1 a) α is a complex root of $z^5 = 1$

i.e. of $z^5 - 1 = 0$

$z = 1$ is clearly a solution, so $(z-1)$ is a factor,

so dividing:

$$(z-1)(\underbrace{z^4+z^3+z^2+z+1}_{f(z)}) = 0$$

Since α is a root and is $\neq 1$, $f(\alpha)$ must be zero -

i.e. $\alpha^4+\alpha^3+\alpha^2+\alpha+1 = 0$

or $\underline{\underline{\alpha^4+\alpha^3+\alpha^2+\alpha = -1}}$. as required.

Since the original eq^n ① has real coeffs,

if α is a root then so is $\bar{\alpha}$ (conjugate)

and this is α^4 : similarly α^2 and α^3.

But this isn't quite what's wanted.

Consider the eq^n with $(\alpha+\alpha^4)$ and $(\alpha^2+\alpha^3)$ as roots:

$$(z-(\alpha+\alpha^4))(z-(\alpha^2+\alpha^3)) = 0$$

$$z^2 - z(\alpha+\alpha^2+\alpha^3+\alpha^4) + (\alpha+\alpha^4)(\alpha^2+\alpha^3) = 0$$

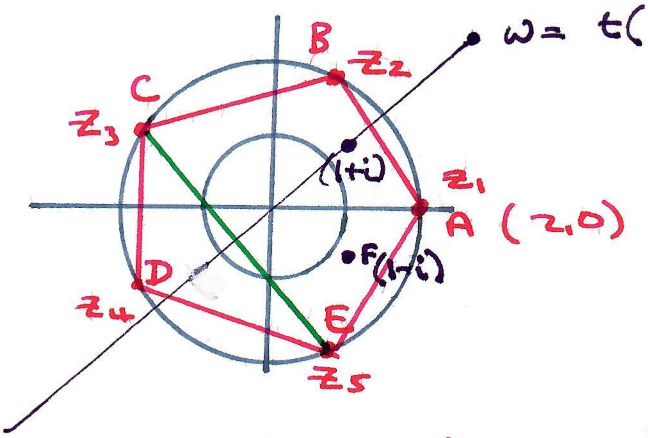
$$z^2 + z + (\underbrace{\alpha^3+\alpha^6+\alpha^4+\alpha^7}_{=\alpha^3+\alpha+\alpha^4+\alpha^2}) = 0$$

$$z^2 + z - 1 = 0$$

b) $z^5 - 32 = 0$

The roots are 2, $2\,\text{cis}\left(\frac{2\pi}{5}\right) \cdots \left(\frac{4\pi}{5}\right) \cdots \left(\frac{6\pi}{5}\right) \cdots \left(\frac{8\pi}{5}\right)$

$z_1 \quad z_2 \quad z_3 \quad z_4 \quad z_5$



ABCDE is a regular pentagon (not drawn perfectly)

By simple geometry $|CE| = |BE|$

and since B and E are conjugates with $B(x+ib)$

this gives $|BE| = 2b$ so $|CE| = 2b$.

F is the point $(1,-1)$ i.e the number $1-i$

Distance $AW + FW$

$$= \sqrt{(2-t)^2 + (0-t)^2} + \sqrt{(1-t)^2 + (-1-t)^2}$$

$$= \sqrt{4 - 4t + t^2 + t^2} + \sqrt{1 - 2t + t^2 + 1 + 2t + t^2}$$

$$= \sqrt{4 - 4t + 2t^2} + \sqrt{2t^2 + 2}$$

$$= \sqrt{2}\left(\sqrt{2 - 2t + t^2} + \sqrt{t^2 + 1}\right)$$

$$\frac{d(AW+FW)}{dt} = \sqrt{2}\left(\frac{1}{2}\frac{2t - 2}{\sqrt{2 - 2t + t^2}} + \frac{1}{2}\frac{2t}{\sqrt{t^2+1}}\right)$$

The answer published doesn't do the differentiation correctly - but this leads to

$$\frac{2t-2}{\sqrt{2-2t+t^2}} = \frac{2t}{\sqrt{t^2+1}}$$

$$(2t-2)\sqrt{t^2+1} = 2t\sqrt{2-2t+t^2}$$

Squaring both sides:

$$4(t-1)^2(t^2+1) = 4t^2(2-2t+t^2)$$

$$(4t^2-8t+4)(t^2+1) = 4t^2(2-2t+t^2)$$

$$\cancel{4t^4}-\cancel{8t^3}+\cancel{4t^2}+\cancel{4t^2}-8t+4 = \cancel{8t^2}-\cancel{8t^3}+\cancel{4t^4}$$

$$-8t+4=0$$

$$t=\tfrac{1}{2}.$$

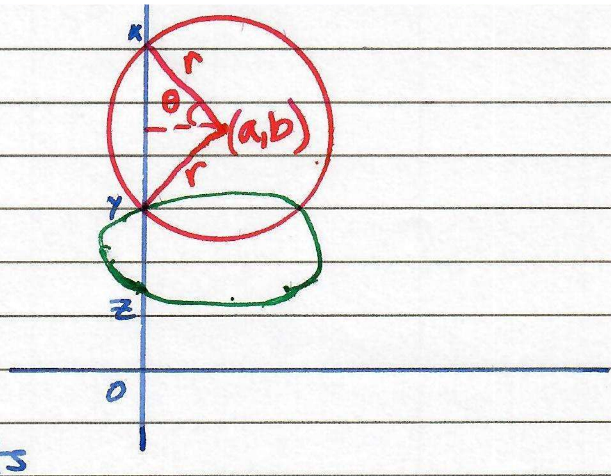
(and miraculously this is also the published answer)

Assuming (allowed) this gives a minimum:

$$AW+FW = \sqrt{\left(\tfrac{3}{2}\right)^2+\tfrac{1}{4}} + \sqrt{\tfrac{1}{4}+\left(\tfrac{3}{2}\right)^2}$$

$$= 2\sqrt{\frac{10}{4}} = \underline{\underline{\sqrt{10}}}.$$

1 a) This is actually an easy question dressed up as a hard one – and nothing to do with the ellipse at all.



