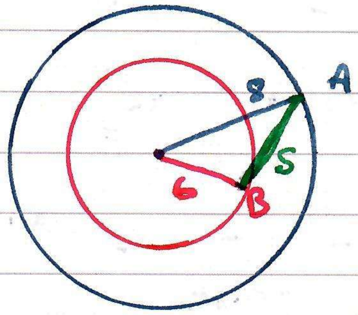
NZQA Scholarship Calculus

2006 1)a)



Measure time t in hours, and coords in usual x, y

Position of A: $x = 8\sin(2\pi t)$

$y = 8\cos(2\pi t)$

Position of B: $x = 6\sin\left(\frac{2\pi t}{12}\right) = 6\sin\left(\frac{\pi t}{6}\right)$

$y = 6\cos\left(\frac{2\pi t}{12}\right) = 6\cos\left(\frac{\pi t}{6}\right)$

So $s = \sqrt{\left(8\sin(2\pi t) - 6\sin\left(\frac{\pi t}{6}\right)\right)^2 + \left(8\cos(2\pi t) - 6\cos\left(\frac{\pi t}{6}\right)\right)^2}$

$= \sqrt{8^2\left(\sin^2(2\pi t) + \cos^2(2\pi t)\right) + 6^2\left(\sin^2\left(\frac{\pi t}{6}\right) + \cos^2\left(\frac{\pi t}{6}\right)\right)^2 - 96\left(\sin(2\pi t)\sin\left(\frac{\pi t}{6}\right) + \cos(2\pi t)\cos\left(\frac{\pi t}{6}\right)\right)}$

$$= \sqrt{100 - 96\left(\cos\left(2\pi - \frac{\pi}{6}\right)t\right)} = 2\sqrt{25 - 24\cos\left(2\pi - \frac{\pi}{6}\right)t}$$

So $$\frac{ds}{dt} = \frac{2}{2}\,\frac{\left(+24\sin\left(2\pi - \frac{\pi}{6}\right)t\right)\left(2\pi - \frac{\pi}{6}\right)}{\sqrt{25 - 24\cos\left(2\pi - \frac{\pi}{6}\right)t}}$$

$$= \frac{24\left(2\pi - \frac{\pi}{6}\right)\sin\left(2\pi - \frac{\pi}{6}\right)t}{\sqrt{25 - 24\cos\left(2\pi - \frac{\pi}{6}\right)t}}$$

When $t = 9$, $\left(2\pi - \frac{\pi}{6}\right)t = \frac{11}{6} \times 9\pi$ which can be considered $\frac{\pi}{2}$ for the trig. functions

So $$\frac{ds}{dt}(9) = \frac{24\left(\frac{11}{6}\pi\right)\overset{1}{\cancel{\sin\left(\frac{\pi}{2}\right)}}}{\sqrt{25 - 24\underset{0}{\cancel{\cos\left(\frac{\pi}{2}\right)}}}}$$

$$= \frac{24 \times 11}{6 \times 5}\pi = 27.64 \text{ cm/hour}$$

$$= \frac{11}{75}\pi = 0.4607 \text{ cm/min.}$$

(Published answers use an argument about the angle between the hands - but $\frac{d}{dt}$ with this feels slippery).

NZQA Scholarship Calculus
2006 1) b)

$$\ln(1+x) \approx A + Bx + Cx^2 \quad ①$$

$$\frac{d}{dx}: \quad \frac{1}{1+x} = B + 2Cx \quad ②$$

$$\frac{d^2}{dx^2}: \quad \frac{-1}{(1+x)^2} = 2C \quad ③$$

Using $x=0$: ①: $0 = A$
②: $1 = B$
③: $-1 = 2C$ $C = \frac{-1}{2}$

So back in ①: $\ln(1+x) \approx x - \frac{x^2}{2}$ as required

(this is assuming differentiating across $\approx$ is OK ...
... personally I'd want more $\lim_{\rightarrow}$ etc)

In $\left(1+\frac{1}{2n}\right)^{n+3} < \left(1+\frac{1}{n}\right)^{n-1}$ we're presumably using the approximation. Since ln is continuously increasing on x (n) we can take ln:

$$(n+3)\ln\left(1+\frac{1}{2n}\right) < (n-1)\ln\left(1+\frac{1}{n}\right)$$

Re-writing these in terms of the approximation gives:

$$(n+3)\left(\frac{1}{2n} - \frac{1}{2}\left(\frac{1}{2n}\right)^2\right) < (n-1)\left(\frac{1}{n} - \frac{1}{2}\frac{1}{n^2}\right)$$

$$\frac{1}{2n}(n+3)\left(1-\frac{1}{4n}\right) < \frac{1}{n}(n-1)\left(1-\frac{1}{2n}\right)$$

