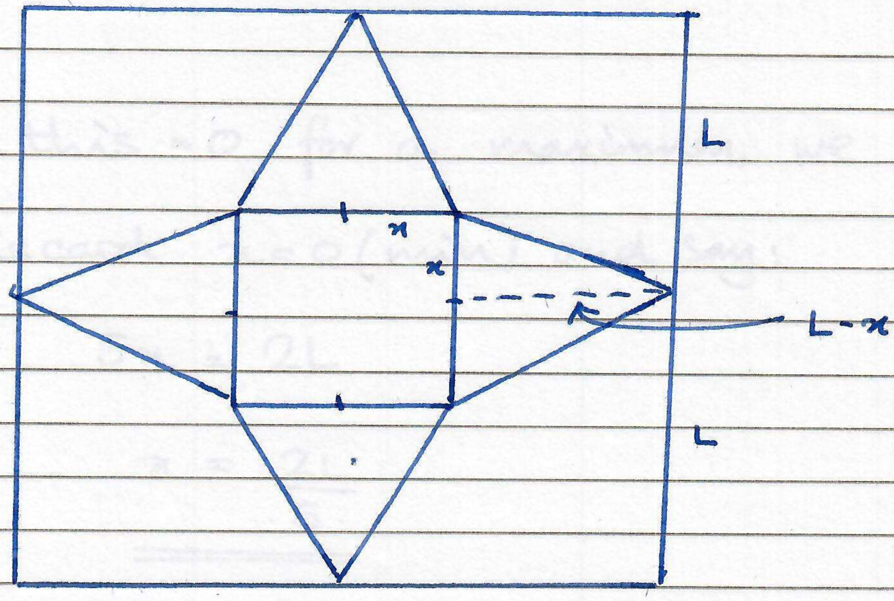
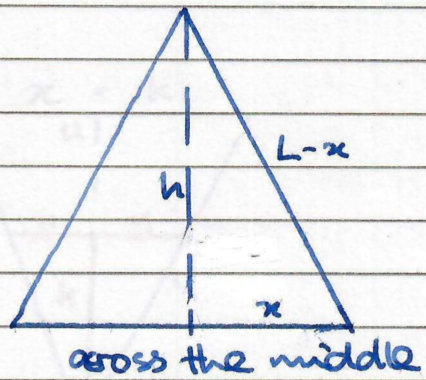
1)





$$h = \sqrt{(L-x)^2 - x^2}$$

$$= \sqrt{L^2 - 2xL}$$

$$V = \frac{1}{3}(4x^2)\sqrt{L^2 - 2xL}$$

$$\frac{dV}{dx} = \frac{4}{3}\left(x^2 \cdot \frac{1}{2} \frac{-2L}{\sqrt{L^2-2xL}} + 2x\sqrt{L^2-2xL} \right)$$

$$= \frac{4}{3}\left(\frac{-x^2L + 2x(L^2-2xL)}{\sqrt{L^2-2xL}} \right)$$

$$= \frac{4}{3} \frac{L}{\sqrt{L^2-2xL}} \left(-x^2 - 4x^2 + 2xL\right)$$

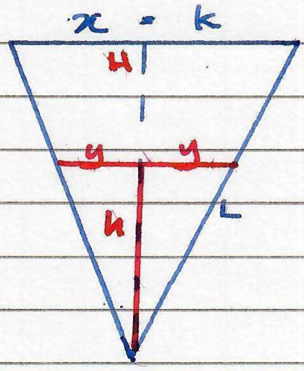
$$= \frac{4}{3} \frac{L}{\sqrt{L^2-2xL}} x(-5x + 2L)$$

Since this $=0$ for a maximum, we can discard $x=0$ (min) and say:

$$5x = 2L$$

$$x = \frac{2L}{5}.$$

Cross-section through the centre:



Presumably the question doesn't imply $k = \frac{2L}{5}$...