Consider the points X and Y – these are both on the circle – and the mapping puts point X onto point Y:

$$\text{ie} \quad OY = \tfrac{1}{2}OX.$$

Using polar coords as shown:

X is point $(0,\ b + r\sin\theta)$

Y is point $(0,\ b - r\sin\theta)$

and also $(0,\ \tfrac{1}{2}(b + r\sin\theta))$

So $\quad b - r\sin\theta = \dfrac{b}{2} + \dfrac{r}{2}\sin\theta$

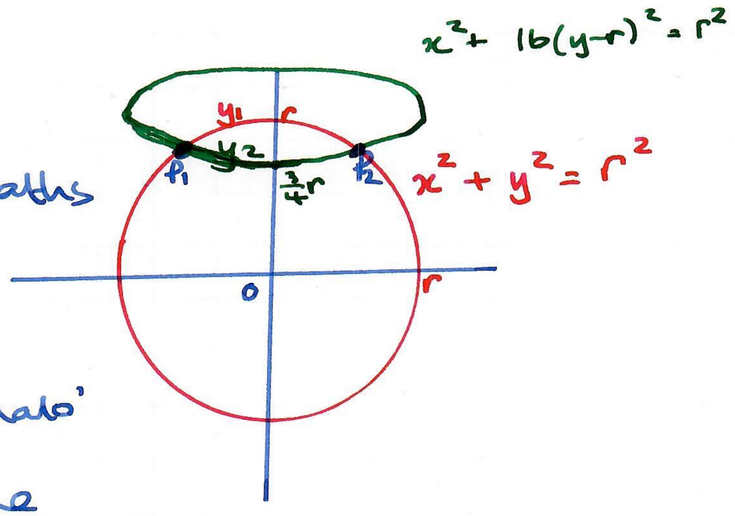
$$\frac{b}{2} = \frac{3r}{2}\sin\theta$$

$$b = 3r\sin\theta$$

$$\begin{aligned} b^2 &= 9r^2\sin^2\theta \\ &= 9r^2(1-\cos^2\theta) \\ &= 9r^2\left(1-\frac{a^2}{r^2}\right) \\ &= \underline{9(r^2-a^2)} \end{aligned}$$

as required.

2b) Once again, the diagram's drawn rather unhelpfully. But looking at part (b) perhaps explains how some people (overcomplicating it) approached Part (a). (The two parts are in fact not connected - and part (b) is much harder.)



Once you bash the maths a bit it's clear what's going on: the ellipse is a 'halo' sitting on top of the circle, shrunk vertically by a factor of $\frac{1}{4}$.

Firstly find points P_1 and P_2:

$$\cancel{x^2}+16(y-r)^2 = r^2 = \cancel{x^2}+y^2$$

$$16(y-r)^2 = y^2$$

$$y-r = \pm\frac{y}{4}$$

Clearly $y<r$ so
$$r-y = \frac{y}{4}$$
$$r = \frac{5y}{4}$$
$$y = \frac{4}{5}r.$$

Then $x^2 = r^2 - \frac{4^2}{5^2} r^2 = \frac{3^2}{5^2} r^2$

So $x = \pm \frac{3}{5} r$

(So we're dealing with nice 3,4,5 Δs.)

Now find expressions for y_1 and y_2 on $\left[-\frac{3}{5}r, \frac{3}{5}r\right]$, paying careful attention to signs.

For y_1: $y_1 = \sqrt{r^2 - x^2}$ +ve root taken always.

For y_2: $(y_2 - r)^2 = \frac{r^2 - x^2}{16}$

$$y_2 - r = \pm \frac{1}{4}\sqrt{r^2 - x^2}$$

y_2 has to be less than r so

$y_2 = r - \frac{1}{4}\sqrt{r^2 - x^2}$ +ve root taken from here, but use the - sign

So $y_1 - y_2 = \sqrt{r^2 - x} - \left(r - \frac{1}{4}\sqrt{r^2 - x^2}\right)$

$$= \frac{5}{4}\sqrt{r^2 - x^2} - r.$$

And now we can integrate to find the overlap area:

$$I = \int_{-\frac{3}{5}r}^{\frac{3}{5}r} \left(\frac{5}{4}\sqrt{r^2 - x^2} - r\right) dx.$$

$$= \frac{5}{4}\int_{-\frac{3}{5}r}^{\frac{3}{5}r}\sqrt{r^2-x^2}\,dx \quad - \left[\, r\,x \,\right]_{-\frac{3}{5}r}^{\frac{3}{5}r}$$

$$= I_1 \qquad - \left(\frac{3}{5}+\frac{3}{5}\right)r^2$$

To calculate I_1:

