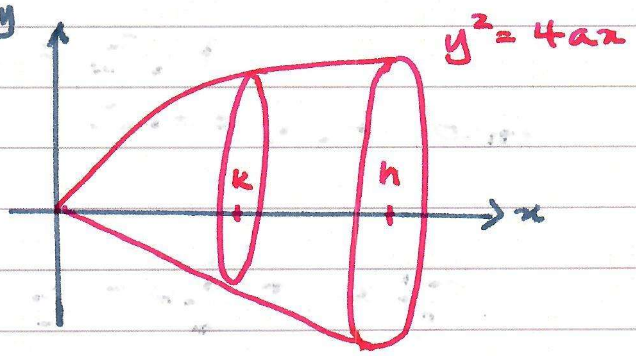
NZQA Scholarship Calculus
2007 1) a)



$$V = \int_0^h \pi y^2 \, dx$$

$$= \int_0^h \pi \, 4ax \, dx$$

$$= \frac{4}{2} \pi a \left[x^2 \right]_0^h$$

$$= \underline{\underline{2\pi a h^2}}$$

b) Surface area $S = \int_0^h y \sqrt{1 + \left(\frac{dy}{dx}\right)^2} \, dx.$

Differentiating implicitly (seems a good start)

$$2y \frac{dy}{dx} = 4a$$

So $\frac{dy}{dx} = \frac{2a}{y} = \frac{2a}{\sqrt{4ax}} = \sqrt{\frac{a}{x}}.$

So $\qquad 1+\left(\frac{dy}{dx}\right)^2 = 1 + \frac{\cancel{4}a}{x}$

and $\qquad \sqrt{1+\left(\frac{dy}{dx}\right)^2} = \sqrt{1 + \frac{\cancel{4}a}{x}}$

and $\qquad y\sqrt{1+\left(\frac{dy}{dx}\right)^2} = \sqrt{4ax}\sqrt{1+\frac{\cancel{4}a}{x}}$

$$= \sqrt{4ax + 4\cancel{a}a^2}$$

$$= 2\sqrt{ax + \cancel{4}a^2}$$

So we need to find

$$S = \int_0^h 2\sqrt{ax+a^2}\,dx.$$

Let $ax + a^2 = u$: then $a\,dx = du$ and $dx = \frac{du}{a}$

$S = \frac{2}{a}\int u^{\frac{1}{2}}\,du$ with limits $\quad x=0: \ u = a^2$

$\qquad\qquad\qquad\qquad\qquad\qquad x=h: \ u = a(a+h)$

Then $S = \frac{2}{a}\left[\frac{2}{3}u^{3/2}\right]_{a^2}^{a(a+h)}$

$= \frac{4}{3a}\left(a^{3/2}(a+h)^{3/2} - a^{3/2}a^{3/2}\right)$

$= \frac{4a^{\frac{1}{2}}}{3}\left((a+h)^{3/2} - a^{3/2}\right)$

$= \frac{4\sqrt{a}}{3}\left((a+h)^{3/2} - a^{3/2}\right)$

(Not sure if this includes the flat end, but by now I've given up caring.)

c) This looks a stinker:

$V = 2\pi a h^2$

$\frac{dV}{dt} = \lambda S$ for some λ.

$\frac{dV}{dt} = \frac{dV}{dh}\cdot\frac{dh}{dt} = 4\pi a h\frac{dh}{dt}$

So in general

$$4\pi a h \frac{dh}{dt} = \lambda S$$

$$= \lambda \, \frac{4}{3}\sqrt{a}\left((a+h)^{3/2} - a^{3/2}\right)$$

So when $h = 8a$,

$$4\pi a \times 8a \times 0.26 = \lambda \, \frac{4}{3}\sqrt{a}\left((9a)^{3/2} - a^{3/2}\right)$$

$$= \lambda \, \frac{4}{3}\sqrt{a}\left(\cancel{728}\,26\, a^{3/2}\right)$$

$$32\pi \cancel{a^2} \times 0.26 = \lambda . \frac{4}{3} . \cancel{728}\,26 . \cancel{a^2}$$

$$\lambda = \frac{32\pi \times 0.26 \times 3}{4 \times \cancel{728}\,26} = \cancel{0.02598}\ \frac{24}{100}\pi$$