The full height of the pyramid is

$$H = \sqrt{(L-k)^2 - k^2} = \sqrt{L^2 - 2kL}$$

So $\frac{y}{h} = \frac{k}{\sqrt{L^2-2kL}} = \frac{k}{H}$ (note H is constant)

$$y = \frac{kh}{H}.$$

So the water surface is $4y^2 = \frac{4k^2h^2}{H^2}$

So the volume of the ~~cone~~ pyramid of water is

$$\frac{1}{3} h \left(\frac{4k^2h^2}{H^2} \right)$$

$$V = \frac{4}{3} \frac{k^2}{H^2} h^3$$

Diff. wrt t:

$$\frac{dV}{dt} = \frac{4}{3} \frac{k^2}{H^2} \cdot 3h^2 \frac{dh}{dt} = \frac{4k^2}{H^2} h^2 \frac{dh}{dt}$$

and we know $\frac{dV}{dt} = \frac{k^2}{3L}$

So $$\frac{dh}{dt} = \frac{\cancel{k^2}}{3L} \times \frac{H^2}{4\cancel{k^2}h^2} = \frac{H^2}{12Lh^2}$$

we are given a condition that $\frac{dh}{dt} > \frac{L-2k}{h+1}$

So $$\frac{H^2}{12Lh^2} > \frac{L-2k}{h+1}$$

$$(h+1)H^2 > 12Lh^2 (L-2k)$$

$$(h+1)(L^2 - 2kL) > 12Lh^2(L-2k)$$

Cancelling $L-2k$ (since this is >0):

$$\cancel{k}(h+1) > 12\cancel{k}h^2$$

$$(h+1) > 12h^2$$

So $\quad 12h^2 - h - 1 > 0$

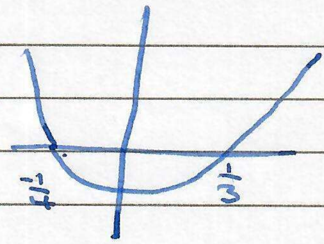
Solve $\quad 12h^2 - h - 1 = 0:$

$$(3h-1)(4h+1)$$

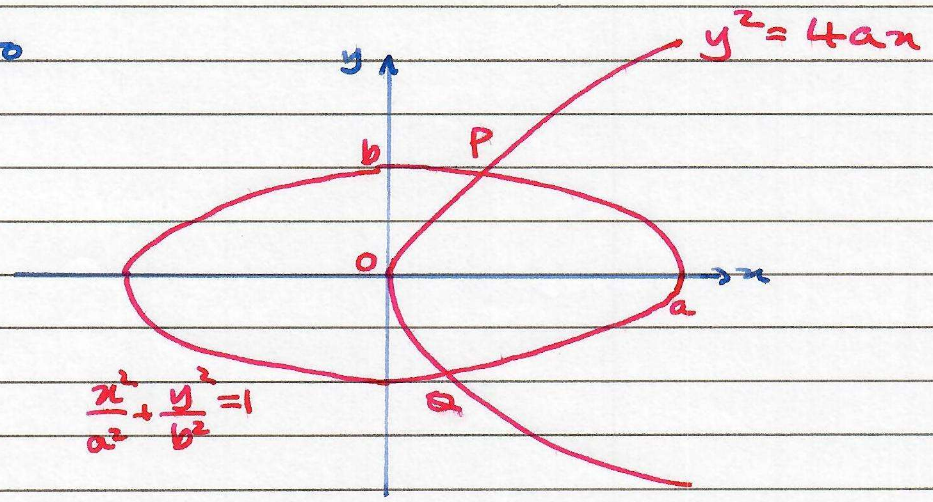
So $\quad h = \frac{1}{3}$ or $-\frac{1}{4}$ (discard)