Mult by $8n^2$ (ok since this is +ve)

$$(n+3)(4n-1) < 4(n-1)(2n-1)$$

$$4n^2 + 11n - 3 < 8n^2 - 12n + 4$$

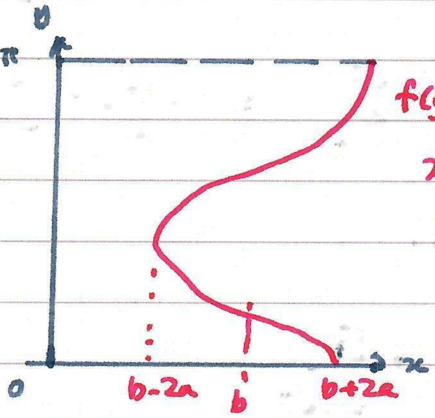
$$0 < 4n^2 - 23n + 7$$

This has roots $\quad n = \dfrac{23 \pm \sqrt{23^2 - 4 \times 28}}{8}$

and the +ve root $(n>0)$ is 5.428.

So the value of n required is $\underline{\underline{6}}$.

NZQA Scholarship Calculus
2006 1) c)



Volume of the vase obtained by revolution is

$$V = \int_0^{\pi} \pi x^2 \, dy$$

This is an expression and appears to answer the question - though presumably they want more.

$$= \pi \int_0^{\pi} \left(2a\cos(2y) + b\right)^2 dy$$

$$= \pi \int \left(4a^2\cos^2(2y) + 4ab\cos(2y) + b^2\right) dy$$

And $\cos(4y) = \cos^2(2y) - \sin^2(2y)$

$= 2\cos^2(2y) - 1$

So $\cos^2(2y) = \frac{1}{2}\left(1 + \cos(4y)\right)$

$$\text{So } V = \pi \int_0^{\pi} \left(\frac{4a^2}{2}(1+\cos(4y)) + 4ab\cos(2y) + b^2 \right) dy$$

$$= \pi \left[2a^2 y + \frac{2a^2}{4}\sin 4y + \frac{4ab}{2}\sin 2y + b^2 y \right]_0^{\pi}$$

$$= \pi \left((2a^2+b^2)\pi + \cancel{\frac{a^2}{2}\sin 4\pi} + \cancel{2ab\sin 2\pi} - (2a^2+b^2)0 - \cancel{\frac{a^2}{2}\sin 0} - \cancel{2ab\sin 0} \right)$$

$$= \underline{\underline{\pi^2(2a^2+b^2)}}$$

NZQA Scholarship Calculus
2006 2) a)

To find $\cos\left(\frac{\pi}{8}\right)$ use $\cos 2x = \cos^2 x - \sin^2 x$
$= 2\cos^2 x - 1$

So $\cos x = \sqrt{\dfrac{\cos 2x + 1}{2}}$

Using $x = \frac{\pi}{8}$, $2x = \frac{\pi}{4}$ and $\cos\left(\frac{\pi}{4}\right) = \frac{1}{\sqrt{2}}$

So $\cos\left(\frac{\pi}{8}\right) = \sqrt{\dfrac{\frac{1}{\sqrt{2}} + 1}{2}} = \sqrt{\dfrac{1+\sqrt{2}}{2\sqrt{2}}}$

$= \sqrt{\dfrac{\sqrt{2}+2}{4}} = \dfrac{\sqrt{2+\sqrt{2}}}{2}$ as required.

$z^2 = \dfrac{1+7i}{-3+4i} = \dfrac{(1+7i)(-3-4i)}{(-3+4i)(-3-4i)}$

$= \dfrac{-3+28-21i-4i}{3^2+4^2}$

$=$

$$= \frac{25 - 25i}{25} = 1 - i$$

So $z^2 = \sqrt{2}\operatorname{cis}\left(-\frac{\pi}{4}\right)$ in polar terms.