$$=$$

$$\text{So } \frac{dV}{dt} = \cancel{0.0259358}\ \frac{24}{100}\pi \quad (= 0.7540)$$

$$= \cancel{0.02598} \times \frac{4}{3}\sqrt{a}\left((a+h)^{3/2} - a^{3/2}\right) \times \frac{24\pi}{100}$$

when $h = 3a$:

$$\frac{dV}{dt} = \cancel{0.02598} \times \frac{4}{3}\sqrt{a}\left((4a)^{3/2} - a^{3/2}\right) \times \frac{6\pi}{25}$$

$$= \frac{4}{3}\sqrt{a}\left(7a^{3/2}\right) \times \frac{6\pi}{25} = \frac{28}{3} a^2 . \frac{6\pi}{25} = \frac{56\pi a^2}{25}$$

(mm³/s, pres.)

2) a) $$10\pi \cos\left(\frac{\pi x}{2}\right) + 10\pi \cos\left(\frac{2\pi x}{3}\right) = 0$$

using standard identity:

$$20\pi \cos\left(\frac{7\pi x}{12}\right)\cos\left(\frac{\pi x}{12}\right) = 0$$

So $\cos\left(\frac{7\pi x}{12}\right) = 0$: $\frac{7\pi}{12}x = \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}, \ldots$

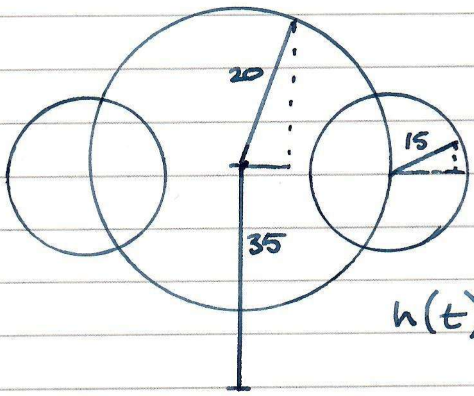
or $\cos\left(\frac{\pi x}{12}\right) = 0$: $\frac{\pi x}{12} = \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}, \ldots$

$$x = \frac{6}{7}, \frac{18}{7}, \frac{30}{7}, \underbrace{\frac{42}{7}}_{=6}, \cancel{\frac{54}{7}}$$

or $x = 6, \cancel{18}, \cancel{30}, \cancel{42}, \cancel{54}$

i.e. $\underline{\underline{x = \frac{6}{7}, \frac{18}{7}, \frac{30}{7} \text{ or } 6}}$ for $0 \leqslant x \leqslant 2\pi$.

2) b) i)

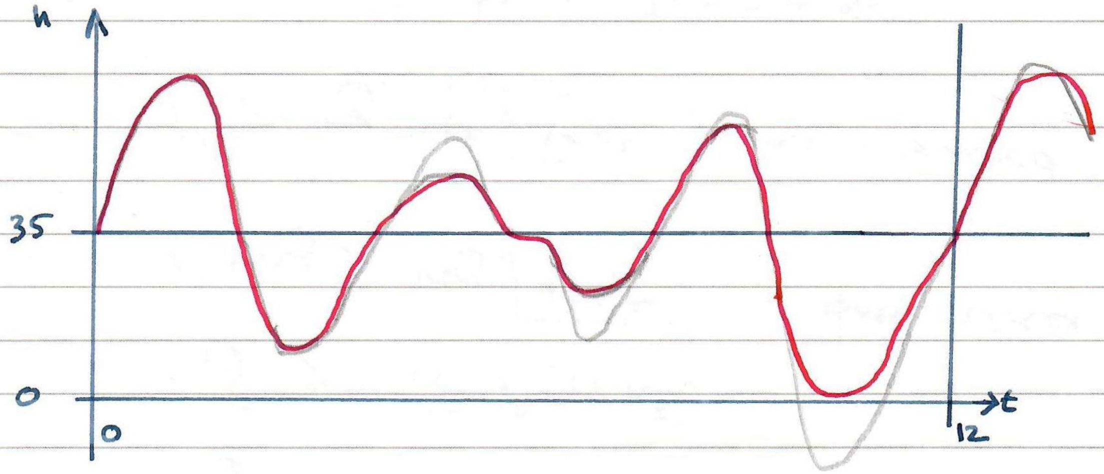


This question's easier than it looks, once you've figured it out.

