is $\int (\ln x)^x p'(x)\,dx$

so this is $\int (\ln x)^x\left(\ln(\ln x) + \frac{1}{\ln x}\right)dx$ —— ②

Now... try $h(x) = f(x)^{g(x)} = (\ln x)^x$

(ie. $f(x) = \ln x \quad g(x) = x$)

Then we know from (a):

$$\frac{d}{dx}\left((\ln x)^x\right) = (\ln x)^x\left[\ln(\ln x) + \frac{\cancel{x}}{\cancel{x}}\frac{1}{\ln x}\right]$$
$$= (\ln x)^x\left(\ln(\ln x) + \frac{1}{\ln x}\right)$$

which, amazingly, is what we have in ②.

So $\underline{\underline{h(x) = (\ln x)^x + c}}$

NZQA Scholarship Calculus
2008 4)c)

$$g(x) = -(2x-a)^4 + bx + c$$

If $(x-1)$ is a divisor then $x=1$ is a root:

$$-(2-a)^4 + b + c = 0$$

$$(2-a)^4 = b + c.$$

To factorise $g(x)$:

$$g(x) = -(2x-1)^4 + 8x - 7$$

$$= -(16x^4 - 32x^3 + 24 - 8x + 1) + 8x - 7$$

$$= -16x^4 + 32x^3 - 24x^2 + 16x - 8$$

$$= -8(2x^4 - 4x^3 + 3x^2 - 2x + 1)$$

$$= -8(x-1)(2x^3 - 2x^2 + x - 1)$$

(by inspection)

and this has another $(x-1)$ factor:

$$= -8(x-1)(x-1)(2x^2 + 1)$$

So $x = 1$ (twice) or $\pm\sqrt{\frac{1}{2}}\, i$

NZQA Scholarship Calculus
2008 5) a)

$x = \sec\theta - 1$ $\quad -\pi < \theta < \pi, \quad \theta \neq \pm\frac{\pi}{2}$
$y = 2\tan\theta + 2$

Guessing ... it's either something like a cardioid or a conic - and it looks like assymptotes popup...

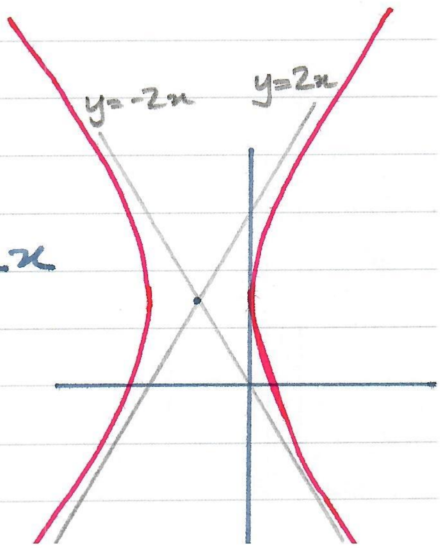
$x = \sec\theta - 1$ $\qquad y = 2\tan\theta + 2$
$(x+1) = \sec\theta$ $\qquad y - 2 = 2\tan\theta$
$(x+1)^2 = \sec^2\theta$ $\qquad \frac{(y-2)^2}{4} = \tan^2\theta$

oh... and the parameter scales uniformly (not eg at double the rate) ... and this looks like a nice $\tan^2\theta = \sec^2\theta - 1$