So to satisfy the inequality:

$$0 < h < \frac{1}{3} \quad (\text{cm})$$



Question Two



for the parabola,

$$2y\frac{dy}{dx} = 4a. \qquad \frac{dy}{dx} = \frac{2a}{y}$$

for the ellipse,

$$\frac{2x}{a^2} + \frac{1}{b^2}2y\frac{dy}{dx} = 0$$

$$\frac{dy}{dx} = \frac{-2x}{2a^2}\frac{b^2}{y}$$

The two gradients are perp. at P, so their product is -1:

$$\frac{2a}{y} = \frac{-\cancel{2}a^2y}{-\cancel{2}xb^2} = \frac{a^2y}{xb^2}$$

$$\text{So } 2ax = \frac{a^2y^2}{b^2}$$

But $2ax = \frac{y^2}{2}$ so $\frac{\cancel{y^2}}{2} = \frac{a^2\cancel{y^2}}{b^2}$ $\therefore$ $\underline{\underline{b^2 = 2a^2}}$ as required.

ii)

Substitute into both curves to find P:

$$\frac{x^2}{a^2} + \frac{y^2}{2a^2} = 1$$

$$y^2 = 4ax$$

So: $$\frac{x^2}{a^2} + \frac{\cancel{4}^{2}ax}{\cancel{2}a^2} = 1$$

$$x^2 + 2ax - a^2 = 0$$

$$x = \frac{-2a \pm \sqrt{4a^2 + 4a^2}}{2}$$

$$= a(\pm\sqrt{2} - 1)$$

$$= a(\sqrt{2} - 1)$$ (-ve sol^n wouldn't be right for the parabola)

So $x^2 = a^2(\sqrt{2}-1)^2 = a^2(2+1) - 2\sqrt{2}a^2$

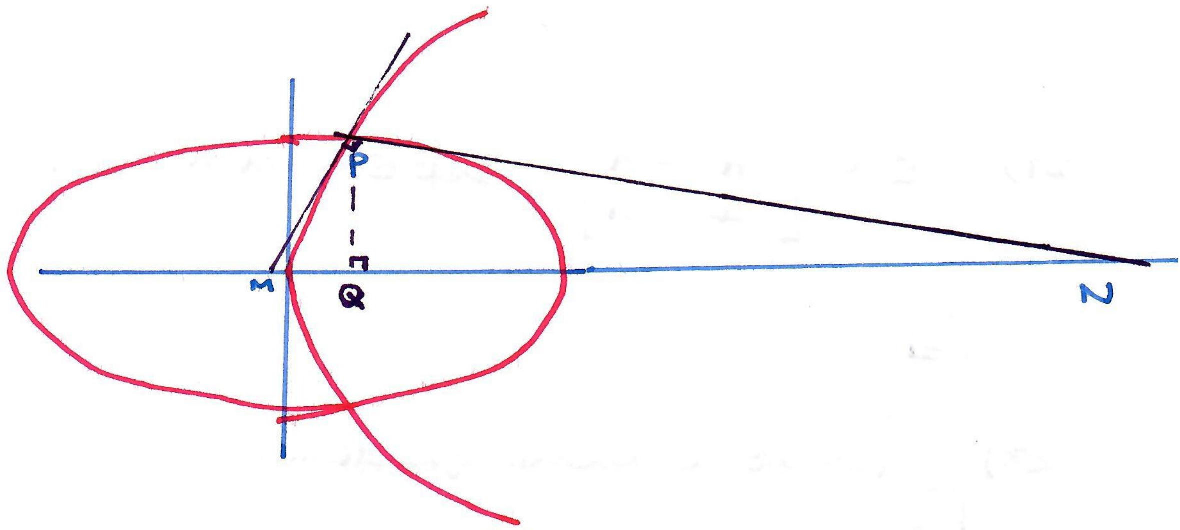
$y^2 = 4ax = 4a^2(\sqrt{2}-1)$

$x^2 + y^2 = 3a^2 - 2\sqrt{2}a^2 + 4a^2\sqrt{2} - 4a^2$

$= a^2(2\sqrt{2} - 1)$

So $|OP| = \sqrt{x^2+y^2} = a\sqrt{2\sqrt{2}-1}$

b)



P is $\left(a(\sqrt{2}-1),\ 2a\sqrt{(\sqrt{2}-1)} \right)$

grad PN $= -\dfrac{2xb^2}{2a^2y} = \dfrac{2a^2a(\sqrt{2}-1)}{a^2 2a\sqrt{\sqrt{2}-1}}$