$$h(t) = 35 + 20\sin\left(\frac{2\pi t}{4}\right) + 15\sin\left(\frac{2\pi t}{3}\right)$$

2π - has to go round that far...
... in 4 or 3 as the period

2)b) ii) By contrast, this part is hard - because we don't have access to the graph. which is actually:



Differentiate:

$$h'(t) = 20.\frac{\pi}{2}\cos\left(\frac{\pi t}{2}\right) + 15.\frac{2\pi}{3}\cos\left(\frac{2}{3}\pi t\right)$$

$= 0$ for a local max or min.

Using the expression in (2a) this gives

$t = \frac{6}{7}, \frac{18}{7}, \frac{30}{7}$ or 6 (not sure about higher values)

and from the graph the highest occurs when $t = \frac{6}{7}$

i.e. $h\left(\frac{6}{7}\right) = 35 + 20\sin\left(\frac{6}{14}\pi\right) + 15\sin\left(\frac{12}{21}\pi\right)$

$= \underline{\underline{69.1224\text{ m}}}.$

(Published answers go into h''(t) etc but don't really deal with the overall graph)

$y_{A_1} = 2a\sqrt{\frac{2b}{3a}} = 2\sqrt{2}\sqrt{\frac{ba}{3}}$ $\quad y_{\cancel{B}_2} = 2\sqrt{2a}\sqrt{\frac{b}{3a}} = 2\sqrt{2}\sqrt{\frac{ba}{3}}$

$y_{B_1} = -2a\sqrt{\frac{2b}{3a}} = -2\sqrt{2}\sqrt{\frac{ba}{3}}$ $\quad y_{B_2}$ similarly.

So A and B are $\left(\frac{5}{3}b, \pm 2\sqrt{2}\sqrt{\frac{ab}{3}}\right)$

or $\left(\frac{5b}{3}, \pm 2\sqrt{\frac{2ab}{3}}\right)$

b) This could be an integration task, but I suspect we can do it with less work.

C_1 cuts the x-axis when $y = 2at = 0$ so $t = 0$.
So $x = b$.

C_2 - — — — — $\sqrt{2}as = 0$ so $s = 0$
So $x = 3b$.

So since both curves are parabolas and one is a stretch / reflection of the other, ratio required =

$$\frac{3b - \frac{5b}{3}}{\frac{5b}{3} - b} = \frac{\frac{4}{3}}{\frac{2}{3}} = 2:1$$

(The answers make it much harder, using integration.)

3) a) Hard things here are keeping track of all the functions, remembering the difference between $\sin x$ and x when you're dealing with radians, and a lot of reflection in $\frac{\pi}{2}$, π and 2π.

☹ <u>And</u> The graph paper in the question has <u>six</u> minor divisions for each major one!

$$f(x) = 3\sin 2x$$
$$g(x) = 2\cos x$$
$$h(x) = k + 2\cos x = k + g(x)$$

when $f(x)$ and $h(x)$ (and $g(x)$) have the same gradient:

$$f'(x) = 6\cos 2x = -2\sin x = h'(x)$$

And $\cos 2x = \cos^2 x - \sin^2 x$
$= 1 - 2\sin^2 x.$

So $6(1 - 2\sin^2 x) = -2\sin x$

$$12\sin^2 x - 2\sin x - 6 = 0$$

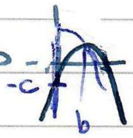
$$6\sin^2 x - \sin x - 3 = 0$$

$$\sin x = \frac{1 \pm \sqrt{73}}{2} = 0.795 \text{ or } -0.679$$

This gives as values of $\underline{x}$: 0.919
2.226
(−0.680)
(See it all plotted on the printed exam paper.) 3.822
5.603

b) $P(x) = -a(x-b)^2 - c$

$a > 0$ so
$b > 0$
$c > 0$



We know $-a(2.575 - b)^2 - c$ $= f(2.575)$
$= 3\sin(2 \times 2.575)$
$= 3\sin(5.150)$
$= -2.7173$

radians!!