. Let $x = r\sin u$

Then $dx = r\cos u\,du$

and $r^2 - x^2 = r^2 - r^2\sin^2 u$

$= r^2(1-\sin^2 u) \quad = r^2\cos^2 u$

also $\cos 2u = \cos^2 u - \sin^2 u$

$= 2\cos^2 u - 1$

So $\cos^2 u = \frac{1}{2}(1+\cos 2u)$

So $I_1 = \frac{5}{4}\int r\cos u \,.\, r\cos u\,du$

$= \frac{5}{4}r^2\int \cos^2 u\,du$

$= \frac{5}{4}r^2 . \frac{1}{2}\int (1+\cos 2u)\,du$

with limits : $x = -\frac{3}{5}r = r\sin u$: $u = \sin^{-1}\left(\frac{-3}{5}\right)$

$= -\sin^{-1}\left(\frac{3}{5}\right)$

$x = \frac{3}{5}r = r\sin u$: $u = \sin^{-1}\left(\frac{3}{5}\right)$

So $I_1 = \frac{5r^2}{8}\left[u + \frac{1}{2}\sin 2u \right]_{-\sin^{-1}\left(\frac{3}{5}\right)}^{\sin^{-1}\left(\frac{3}{5}\right)}$

$$= \frac{5r^2}{8} \cdot 2\sin^{-1}\left(\frac{3}{5}\right) + \frac{5r^2}{16}\left(\sin 2\left(\sin^{-1}\frac{3}{5}\right) + \sin 2\left(\sin^{-1}\frac{3}{5}\right)\right)$$

$$= \frac{5}{4}r^2\sin^{-1}\left(\frac{3}{5}\right) + \frac{5r^2}{4}\sin\left(\sin^{-1}\frac{3}{5}\right)\cos\left(\sin^{-1}\frac{3}{5}\right)$$

$$= \frac{5}{4}r^2\sin^{-1}\left(\frac{3}{5}\right) + \frac{\cancel{5}r^2}{\cancel{4}} \times \frac{3}{\cancel{5}} \cdot \frac{\cancel{4}}{5}$$

$$= \frac{5}{4}r^2\sin^{-1}\left(\frac{3}{5}\right) + \frac{3}{5}r^2$$

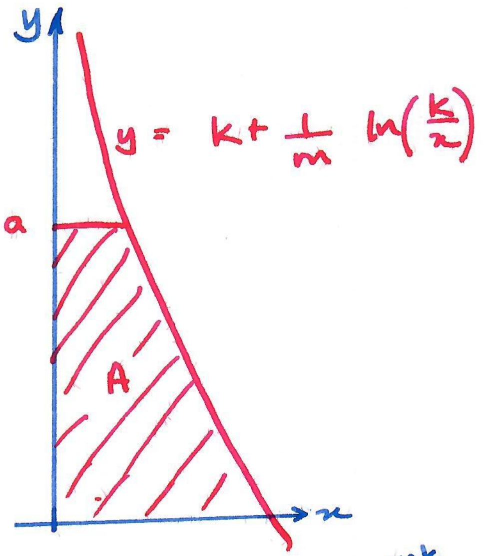
So the final integral $I = I_1 - \frac{6}{5}r^2$

$$= \frac{5}{4}r^2\sin^{-1}\left(\frac{3}{5}\right) - \frac{3}{5}r^2$$

$(= 0.2044\, r^2)$

(No wonder we only did the first part for Saturdaymaths.)

3) a)



i) Max. possible area is $\frac{k}{m}e^{mk}$ (thank you).

Presumably this question is really about dy rather than dx, so re-write the blade eq^n as

$$m(y-k) = \ln\left(\frac{k}{x}\right)$$

and then $\frac{k}{x} = e^{m(y-k)}$

or $x = ke^{-m(y-k)}$

Then the area required is

$$\int_0^a x\,dy = k\int_0^a e^{-m(y-k)}\,dy$$

$$= k\left[\frac{1}{-m}e^{-m(y-k)}\right]_0^a$$

$$= \frac{-k}{m}\left(e^{-m(a-k)} - e^{+mk}\right)$$

$$= \frac{-k}{m}e^{mk}\left(e^{-ma} - 1\right)$$

The minus signs are a bit troubling, but ... this has to be equal to $\frac{P}{100} \frac{k}{m} e^{mk}$.

So $\frac{P}{100} = 1 - e^{-ma}$

$e^{-ma} = 1 - \frac{P}{100} = \frac{100-P}{100}$

$e^{ma} = \frac{100}{100-P}$

$ma = \ln\left(\frac{100}{100-P}\right)$

$a = \frac{1}{m} \ln\left(\frac{100}{100-P}\right)$ which is about the simplest form.

ii) again $y = k + \frac{1}{m} \ln\left(\frac{k}{x}\right)$

So $x = k e^{-m(y-k)}$

$$V = \int_0^a \pi x^2 \, dy$$

$$= \pi k^2 \int e^{-2m(y-k)} \, dy$$

$$= \left[\frac{\pi k^2}{-2m} \left(e^{-2m(y-k)} \right) \right]_0^a$$

$$= \frac{-\pi k^2}{2m}\left(e^{-2m(a-k)} - e^{-2m(-k)}\right)$$

$$= \frac{-\pi k^2}{2m} e^{2mk}\left(e^{-2ma} - 1\right)$$

So $\dfrac{V}{A} = \dfrac{\frac{-\pi k^2}{2m} e^{2mk}\left(e^{-2ma} - 1\right)}{\frac{-k}{m} e^{mk}\left(e^{-ma} - 1\right)}$

Let $e^{-ma} = u$.