$= \sqrt{\sqrt{2}-1}.$ Ⓐ

grad PM $= \dfrac{-1}{\sqrt{\sqrt{2}-1}} = \dfrac{-\sqrt{\sqrt{2}-1}}{\sqrt{2}-1}$

$= \dfrac{-\sqrt{\sqrt{2}-1}\ (\sqrt{2}+1)}{(\sqrt{2}-1)(\sqrt{2}+1)}$

$= -\dfrac{\sqrt{\sqrt{2}-1}\ \sqrt{\sqrt{2}+1}\ \sqrt{\sqrt{2}+1}}{1}$

$= -\sqrt{1}\sqrt{\sqrt{2}+1}$

$= -\sqrt{\sqrt{2}+1}$ Ⓑ

Check Ⓐ×Ⓑ $= -1$ ✓.

Drop a perpendicular from P to the x-axis.

Then $|PQ| = 2a\sqrt{\sqrt{2}-1}$

(I really hate this question)

$$\frac{PQ}{MQ} = \text{grad } PM = -\sqrt{\sqrt{2}+1}$$

So $$|MQ| = \frac{|PQ|}{\sqrt{\sqrt{2}+1}}$$

And $$\frac{|PQ|}{|QN|} = |\text{grad } PN| = \sqrt{\sqrt{2}-1}$$

$$|QN| = \frac{|PQ|}{\sqrt{\sqrt{2}-1}}.$$

So $$MN = |PQ|\left(\frac{1}{\sqrt{\sqrt{2}+1}} + \frac{1}{\sqrt{\sqrt{2}-1}}\right)$$

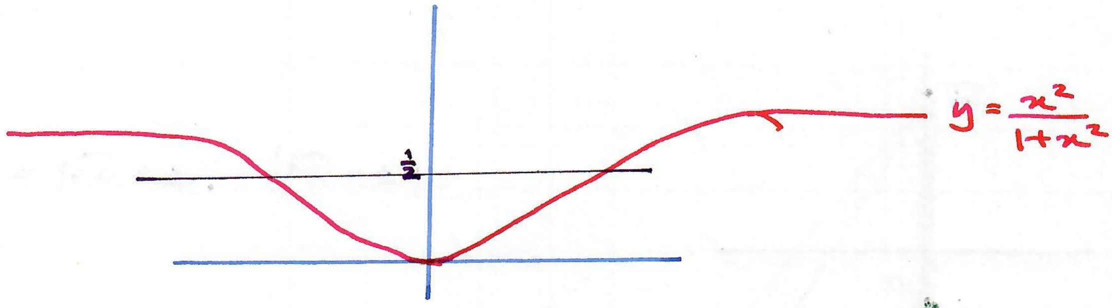
$$= 2a\sqrt{\sqrt{2}-1}\left(\frac{1}{\sqrt{\sqrt{2}+1}} + \frac{1}{\sqrt{\sqrt{2}-1}}\right)$$

$$= 2a\sqrt{\sqrt{2}-1}\left(\frac{\sqrt{\sqrt{2}-1} + \sqrt{\sqrt{2}+1}}{\sqrt{2-1}}\right)$$

$$= \frac{2a}{1}\left(\sqrt{2}-1 + \sqrt{2-1}\right)$$

$$= 2a(\sqrt{2}-1+1) = \underline{\underline{2a\sqrt{2}}} \quad \text{(amazingly).}$$

ii)