The question gets ever meaner but leads eventually to a parabola well away from where you might expect. And when it says 'exactly one point of intersection' it actually means touches, tangentially – to both $f(x)$ and $g(x)$.

I've drawn the curve on the exam paper and I've give the answer – but I'm not going to bother with all the calculator-bashing:

$a = 2.244 \quad b = \pi \quad c = 2.$

c) Part c likewise. $1.68 < k < 4.4887.$

Altogether an appalling question.

4) a)

$$z = \underbrace{\sqrt{\tfrac{1}{2}\left(a+\sqrt{(a^2+b^2)}\right)}}_{X} + i\underbrace{\sqrt{\tfrac{1}{2}\left(-a+\sqrt{(a^2+b^2)}\right)}}_{Y}$$

$$z^2 = X^2 - Y^2 + 2XYi$$

$$= \tfrac{1}{2}\left(a+\sqrt{a^2+b^2}\right) - \tfrac{1}{2}\left(-a+\sqrt{a^2+b^2}\right)$$

$$+ 2i\sqrt{\tfrac{1}{2^2}\left(a+\sqrt{a^2+b^2}\right)\left(-a+\sqrt{a^2+b^2}\right)}$$

$$= a + i\sqrt{-a^2 + a^2 + b^2}$$

$$= \underline{\underline{a + ib}}$$

b) (They've managed to pick up the poorer colouring of the Mandelbrot set.)

i) $c = 1 + \frac{i}{2}$

$$f^1(0) = 1 + \frac{i}{2}$$

$$f^2(0) = \left(1+\frac{i}{2}\right)^2 + 1 + \frac{i}{2} = 1 - \frac{1}{4} + i + 1 + \frac{i}{2} = 1\tfrac{3}{4} + \tfrac{3}{2}i$$

Clearly this has modulus > 2, so already this shows c is not in the set.

ii) $c = i$: $\quad f'(0) = i \quad f^2(0) = i^2 + i = -1 + i$

$$f^3(0) = (-1+i)^2 + i = 1 - 1 - 2i + i = -i$$

$$f^4(0) = (-i)^2 + i = -1 + i.$$

Clearly this repeats and all $| \; | < 2$,

[illegible] is in the set.

ii) This looks over-complicated, once again, since we're only talking about $f(0)$.

$$f'(0) = c \quad \text{as given (same as part (a))}$$

$$f^2(0) = c^2 + c$$

$$= a + ib + c, \text{ and using } b^2 = 3a^2:$$

$$= a + ib + \sqrt{\tfrac{1}{2}\left(a + \sqrt{4a^2}\right)} + i\sqrt{\tfrac{1}{2}\left(-a + \sqrt{4a^2}\right)}$$

Assuming +ve roots, this is:

$$a + i\sqrt{3}a + \sqrt{\tfrac{1}{2}3a} + i\sqrt{\tfrac{1}{2}a}$$

So if $a = \frac{1}{8}$, $\sqrt{a} = \frac{1}{\sqrt{8}} = \frac{1}{2\sqrt{2}}$. Rearranging:

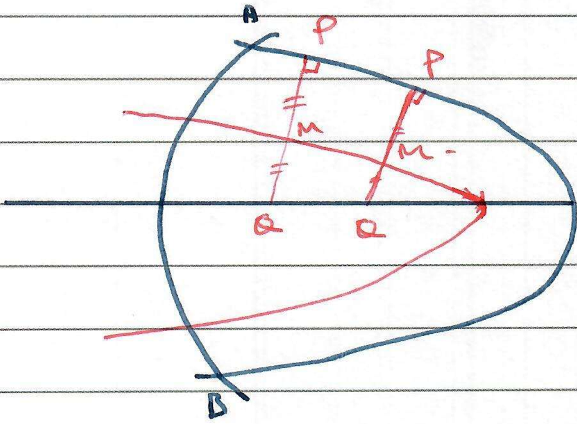
$$f^2(0) = \left(\frac{1}{8} + \sqrt{\frac{3}{16}}\right) + i\left(\frac{\sqrt{3}}{8} + \sqrt{\frac{1}{16}}\right)$$

$$= \left(\frac{1}{8} + \frac{\sqrt{3}}{4}\right) + i\left(\frac{\sqrt{3}}{8} + \frac{1}{4}\right) = \frac{1}{8}\left(1 + 2\sqrt{3}\right) + \frac{i}{8}\left(\sqrt{3} + 2\right)$$