Then $\dfrac{V}{A} = \dfrac{\pi k}{2} e^{mk} \dfrac{(u^2-1)}{(u-1)}$

$$= \frac{\pi k}{2} e^{mk}(u+1)$$

and this must be $< e^{mk}$

So $\dfrac{\pi k}{2}(u+1) < 1$

$$u+1 < \frac{2}{\pi k}$$

$$e^{-ma} = u < \frac{2 - \pi k}{\pi k}$$

$$ma < \ln \frac{\pi k}{2 - \pi k}$$

$$\underline{\underline{a < \frac{1}{m} \ln\left(\frac{\pi k}{2 - \pi k}\right)}}$$

as required.

3b) $y = x \sin nx + \frac{1}{n} \cos nx$

Using the product rule:

$$\frac{dy}{dx} = nx \cos nx + \sin nx + \frac{\cancel{n}}{\cancel{n}}(-\sin nx)$$

$$= nx \cos nx.$$

$$I_n = \int_0^{\pi} \left(\frac{\pi}{2}-x\right) \sin\left(\left(n+\tfrac{1}{2}\right)x\right) \operatorname{cosec}\left(\frac{x}{2}\right) dx$$

$$-I_{n-1} \quad \int_0^{\pi} \left(\frac{\pi}{2}-x\right) \sin\left(\left(n-\tfrac{1}{2}\right)x\right) \operatorname{cosec}\left(\frac{x}{2}\right) dx$$

$$= \int_0^{\pi} \left(\frac{\pi}{2}-x\right) \operatorname{cosec}\left(\frac{x}{2}\right) \underbrace{\left[\sin\left(n+\tfrac{1}{2}\right)x - \sin\left(n-\tfrac{1}{2}\right)x\right]}_{2\cos\frac{2n}{n}x \ \sin\frac{\frac{1}{2}+\frac{1}{2}}{2}x} dx$$

oh, ho, ho...

$$\int_0^{\pi} \left(\frac{\pi}{2}-x\right) \frac{1}{\cancel{\sin\left(\frac{x}{2}\right)}} 2\cos nx \ \cancel{\sin\left(\frac{x}{2}\right)} dx$$

$$= \int_0^{\pi} \left(\frac{\pi}{2}-x\right) 2\cos nx \ dx$$

$$= \int_0^{\pi} \pi \cos nx - 2\int_0^{\pi} x\cos nx \ dx$$

$$= \left[\frac{\pi}{n} \sin nx\right]_0^{\pi} - \frac{2}{n}\int_0^{\pi} nx \cos nx \ dx$$

$$= 0 - 0 - \frac{2}{n}\left[x \sin nx + \frac{1}{n}\cos nx \right]_0^{\pi}$$

$$= -\frac{2}{n}\left(\cancel{\pi \sin n\pi} + \frac{1}{n}\cos n\pi - \cancel{\pi \sin 0} - \frac{1}{n}\cos 0 \right)$$

$$= \frac{-2}{n^2}\left((-1)^n - 1 \right)$$

$= 0$ when n is even

$\frac{4}{n^2}$ when n is odd

So if $I_0 = 0$:

$$I_1 = 0 + \frac{4}{1^2} = \frac{4}{1^2}$$

$$I_2 = 4 + 0 = \frac{4}{1^2}$$

$$I_3 = \frac{4}{1^2} + \frac{4}{3^2}$$

$$I_4 = \frac{4}{1^2} + \frac{4}{3^2} + 0$$

$$I_5 = \frac{4}{1^2} + \frac{4}{3^2} + \frac{4}{5^2}$$

Clearly when n is odd

$$I_n = 4\left(\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \cdots \frac{1}{n^2} \right)$$

when n is even

$$I_n = 4\left(\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \cdots + \frac{1}{(n-1)^2} \right)$$

(Could prove by induction ... but the answer-sheet doesn't seem to <u>require</u> it.)

4) a) i) This part explores a parameterised approach to $\frac{d^2y}{dx^2}$ — which isn't what you think.

$$x = f(t), y = g(t)$$

$$\frac{dy}{dt} = \frac{dy}{dt} \cdot \frac{dt}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}}$$

By the quotient rule,

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{\frac{dx}{dt}\frac{d}{dx}\left(\frac{dy}{dt}\right) - \frac{dy}{dt}\frac{d}{dx}\left(\frac{dx}{dt}\right)}{\left(\frac{dx}{dt}\right)^2}$$

So discarding the denominator:

$$0 = \frac{dx}{dt} \cdot \frac{d}{dx}\left(\frac{dy}{dt}\right) - \frac{dy}{dt}\frac{d}{dx}\left(\frac{dx}{dt}\right)$$

$$\frac{dx}{dt} \cdot \frac{d}{dx}\left(\frac{dy}{dt}\right) = \frac{dy}{dt} \cdot \frac{d}{dx}\left(\frac{dx}{dt}\right)$$

and the $\frac{d}{dx}$ terms are given (using the chain rule) by:

$$\frac{dx}{dt} \cdot \frac{d}{dt}\left(\frac{dy}{dt}\right) \cdot \frac{dt}{dx} = \frac{dy}{dt} \cdot \frac{d}{dt}\left(\frac{dx}{dt}\right) \cdot \frac{dt}{dx}$$

Cancelling $\frac{dt}{dx}$ and noting $\frac{d}{dt}\left(\frac{dy}{dt}\right) = \frac{d^2y}{dt^2}$, we get

$$\frac{dx}{dt} \cdot \frac{d^2y}{dt^2} = \frac{dy}{dt} \cdot \frac{d^2x}{dt^2}$$ as required.