$$\therefore \quad z = \pm \sqrt[4]{2}\operatorname{cis}\left(-\frac{\pi}{8}\right)$$

$$= \pm \sqrt[4]{2}\left(\cos\left(-\frac{\pi}{8}\right) + i\sin\left(-\frac{\pi}{8}\right)\right)$$

$$\cos\left(-\frac{\pi}{8}\right) = \cos\left(\frac{\pi}{8}\right) = \frac{\sqrt{2+\sqrt{2}}}{2}$$

$$\sin\left(-\frac{\pi}{8}\right) = -\sin\left(\frac{\pi}{8}\right) = -\sqrt{1-\cos^2\left(\frac{\pi}{8}\right)}$$

$$= -\sqrt{1-\left(\frac{2+\sqrt{2}}{4}\right)} = -\sqrt{\frac{4-2-\sqrt{2}}{4}}$$

$$= -\frac{\sqrt{2-\sqrt{2}}}{2}$$

So putting it all together

$$z = \pm \sqrt[4]{2}\left(\frac{\sqrt{2+\sqrt{2}}}{2} - i\,\frac{\sqrt{2-\sqrt{2}}}{2}\right)$$

$$= \frac{\sqrt{2\sqrt{2}+2}}{2} - i\,\frac{\sqrt{2\sqrt{2}-2}}{2}$$

or (negative)

$$-\frac{\sqrt{2\sqrt{2}+2}}{2} + i\,\frac{\sqrt{2\sqrt{2}-2}}{2}$$

which is in the required form.

b) $f(x) = x^3 - 3x^2 - x + 2$: rotate $180°$ around $(1, -1)$

one version of the answer gets into inflexion points... but the other says:

Shift the centre to $(1, -1)$:

$$f(x) = (x+1)^3 - 3(x+1)^2 - (x+1) + 2 + 1$$

Rotate, by replacing $f(x)$ by $-f(x)$, x by $-x$:

$$f(x) = (1-x)^3 + 3(x-1)^2 - (x-1) - 3$$

Shift back: $g(x) = (x-2)^3 + 3(x-2)^2 - (x-2) - 3 - 1$

$$= x^3 - 3x^2 - x + 2.$$

NZQA Scholarship Calculus
2006 3) a)

$y = \sin(\ln x)$

$\frac{dy}{dx} = \frac{\cos(\ln x)}{x}$ (chain rule)

$\frac{d^2y}{dx^2} = \frac{-x\sin x(\ln x)/x - \cos(\ln x)}{x^2}$ (quotient + chain rules)

$= -\frac{\cos(\ln x) + \sin(\ln x)}{x^2}$

So in the given equation,

$x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} + y = k$

$= \frac{x^2}{x^2}(-\cos(\ln x) - \sin(\ln x)) + \frac{x}{x}\cos(\ln x)$
$+ \sin(\ln x))$

$= 0$. So $\underline{\underline{k = 0}}$. (this seems a bit pointless)

So on the given equation:

$$\frac{d\left(x^2\frac{dy}{dx}\right)}{dx} = x\frac{dy}{dx} - y + 5$$

and the LHS becomes: (using product rule)

$$x^2\frac{d^2y}{dx^2} + 2x\frac{dy}{dx} = x\frac{dy}{dx} - y + 5.$$

i.e. $$x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} + y = 5.$$

Pres. were meant to say LHS = 0 has a sol^n $y = \sin(\ln x)$, so now look for a sol^n LHS = 5 which can have a simple form.

Clearly $y = 5$ works, so we can say a solution is:

$$y = \sin(\ln x) + 5.$$

3) b) $\sin(\ln x) = a \qquad \cos(\ln y) = \frac{1}{b}$

Find $\tan(\ln xy) . \tan\left(\ln\left(\frac{x}{y}\right)\right)$

This is $\tan(\ln x + \ln y) . \tan(\ln x - \ln y)$

$$= \frac{\tan(\ln x) + \tan(\ln y)}{1 - \tan(\ln x)\tan(\ln y)} \times \frac{\tan(\ln x) - \tan(\ln y)}{1 + \tan(\ln x)\tan(\ln y)}$$

$$= \frac{\tan^2(\ln x) - \tan^2(\ln y)}{1 - \tan^2(\ln x)\tan^2(\ln y)}$$

Also $\sin^2(\ln x) = a^2$ so $\cos^2(\ln x) = 1 - a^2$

$\therefore \tan^2(\ln x) = \frac{a^2}{1-a^2}$

$\cos^2(\ln y) = \frac{1}{b^2}$ so $\sin^2(\ln y) = 1 - \frac{1}{b^2}$

$\therefore \tan^2(\ln y) = \dfrac{1 - \frac{1}{b^2}}{\frac{1}{b^2}} = \dfrac{\frac{b^2-1}{b^2}}{\frac{1}{b^2}} = b^2 - 1$