Not sure why the question bothers shifting x by 1: we'll shift it back again

For the solid of revolution, volume is given by

$$V = \int_0^{\frac{1}{2}} \pi x^2 \, dy$$

Find x^2: $\quad y = \frac{x^2}{1+x^2}$

$$y + x^2 y = x^2$$

$$x^2(1-y) = y$$

$$x^2 = \frac{y}{1-y}$$

So $V = \pi \int_0^{\frac{1}{2}} \frac{y}{1-y} \, dy$

Rearrange $\frac{y}{1-y} = \frac{(y-1)+1}{1-y} = \frac{1}{1-y} - 1$

So $V = \pi \int_0^{\frac{1}{2}} \left(\frac{1}{1-y} - 1 \right) dy$

3) a) i) $y = \dfrac{x^2}{1+x^2}$

$$\frac{dy}{dx} = \frac{2x(1+x^2) - x^2(2x)}{(1+x^2)^2}$$

$$= \frac{2x}{(1+x^2)^2}$$

when $x = 1$ this is $\dfrac{2 \cdot 1}{(1+1)^2} = \dfrac{2}{4} = \dfrac{1}{2}$ as given.

To find the point required:

$$\frac{2x}{(1+x^2)^2} = \frac{1}{2}$$

$$4x = (1+x^2)^2$$

So $x^4 + 2x^2 - 4x + 1 = 0$

Happily we know $x=1$ is a solⁿ, so $(x-1)$ is a factor:

$$(x-1)(x^3 + x^2 + 3x - 1) = 0$$

So the other root is a root of

$$x^3 + x^2 + 3x - 1 = 0$$

when $x = \frac{1}{4}$ this is $\frac{1}{64} + \frac{1}{16} + \frac{3}{4} - 1 < 0$

$x = \frac{1}{2}$ this is $\frac{1}{8} + \frac{1}{4} + \frac{3}{2} - 1 > 0$

So there is a root $\frac{1}{4} < x < \frac{1}{2}$ as required.

$$= \pi \left[-\ln|(1-y)| - y \right]_0^{\frac{1}{2}}$$

$$= \pi \left(-\ln\left(\tfrac{1}{2}\right) + \underbrace{\ln(1)}_{=0} - \tfrac{1}{2} + 0 \right)$$

$$= \pi \left(-\ln\left(\tfrac{1}{2}\right) - \tfrac{1}{2} \right)$$

$$= \pi \left(\ln(2) - \tfrac{1}{2} \right) \quad (\text{units}^3)$$

b) i) Consider $I = \int_0^a f(a-x)\,dx$

Let $a - x = u$

then $dx = -du$

and limits are $x = a: \; u = 0$

$x = 0: \; u = a.$

$$\text{so } I = -\int_a^0 f(u)\,du = -\left(-\int_0^a f(u)\,du \right) = \int_0^a f(u)\,du$$

which is just $\int_0^a f(x)\,dx.$ as required.

ii) using (b(i)):

$$\int_0^{\frac{\pi}{2}} \frac{\sin^n x}{\sin^n x + \cos^n x}\,dx = \int_0^{\frac{\pi}{2}} \frac{\sin^n\left(\frac{\pi}{2}-x\right)\,dx}{\sin^n\left(\frac{\pi}{2}-x\right) + \cos^n\left(\frac{\pi}{2}-x\right)}$$

$= I$

and this is

$$\int_0^{\frac{\pi}{2}} \frac{\cos^n(x)}{\cos^n(x) + \sin^n(x)} \, dx$$

$$= J$$

Since $\int_0^{\frac{\pi}{2}}$ is associative, this gives

$$I + J = \int_0^{\frac{\pi}{2}} \frac{\sin^n x + \cos^n x}{\sin^n x + \cos^n x} \, dx$$

$$= \int_0^{\frac{\pi}{2}} 1 \, dx$$

$$= x \Big|_0^{\frac{\pi}{2}} = \frac{\pi}{2}.$$

So $\underline{\underline{I = \frac{\pi}{4}}}$ $(= J)$

4) a) Simplest approach here is just to blast into it...

$v = 1 + i$

$z = x + iy$

$z - v = (x-1) + i(y-1)$

$vz = (1+i)(x+iy) = (x-y) + i(x+y)$

So $|z-v|^2 = (x-1)^2 + (y-1)^2 = x^2 - 2x + 1 + y^2 - 2y + 1$

$|vz|^2 = (x-y)^2 + (x+y)^2 = x^2 - \cancel{2xy} + y^2$
$+ x^2 + \cancel{2xy} + y^2$
$= 2x^2 + 2y^2$

Since the moduli are all +ve real numbers we can equate the squares...