4)a) ii) At the points of inflexion, $\frac{d^2y}{dx^2} = 0$

So using part (i):

$$x = a\cos t + \tfrac{1}{2}b\cos 2t$$

$$\frac{dx}{dt} = -a\sin t - b\sin 2t$$

$$\frac{d^2x}{dt^2} = -a\cos t - 2b\cos 2t$$

$$y = a\sin t + \tfrac{1}{2}b\sin 2t$$

$$\frac{dy}{dt} = a\cos t + b\cos 2t$$

$$\frac{d^2y}{dt^2} = -a\sin t - 2b\sin 2t$$

So part (i) gives:

$$+(a\sin t + b\sin 2t)(a\sin t + 2b\sin 2t)$$

$$= -(a\cos t + b\cos 2t)(a\cos t + 2b\cos 2t)$$

$$\therefore \quad a^2\sin^2 t + ab\sin 2t\sin t + 2ab\sin 2t\sin t + 2b^2\sin^2 2t$$

$$= -a^2\cos^2 t - ab\cos 2t\cos 2t - 2ab\cos 2t\cos t - 2b^2\cos^2 2t$$

$$a^2 + 2b^2 = -3ab(\sin 2t\sin t + \cos 2t\cos t)$$

$$= -3ab\cos(2t - t) \quad \text{(standard identity)}$$

$$= -3ab\cos t. \quad \text{So } \cos t = -\frac{a^2+2b^2}{3ab}$$

as required.

(Ideally we'd check $\frac{dy}{dx}$ actually turned direction, but...)

b) Apparently this is a new function... and there's no parameterising.

$y = f(x)$ Actually it doesn't seem to make sense.

Let $\dfrac{dy}{dx} = z$

Then $\dfrac{d^2y}{dx^2} = \dfrac{dz}{dx} = \dfrac{dz}{dy}\dfrac{dy}{dx}$

$\dfrac{dx}{dy} \cdot \dfrac{d^2y}{dx^2} = \dfrac{dz}{dy} = k\dfrac{dy}{dx}. = kz.$

So $\dfrac{dz}{dy} = kz$

but

So $\dfrac{dz}{dy}\dfrac{dy}{dx} = kz^2$

$\dfrac{dz}{dx} = kz^2$ (this seems very crazy)

$\dfrac{dz}{z^2} = k\,dx$

$z^{-2}dz = k\,dx$

$\dfrac{-1}{z} = kx + c$

At $(0,1)$: $\dfrac{-1}{1} = k.0 + c$ so $c = -1$. So $\dfrac{-1}{z} = kx - 1$

So $z = \dfrac{1}{1-kx} = \dfrac{dy}{dx}$: $y = \dfrac{-1}{k}\ln|(1-kx)| + c$

Subs $(0,1)$: $1 = 0 + c$: $c = 1$, $\underline{y = 1 - \frac{1}{k}\ln|1-kx|}$

5) a) $\cos(2A+B) = \cos 2A \cos B - \sin 2A \sin B$ (standard identity)

So $\cos 3\theta = \cos 2\theta \cos\theta - \sin 2\theta \sin\theta$

$= (\cos^2\theta - \sin^2\theta)\cos\theta - 2\sin\theta\cos\theta\sin\theta$

$= \cos^3\theta \quad - 3\sin^2\cos\theta$

$= \cos^3\theta - 3(1-\cos^2\theta)\cos\theta$

$= \cos^3\theta - \frac{3}{\cos\theta} + 3\cos^3\theta$

$= 4\cos^3\theta - 3\cos\theta$

So $\frac{1}{4}\cos 3\theta = \cos^3\theta - \frac{3}{4}\cos\theta$ as required.

To solve $27x^3 - 9x = 1$

Let $x = \frac{2}{3}\cos\theta$:

$27\left(\frac{8}{27}\right)\cos^3\theta - 9\left(\frac{2}{3}\right)\cos\theta = 1$

$8\cos^3\theta - 6\cos\theta = 1$

$4\cos^3\theta - 3\cos\theta = \frac{1}{2}$

So using the identity:

$\cos 3\theta = \frac{1}{2}$

$3\theta = \frac{\pi}{3}, \quad \frac{5\pi}{3}, \quad \frac{7\pi}{3}$

$\theta = \frac{\pi}{9}, \quad \frac{5\pi}{9}, \quad \frac{7\pi}{9}$

So $x = \frac{2}{3}\cos\left(\frac{\pi}{9}\right), \frac{2}{3}\cos\left(\frac{5\pi}{9}\right), \frac{2}{3}\cos\left(\frac{7\pi}{9}\right).$