So the expression is $\dfrac{\frac{a^2}{-a^2+1} - (b^2-1)}{1 - \frac{a^2}{-a^2+1}(b^2-1)}$

$$= \frac{a^2 - (b^2-1)(1-a^2)}{1-a^2 - a^2(b^2-1)}$$

$$= \frac{a^2b^2 - b^2 + 1}{1-a^2b^2}$$

When this $= 1$,

$$a^2b^2 - b^2 + 1 = 1 - a^2b^2$$

$$2a^2b^2 = b^2$$

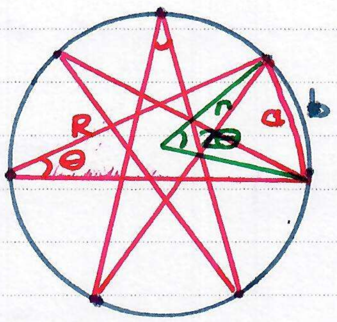
$b \neq 0$, so $a = \pm \frac{1}{\sqrt{2}}$ $\quad \sin(\ln x) = \pm \frac{1}{\sqrt{2}}$

(so $\ln x = \pm \frac{\pi}{4}$

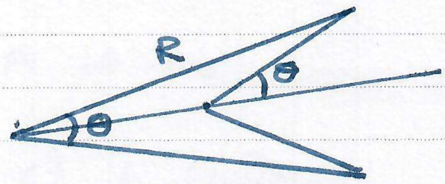
$x = e^{-\frac{\pi}{4}}$ or $e^{\frac{\pi}{4}}$)

NZQA Scholarship Calculus

2006 4) a)



This is a simple 'angle at circumference is half the angle at the centre' problem:



$2\theta = \frac{2\pi}{7}$: $\theta = \frac{2\pi}{2\times7} = \frac{\pi}{7}$. $\frac{\theta}{2} = \frac{\pi}{14}$.

Assume $r = 1$ (this is only a ratio problem)

$\frac{R}{2\times1} = \cos\frac{\pi}{14}$: $R = 2\cos\frac{\pi}{14}$

$a = \frac{2\pi R}{14}$

$b = \frac{2\pi\cdot1}{7}$

So $\frac{a}{b} = \frac{\cancel{2\pi}R}{\cancel{14}_{2}} \times \frac{\cancel{7}}{\cancel{2\pi}}$

$= \frac{2\cos\frac{\pi}{14}}{2}$

$= \cos\frac{\pi}{14} = \cos 12.857^{0}$

$= 0.975$

3) b) $\quad y = f(x)$

$$y\frac{dy}{dx} = 2y + x \qquad \text{——} \; ①$$

To find the $g(x)$ family $y = g(x)$, let the new y have $\dfrac{dy}{dx} = \dfrac{-1}{\frac{dy}{dx}\text{ original}}$:

So from ①, $\quad \dfrac{dy}{dx} = 2 + \dfrac{x}{y}$

And for the $g(x)$ family:

$$\frac{dy}{dx} = \frac{-1}{2 + \frac{x}{y}} = \frac{-1}{\left(\frac{2y+x}{y}\right)} = \frac{-y}{2y+x} \qquad ②$$

~~Using the hint given:~~

$$\cancel{z = \frac{x}{y} \qquad \frac{dz}{dx} = \frac{y - x\frac{dy}{dx}}{y^2}} \qquad \text{——} \; ③$$

~~From ②:~~ $\quad \cancel{x\dfrac{dy}{dx} + 2y\dfrac{dy}{dx} = -y} \qquad \text{——} \; ④$

Using a Cambridge Tripos technique:

$$\frac{dx}{dy} = \frac{2y+x}{-y}$$

$$y\frac{dx}{dy} = -2y - x$$

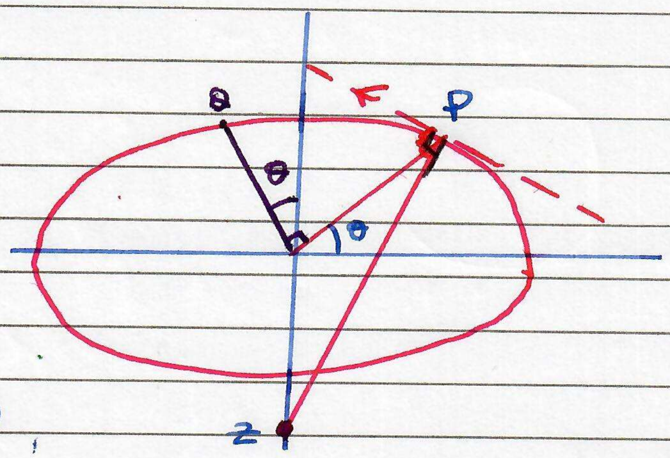
$$\frac{d}{dy}(xy) = y\frac{dx}{dy} + x = -2y.$$