$\cancel{x^2} - 2x + 1 + \cancel{y^2} - 2y + 1 = \cancel{2}x^2 + \cancel{2}y^2$

$x^2 + y^2 + 2x + 2y - 2 = 0$

$(x+1)^2 + (y+1)^2 - 2 - 2 = 0$

$(x+1)^2 + (y+1)^2 = 4.$ ——— ①

This is a ⊙ centre $(1, 1)$ $(1+i)$

radius $= 2$.

ii) Suspect there's a quick way (and/or) a geometric way, but again

$$z - v = (x-1) + i(y-1)$$

$$|z-v|^2 = (x-1)^2 + (y-1)^2 = x^2 - 2x + 1 + y^2 - 2y + 1$$

$$z + v = (x+1) + i(y+1)$$

$$|z+v|^2 = (x+1)^2 + (y+1)^2 = x^2 + 2x + 1 + y^2 + 2y + 1$$

Again equating the $|\ |^2$: (and cancelling terms):

$$-2x - 2y = 2x + 2y$$

$$4(x+y) = 0$$

$$\underline{y = -x.}$$

Subs in ①:

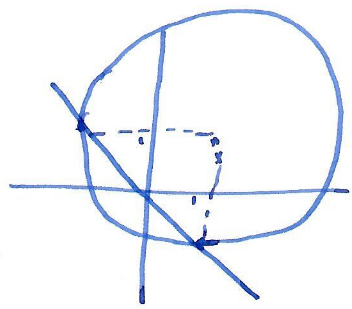
$$(x+1)^2 + (-x+1)^2 = 4$$

$$x^2 + \cancel{2x} + 1 + x^2 - \cancel{2x} + 1 = 4$$

$$2x^2 + 2 = 4$$

$$x^2 + 1 = 2$$

$$x^2 = 1$$



There appear to be 2 points of intersection ... from ①,

when $x = -1$ $y = 1$ i.e. $\underline{\underline{-1+i}}$

$x = 1$ $y = -1$ i.e. $\underline{\underline{1-i}}$

b) $z^5 = 1$ The complex roots are $\text{cis}\left(\frac{2\pi n}{5}\right)$

for $n = 0, 1, 2, \cancel{3, 4}\ -1, -2$

and include $\text{cis}\,\frac{2\pi \cdot 0}{n} = 1.$

Factorising $z^5 - 1 = 0$ gives

$$(z-1)(z^4 + z^3 + z^2 + z + 1) = 0$$

and we know this has factors $\left(z - \text{cis}\,\frac{2\pi n}{5}\right)$

as above.

Multiply the factors together in conjugates:

$$\left(z - \text{cis}\,\frac{2\pi}{5}\right)\left(z - \text{cis}\left(-\frac{2\pi}{5}\right)\right)$$

$$= z^2 - z\,\text{cis}\,\frac{2\pi}{5} - z\,\text{cis}\left(-\frac{2\pi}{5}\right) + \text{cis}\left(\frac{2\pi}{5}\right)\text{cis}\left(-\frac{2\pi}{5}\right)$$

$$= z^2 - z\left(\cos\frac{2\pi}{5} + \cos\left(-\frac{2\pi}{5}\right) + i\sin\frac{2\pi}{5} + i\sin\left(-\frac{2\pi}{5}\right)\right) + \cos\left(\frac{2\pi}{5}\right)\cos\left(-\frac{2\pi}{5}\right) - \sin\left(\frac{2\pi}{5}\right)\sin\left(-\frac{2\pi}{5}\right)$$

$$= z^2 - \left(2\cos\frac{2\pi}{5}\right)z + 1$$

Similarly $\left(z - \text{cis}\,\frac{4\pi}{5}\right)\left(z - \text{cis}\left(-\frac{4\pi}{5}\right)\right)$

$$= z^2 - \left(2\cos\frac{4\pi}{5}\right)z + 1.$$

Since these 4 factors multiply to give 1 as the coeff for z^4, we know they give

$$z^4+z^3+z^2+z+1 = \left(z^2-\left(2\cos\frac{2\pi}{5}\right)z+1\right)\left(z^2-\left(2\cos\frac{4\pi}{5}\right)z+1\right)$$