Since these are the roots of

$$27x^3 - 9x - 1 = 0$$

we know their product $= \frac{-(-1)}{27}$:

$$\frac{2}{3}\cos\left(\frac{\pi}{9}\right)\frac{2}{3}\cos\left(\frac{5\pi}{9}\right)\frac{2}{3}\cos\left(\frac{7\pi}{9}\right) = \frac{1}{27}$$

$$\cos\left(\frac{\pi}{9}\right)\cos\left(\frac{5\pi}{9}\right)\cos\left(\frac{7\pi}{9}\right) = \frac{1}{8}$$

And $\cos\frac{3\pi}{9} = \cos\frac{\pi}{3} = \frac{1}{2}$

So $\cos\left(\frac{\pi}{9}\right)\cos\left(\frac{3\pi}{9}\right)\cos\left(\frac{5\pi}{9}\right)\cos\left(\frac{7\pi}{9}\right) = \frac{1}{2}\cdot\frac{1}{8} = \underline{\frac{1}{16}}$

5b) The term in the sum is

$$u_k = \frac{2k-1}{k(k+1)(k+2)} \qquad \text{——— ①}$$

Separating the factors in the denominator, this equals $\frac{A}{k} + \frac{B}{k+1} + \frac{C}{k+2}$ for some A, B, C:

$$= \frac{A(k+1)(k+2) + Bk(k+2) + Ck(k+1)}{k(k+1)(k+2)}$$

$$= \frac{A(k^2+3k+2) + B(k^2+2k) + C(k^2+k)}{k(k+1)(k+2)}$$

$$= \frac{(A+B+C)k^2 + (3A+2B+C)k + (2A)}{k(k+1)(k+2)} \qquad \text{——— ②}$$

Equating coeffs in ① and ②:

k^2: $\quad A+B+C=0$

k: $\quad 3A+2B+C=2$

1: $\quad 2A = -1 \qquad\qquad A = -\frac{1}{2}$

$$B+C=\frac{1}{2}$$

$$2B+C = 2+\frac{3}{2}=\frac{7}{2}$$

$$B=\frac{6}{2}=3$$

$$C=\frac{1}{2}-3=-\frac{5}{2}$$

So $\quad u_k = \dfrac{-1}{2k} + \dfrac{3}{k+1} - \dfrac{5}{2(k+2)}$

which is exactly the form required.

At this point we could use the standard summation formulae, but theres a better trick. This gives:

$$u_1 = -\frac{1}{2} + \frac{3}{2} - \cancel{\frac{5}{6}}$$

$$u_2 = -\frac{1}{4} + \cancel{\frac{3}{3}} - \cancel{\frac{5}{8}}$$

$$u_3 = \cancel{\frac{-1}{6}} + \cancel{\frac{3}{4}} - \cancel{\frac{5}{10}}$$

$$u_4 = \cancel{\frac{-1}{8}} + \cancel{\frac{3}{5}} - \cancel{\frac{5}{12}}$$

$$\cdots$$

$$u_{n-2} = \cancel{\frac{-1}{2(n-2)}} + \cancel{\frac{3}{n-1}} - \cancel{\frac{5}{2n}}$$

$$u_{n-1} = \cancel{\frac{-1}{2(n-1)}} + \cancel{\frac{3}{n}} - \frac{5}{2(n+1)}$$

$$u_n = \cancel{\frac{-1}{2n}} + \frac{3}{n+1} - \frac{5}{2(n+2)}$$

So the sum is

$$-\frac{1}{2}+\frac{3}{2}-\frac{1}{4}$$

$$-\frac{5}{2(n+1)} + \frac{3}{n+1} - \frac{5}{2(n+2)}$$

$$= \frac{3}{4} + \frac{1}{2(n+1)} - \frac{5}{2(n+2)}$$

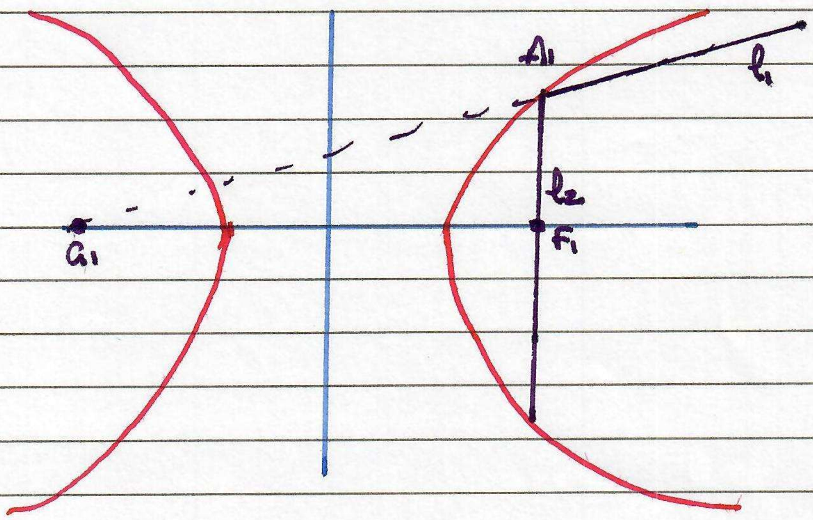
And since the reciprocal terms $\to 0$ as $n \to \infty$,

$$\lim_{n\to\infty} f(n) = \frac{3}{4}$$

as required.

6) a) See ~~[illegible]~~ next sheet.

b)



It's an interesting set-up. Actually the reflected line l_2 <u>always</u> goes through F_1 even if it isn't vertical ... it's a basic property of hyperbolas if l_1 goes through G_1.