We can integrate both sides wrt y:

$$xy = -\frac{2y^2}{2} + c = c - y^2$$

$$xy + y^2 = c$$

$$\underline{\underline{(x+y)y = c}}$$



5) a) Find Z.

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

P: $(a\cos\theta, \ b\sin\theta)$

Direction of motion is the gradient of the ellipse and this is $\frac{dy}{dx}$ ... but we need to work in terms of θ:

$$\frac{dx}{d\theta} = -a\sin\theta \qquad \frac{dy}{d\theta} = b\cos\theta.$$

$$\frac{dy}{dx} = -\frac{b}{a}\frac{\cos\theta}{\sin\theta} = -\frac{b}{a} \text{ when } \theta = \frac{\pi}{4}.$$

So the gradient of the beam (normal) is $\frac{a}{b}$.

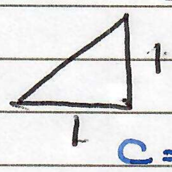
and it passes through P which is $\left(\frac{a}{\sqrt{2}}, \frac{b}{\sqrt{2}}\right)$

Substituting:

θ ϕ

th f

$$y = \frac{a}{b}x + c$$



$$\frac{b}{\sqrt{2}} = \frac{a}{b}\frac{a}{\sqrt{2}} + c \qquad c = \frac{b}{\sqrt{2}} - \frac{a^2}{b\sqrt{2}}$$

So the y-intercept is $\frac{b}{\sqrt{2}} - \frac{a^2}{b\sqrt{2}}$ and since

P is at height $b\cos\theta = \frac{b}{\sqrt{2}}$, this is $\frac{a^2}{b\sqrt{2}} = \frac{\sqrt{2}a^2}{b}$ below P

as required.

For Q, we use the fact $QOP = \frac{\pi}{2}$:

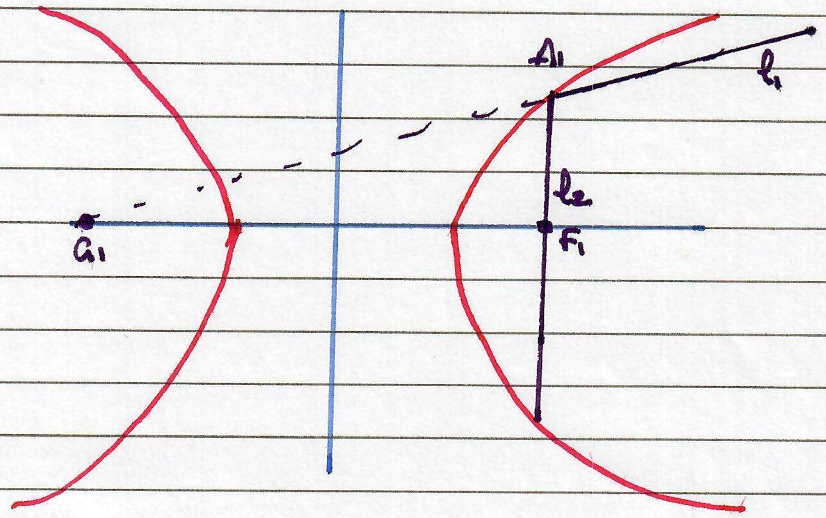
so coords of Q are

$$\left(a\cos\left(\theta+\frac{\pi}{2}\right),\ b\sin\left(\theta+\frac{\pi}{2}\right)\right)$$

$$= \underline{\underline{(-a\sin\theta,\ b\cos\theta)}}$$

3) a) See ~~[illegible]~~ next sheet.

b)



It's an interesting set-up. Actually the reflected line l_2 always goes through F_1 even if it isn't vertical ... it's a basic property of hyperbolas if l_1 goes through G_1.

But here, pres. to make the algebra easy, we're told AF_1 is vertical.