So $|f^2(0)| = \frac{1}{8}\sqrt{(1+2\sqrt{3})^2 + (\sqrt{3}+2)^2} = \frac{1}{8}\sqrt{20 + 8\sqrt{3}} = \underline{\frac{1}{4}\sqrt{5 + 2\sqrt{3}}}$ as required

5) $C_1: \quad x = at^2 + b \qquad y = 2at$

$C_2: \quad x = 3b - as^2 \qquad y = \sqrt{2}\,as$



At A and B: $\qquad at^2 + b = 3b - as^2 \quad$ —— ①

$\qquad 2at = \sqrt{2}\,as \quad$ —— ②

from ②: $\quad s = \sqrt{2}\,t.$

in ①: $\quad at^2 + b = 3b - 2at^2$

$3at^2 = 2b$

$t^2 = \dfrac{2b}{3a}$

$$t = \pm\sqrt{\dfrac{2b}{3a}} \qquad s = \pm 2\sqrt{\dfrac{b}{3a}}$$

So $x_A = a \cdot \dfrac{2b}{3a} + b = \dfrac{5}{3}b \qquad x_B = 3b - 4a\dfrac{b}{3a} = \dfrac{5}{3}b.$

c) I suspect there's an easy geometric way to do this, but... (or at least a standard result. Mr Budden would know it.)

C_2 is the curve $(3b - as^2, \sqrt{2}as)$

Not sure why we're bothering with the parameter still, so:

$$s = \frac{y}{\sqrt{2}a} \qquad s^2 = \frac{y^2}{2a^2}$$

$$x = 3b - \frac{ay^2}{2a^2} = 3b - \frac{y^2}{2a}$$

$$2ax = 6ab - y^2$$

Differentiate implicitly:

$$\cancel{2}a = -\cancel{2}y\frac{dy}{dx}$$

$$\frac{dy}{dx} = -\frac{a}{y}$$

So the normal has gradient $\frac{dy}{dx} = \frac{y}{a} = \frac{\sqrt{2}\cancel{a}s}{\cancel{a}}$

So the normal has eq

$$y = \frac{\sqrt{2}}{2}\, s\, x + c$$

Subs in for P:

$$\sqrt{2as} = \sqrt{2}\, s\,(3b - as^2) + c$$

$$c = \sqrt{2as} - \sqrt{2}\, s\,(3b - as^2)$$

So $y = \sqrt{2}sx - 3\sqrt{2}\,sb + \sqrt{2}\,s^3a + \sqrt{2}\,as$

So when $y = 0$,

$$0 = \sqrt{2}\,sx - 3\sqrt{2}\,sb + \sqrt{2}\,s^3a + \sqrt{2}\,as$$

$$= x - 3b + s^2a + a$$

So point Q is $(3b - a - as^2)\ 0.$

So since M is the midpoint of PQ

M is $\left(\dfrac{3b - as^2 + 3b - a - as^2}{2}, \dfrac{\sqrt{2as} + 0}{2} \right)$

$$= \left(\frac{6b - a - 2as}{2}, \frac{as}{\sqrt{2}} \right)$$

This looks a bit horrid, but if we say it's (x, y) then we have

$$y = \frac{as}{\sqrt{2}} \qquad s = \frac{\sqrt{2}y}{a}.$$

So $x = \cancel{\dfrac{3b - a^2 + 3b - a - \frac{2y^2 a}{a^2}}{2}}$

$$3b - \frac{a}{2} - \frac{2y^2}{a}$$

$$\frac{2y^2}{a} = -x + 3b - \frac{a}{2}$$

$$y^2 = -\frac{a}{2}x + \frac{3ab}{2} - \frac{a^2}{4}$$

$$= -\frac{a}{2}\left(x - 3b + \frac{a}{2}\right)$$

which is the form required with $k = -\frac{1}{2}$, $m = \frac{1}{2}$, $n = -3$.