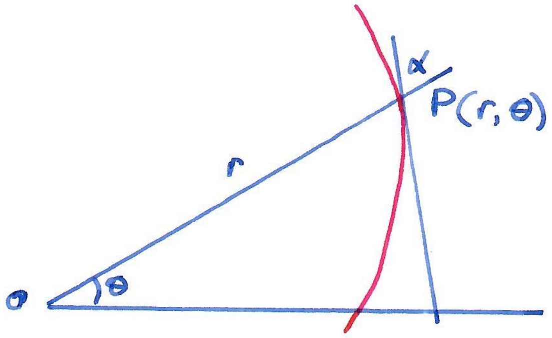
Comparing coeffs of z^2:

$$1 = 1 + \left(2\cos\frac{2\pi}{5}\right)\left(2\cos\frac{4\pi}{5}\right) + 1$$

So $4\cos\frac{2\pi}{5}\cos\frac{4\pi}{5} = 1-2 = -1.$

$$\cos\frac{2\pi}{5}\cos\frac{4\pi}{5} = -\frac{1}{4}$$

5.



$$r = Ae^{k\theta}$$

a) $x = r\cos\theta = Ae^{k\theta}\cos\theta$

$y = r\sin\theta = Ae^{k\theta}\sin\theta$

$$\frac{dx}{d\theta} = A\left(-e^{k\theta}\sin\theta + ke^{k\theta}\cos\theta\right)$$

$$\frac{dy}{d\theta} = A\left(e^{k\theta}\cos\theta + ke^{k\theta}\sin\theta\right)$$

$$\text{So } \frac{dy}{d\theta} = \frac{\cancel{A}}{\cancel{A}} \cdot \frac{\cancel{e}^{k\theta}\cos\theta + k\cancel{e}^{k\theta}\sin\theta}{-\cancel{e}^{k\theta}\sin\theta + k\cancel{e}^{k\theta}\cos\theta}$$

$$= \frac{\cos\theta + k\sin\theta}{k\cos\theta - \sin\theta}$$

$$= \frac{1 + k\tan\theta}{k - \tan\theta} \quad \text{——— ①}$$

b) From the diagram,

$$\frac{dy}{dx} = \tan(\theta + \alpha) = \frac{\tan\theta + \tan\alpha}{1 - \tan\theta\tan\alpha} \quad \text{— ②}$$

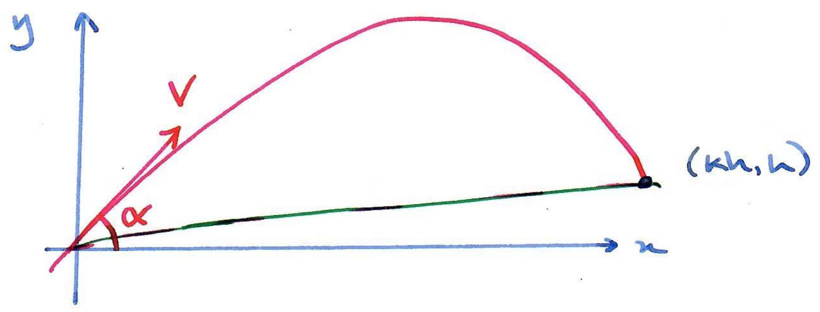
Divide ① through by k and set equal to ②:

$$\frac{\frac{1}{k} + \tan\theta}{1 - \frac{1}{k}\tan\theta} = \frac{\tan\alpha + \tan\theta}{1 - \tan\alpha\tan\theta}$$