But here, pres. to make the algebra easy, we're told AF_1 is vertical.

Repeat Part (a), though with more depth than last week:

$$4x^2 - y^2 - 16xh + 2yh + 15h^2 - 4a^2 = 0 \qquad (1)$$

Complete the squares:

$$4(x-2h)^2 - (y-h)^2 - \cancel{16h^2} + \cancel{h^2} + \cancel{15h^2} - 4a^2 = 0$$

$$4(x-2h)^2 - (y-h)^2 = 4a^2$$

$$\frac{(x-2h)^2}{a^2} - \frac{(y-h)^2}{4a^2} = 1$$

This is standard form of a hyperbola with:

centre $(2h, h)$
'open' to left and right
semi-axis parameters $a = a$, $b = 2a$
eccentricity $e^2 = 1 + \frac{b^2}{a^2} = 1 + 4 = 5$

So $e = \sqrt{5}$.

foci $\pm ae = \pm a\sqrt{5}$ shifted to $(2h, h)$

The given straight line is parallel to

$y = (e^2-1) = 4x$: i.e. its gradient is 4.

So we calculate $\frac{dy}{dx}$ in (1):

$$8x - 2y\frac{dy}{dx} - 16h + 2h\frac{dy}{dx} = 0$$

Subs. $\frac{dy}{dx} = 4$ and the point (p, q):

$$8p - 8q - 16h + 8h = 0$$

$$\Downarrow$$

$$8p - 8q = 8h$$

$$\underline{\underline{p - q = h}} \text{ as required.}$$

b) Now it gets nasty.

(Though Note that A_1F_1 is vertical, which is a help).
(see diagram earlier)

Launch into it:

Foci (G and F) are $(2h \pm a\sqrt{5}, h)$

A_1F_1 is vertical, so $x_{A_1} = 2h + a\sqrt{5}$.

Subs. into the hyperbola:

$$\frac{(a\sqrt{5})^2}{a^2} - \frac{(y-h)^2}{4a^2} = 1$$

$$5 - \frac{(y-h)^2}{4a^2} = 1$$

$$(y-h)^2 = 16a^2$$

$$y - h = \pm 4a$$

So A_1 is the point $(2h + a\sqrt{5}, h + 4a)$

And G_1 is the point $(2h - a\sqrt{5}, h)$

So the gradient of A_1G_1 is:

$$\frac{h+4a-h}{(2h+a\sqrt{5})-(2h-a\sqrt{5})} = \frac{4a}{2a\sqrt{5}} = \frac{2}{\sqrt{5}}.$$

Subs $G_1(2h-a\sqrt{5}, h)$ to get the eq^n of A_1G_1:

$$y-h = \frac{2}{\sqrt{5}}(x-2h+a\sqrt{5})$$

or $$y = \frac{2}{\sqrt{5}}x + h - \frac{4h}{\sqrt{5}} + 2a.$$

c) It's tempting to say 'you already know it', but...

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

and the tangent is

$$\frac{xx_1}{a^2} - \frac{yy_1}{b^2} = 1.$$

Rearranging ②:

$$\frac{yy_1}{b^2} = \frac{xx_1}{a^2} - 1$$

$$y = \frac{xb^2x_1}{a^2y_1} - \frac{b^2}{y_1}$$

So the gradient m is $\frac{b^2x_1}{a^2y_1}$ and 'c' is $-\frac{b^2}{y_1}$

Write the tangent as $y = mx + c$.

Then in ①:

$$\frac{x^2}{a^2} - \frac{(mx+c)^2}{b^2} = 1$$

$$b^2x^2 - a^2(mx+c)^2 = a^2b^2$$

$$(b^2 - a^2m^2)x^2 - 2a^2mcx - a^2(c^2+b^2) = 0$$

'This must have only one solution to be a tangent' (this seems a strange line of attack) so the discriminant must be 0:

$$(2a^2mc)^2 + 4(b^2 - a^2m^2)a^2(c^2+b^2) = 0$$

$$\cancel{4a^4m^2c^2} + 4a^2(c^2b^2 - \cancel{a^2m^2c^2} + b^4 - a^2m^2b^2) = 0$$

$$4a^2b^2(c^2 + b^2 - a^2m^2) = 0$$

So $c^2 = a^2m^2 - b^2$

$$c = \pm\sqrt{a^2m^2 - b^2}$$

So the eq^n of the tangent is ...

$$y = mx \pm \sqrt{a^2m^2 - b^2}$$

... which sounds like we've gone round in circles - except that it's useful in part (d).

d) First off, change coordinates by a straight translation:

$$X = x - 2h \qquad Y = y - h$$

So $\dfrac{X^2}{a^2} - \dfrac{Y^2}{4a^2} = 1$

The given point $\left(\frac{2a}{3} + 2h, h\right)$ becomes $\left(\frac{2a}{3}, 0\right)$

This hyperbola has $b = 2a$, so the tangent at this point is

$$y = mx + \sqrt{a^2m^2 - 4a^2}$$
$$= mx + a\sqrt{m^2 - 4}$$

And since it passes through $\left(\frac{2a}{3}, 0\right)$:

$$0 = \frac{2am}{3} + a\sqrt{m^2 - 4}$$

$$\frac{4m^2}{9} = m^2 - 4$$

$$4m^2 = 9m^2 - 36$$

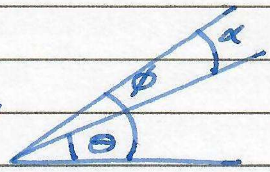
$$5m^2 = 36$$

$$m = \frac{6}{\sqrt{5}}$$ (known to be +ve).