So $(x+1)^2 - 1 = \frac{(y-2)^2}{4}$

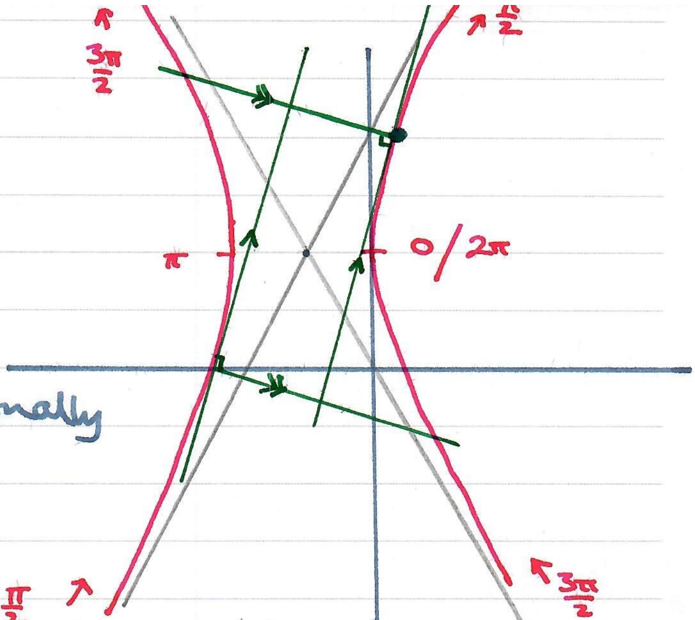
$(x+1)^2 - \frac{(y-2)^2}{4} - 1 = 0$

This is a classic hyperbola



centre $(-1, 2)$
assymptotes $y = \pm 2x$

NZQA Scholarship Calculus
2008 5) b)



In terms of the geometry everything drops out nicely now – particularly noting everything is rotationally symmetric around $(-1, 2)$.

We could even shift the origin to there, though in the event there's no need to.

In terms of exploring how the parameter works, values of θ are marked – and as expected, the sec $\frac{\pi}{2}$ and tan $\frac{\pi}{2}$ values are at ∞.

When $\theta = \frac{\pi}{4}$, $\quad x = \sec\left(\frac{\pi}{4}\right) - 1 = \sqrt{2} - 1$

$$y = 2\tan\left(\frac{\pi}{4}\right) + 2 = 4$$

To find $\frac{dy}{dx}$ do it implicitly or (easier) using parameter:

$$\frac{dy}{dx} = \frac{dy}{d\theta}\frac{d\theta}{dx} = \frac{2\sec^2\theta}{\sec\theta\tan\theta} = 2\frac{\sec\theta}{\tan\theta}$$

$$= \frac{2}{\cos\theta}\frac{\cos\theta}{\sin\theta} = \frac{2}{\sin\theta}$$

So at this point, the gradient is $\frac{2}{1/\sqrt{2}} = 2\sqrt{2}$.

Gradient of the normal at P is the -ve reciprocal,

i.e. $\frac{-1}{2\sqrt{2}}$

and the eq^n of the normal is:

$$y = \frac{-1}{2\sqrt{2}} x + c$$

Subs the point $P(\sqrt{2}-1, 4)$; but with $\frac{y-y_1}{x-x_1}$ form:

$$\cancel{4 = \frac{-1}{2\sqrt{2}}(\sqrt{2}-1) + c} \quad \cancel{= \frac{-1}{2} + \frac{1}{2\sqrt{2}} + c}$$

$$\cancel{c = \frac{9}{2} - \frac{\sqrt{2}}{4}}$$

$$y - 4 = -\frac{1}{2\sqrt{2}}(x - \sqrt{2} + 1)$$

$$-2\sqrt{2}(y-4) = x - \sqrt{2} + 1$$

which helpfully gives

$$x + 1 = -2\sqrt{2}(y-4) + \sqrt{2}$$

So this meets the hyperbola again when

$$(-2\sqrt{2}(y-4) + \sqrt{2})^2 = \frac{(y-2)^2}{4} + 1$$

$$4\left(-2\sqrt{2}(y-4)+\sqrt{2}\right)^2 = (y-2)^2+4$$

after some bashing...

$$4(8y^2-72y+162) = y^2-4y+8$$

$$31y^2-284y+640=0$$

But we know $y=4$ is a root...

$(y-4)(31y-160)=0$: y at point Q is $\underline{\underline{\frac{160}{31}}}$

($\underline{a=160,\ b=31}$ as required)

5)c) (b) was tedious: (c) just involves spinning the geometry through $180°$ around $(-1,2)$.

So y at point R is $\frac{160}{31}-2\left(\frac{160}{31}-2\right)$

$$= \frac{160}{31}-\frac{2}{31}(160-62)$$

$$= \frac{160}{31}-\frac{2\times 98}{31}$$

$$= \underline{\underline{\frac{-36}{31}}}$$