So $\frac{1}{k} = \tan\alpha$

$$\alpha = \tan^{-1}\left(\frac{1}{k}\right)$$

6.) It helps here if you have some idea what a 'conversion' means... and you can get rid of all the 'words'. After that it's mainly bookkeeping



a) $$\frac{d^2x}{dt^2} = 0 \qquad \frac{d^2y}{dt^2} = -g$$

Initially $\frac{dx}{dt} = V\cos\alpha$

So integrating:

$$\frac{d^2x}{dt^2} = 0$$

$$\frac{dx}{dt} = c = V\cos\alpha$$

And initially $\frac{dv}{dt} = \underline{V\sin\alpha}$

So integrating:

$$\frac{d^2y}{dt^2} = -g$$

$$\frac{dy}{dt} = -gt + c = V\sin\alpha \text{ when } t = 0$$

So $\frac{dy}{dt} = \underline{V\sin\alpha - gt}$.

(It's tempting to think we could now do $\frac{dy}{dx}$ .... and perhaps <u>could have done</u> $\frac{d^2y}{dx^2}$ – but that way is disastrous...)

We now have $\dfrac{dx}{dt} = V\cos\alpha$

$x = V\cos\alpha\, t + c$ and again $c = 0$.

$\underline{x = (V\cos\alpha)\,t}$ ——— ①

$\dfrac{dy}{dt} = V\sin\alpha - gt$

$y = (V\sin\alpha)t - g\dfrac{t^2}{2} + c$ and again $c = 0$.

——— ②

From ①: $t = \dfrac{x}{V\cos\alpha}$

So in ②: $y = \dfrac{(V\sin\alpha)x}{(V\cos\alpha)} - \dfrac{g}{2}\left(\dfrac{x^2}{(V\cos\alpha)^2}\right)$

$y = x\tan\alpha - \dfrac{gx^2}{2V^2}\left(\dfrac{1}{\cos\alpha}\right)^2$

$= x\tan\alpha - \dfrac{gx^2}{2V^2}(\sec^2\alpha)$

$= x\tan\alpha - \dfrac{gx^2}{2V^2}(1+\tan^2\alpha)$

(useful form for solving for α):

b) So substituting for (kh, h):

$h = kh\tan\alpha - \dfrac{g(kh)^2}{2V^2}(1+\tan^2\alpha)$

$= khT - \dfrac{g(kh)^2}{2V^2}(1+T^2)$ where $T = \tan\alpha$

$$2hV^2 = 2khV^2T - gk^2h^2 - gk^2h^2T^2$$

$$gk^2h^2T^2 - 2khV^2T + 2hV^2 + gk^2h^2 = 0 \quad \text{—— (3)}$$

and there are 2 solutions if

$$(2khV^2)^2 - 4gk^2h^2(2hV^2 + gk^2h^2) > 0$$

$$\cancel{4k^2h^2}V^4 > \cancel{4}g\cancel{k^2h^2}(2hV^2 + gk^2h^2)$$

$$V^4 > 2hgV^2 + g^2k^2h^2$$

$$V^4 - 2hgV^2 - g^2k^2h^2 > 0.$$

Completing the $\square$:

$$(V^2 - hg)^2 - h^2g^2 - g^2k^2h^2 > 0$$

$$(V^2 - hg)^2 > h^2g^2(1+k^2)$$

i.e. if $V^2 - hg > hg\sqrt{1+k^2}$ (Both sides +ve)

$\underline{\underline{V^2 > hg\left(1 + \sqrt{1+k^2}\right)}}$ as required.

Let the soln's of (3) be $T_1 = \tan\alpha_1$, $T_2 = \tan\alpha_2$

Then $\tan(\alpha_1 + \alpha_2) = \dfrac{T_1 + T_2}{1 - T_1T_2}$

and from polynomial properties: $T_1 + T_2 = \dfrac{2khV^2}{gk^2h^2}$

$1 - T_1T_2 = -\dfrac{2hV^2 + gk^2h^2}{gk^2h^2} + gk^2h^2$

So $\dfrac{T_1+T_2}{1-T_1T_2} = \dfrac{2khV^2}{-2hV^2} = -k$ So $\underline{\underline{\tan(\alpha_1+\alpha_2) = -k}}$ etc.