And the asymptotes of $\dfrac{X^2}{a^2} - \dfrac{Y^2}{4a^2} = 1$

are $Y = \pm \dfrac{2a}{a} X$ or $Y = 2X$ (known to be +ve)

So we know two tangents (of, say, θ and ϕ) and we want the tangent of the angle α between them.



Standard formula:

$$\tan\alpha = \left|\frac{\tan\theta - \tan\phi}{1+\tan\theta\tan\phi}\right|$$

$$= \left|\frac{\frac{6}{\sqrt{5}} - 2}{1+\frac{6}{\sqrt{5}}.2}\right|$$

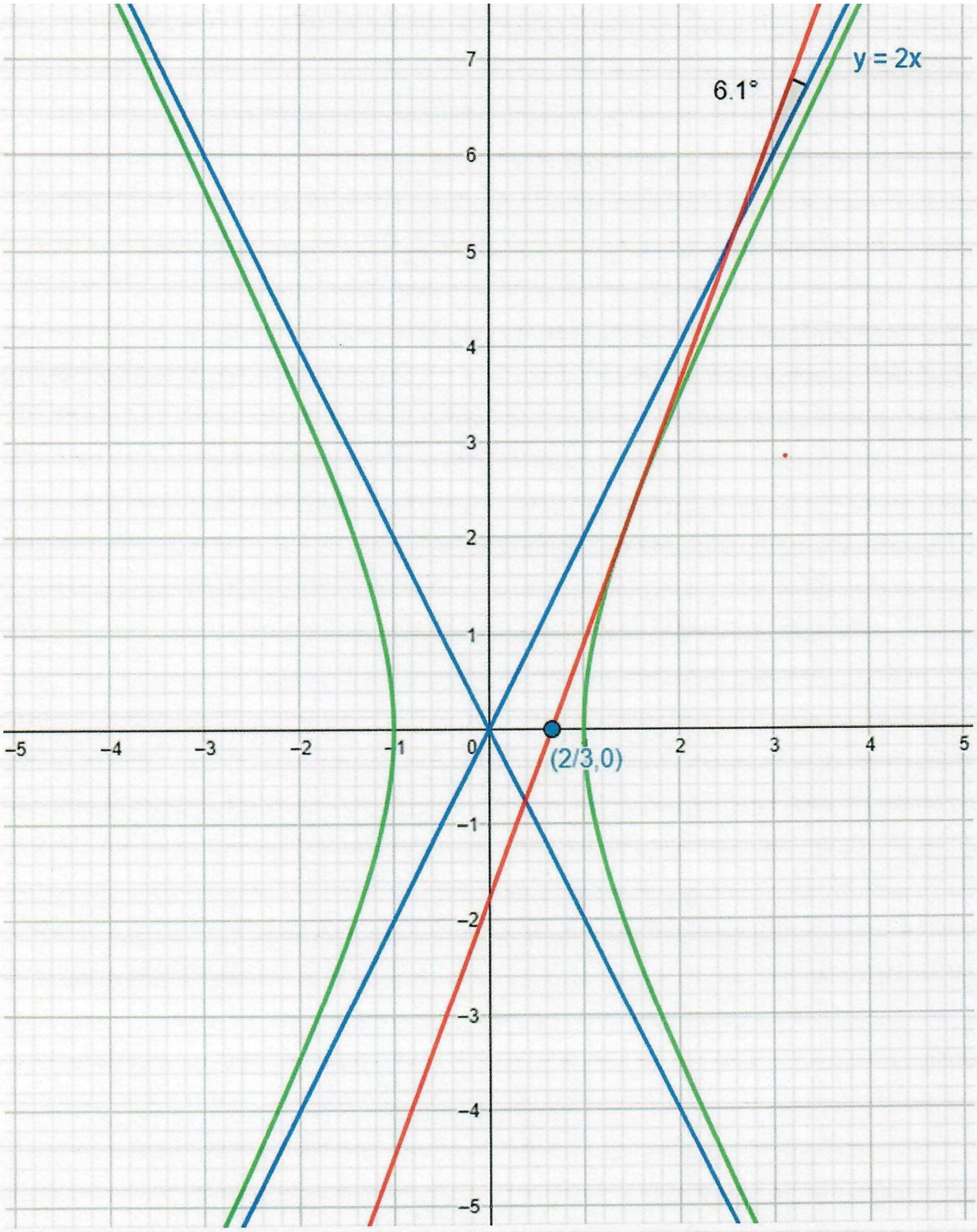
$$= \left|\frac{6-2\sqrt{5}}{\sqrt{5}+12}\right|$$

$$= \left|\frac{(6-2\sqrt{5})(\sqrt{5}-12)}{5-144}\right|$$

$$= \left|\frac{-72-10+30\sqrt{5}}{139}\right|$$

$$= \left|\frac{2(41-15\sqrt{5})}{139}\right|$$

So $\alpha = \tan^{-1}\left(\frac{2(41-15\sqrt{5})}{139}\right)$ as required.


y = 2x
6.1°
(2/3,0)
7
6
5
4
3
2
1
0
−1
−2
−3
−4
−5
−5
−4
−3
−2
−1
2
3
4
5