1) $\ln(x+5) = 5 + \ln x$.

$5 = \ln(e^5)$ so we have:

$\ln(x+5) = \ln(e^5) + \ln x = \ln(e^5 x)$

So $x + 5 = e^5 x$: $x(e^5 - 1) = 5$

$x = \frac{5}{e^5 - 1} = 0.03392 = \underline{\underline{0.034 (3dp)}}$ ✓

2)

$$
\begin{array}{r|lllll}
 & 2x^2 & -2x & -4 & & \\
\hline
(x^2+x+3) & 2x^4 & +0x^3 & +0x^2 & +0x & -27 \\
 & 2x^4 & +2x^3 & +6x^2 & & \\
\hline
 & & -2x^3 & -6x^2 & & \\
 & & -2x^3 & -2x^2 & -6x & \\
\hline
 & & & -4x^2 & +6x & -27 \\
 & & & -4x^2 & -4x & -12 \\
\hline
 & & & & 10x & -15
\end{array}
$$

So Quotient is $2x^2 - 2x - 4$

Remainder is $\underline{\underline{10x - 15}}$. ✓

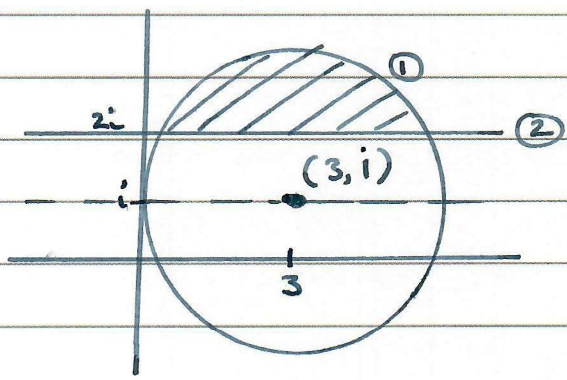
3) $|z - 3 - i| \leqslant 3 \Rightarrow |x - 3 + i(y-1)| \leqslant 3$

so $(x-3)^2 + (y-1)^2 \leqslant 9.$ ①

$|z| \geqslant |z - 4i| \Rightarrow (x^2 + y^2) \geqslant x^2 + (y-4)^2$

$y^2 \geqslant y^2 - 2\cdot 4y + 16$

$8y \geqslant 16 \qquad y \geqslant 2.$ ②



In the shaded region both conditions are true. ✓

4) $x = \dfrac{\cos\theta}{2 - \sin\theta} \qquad y = \theta + 2\cos\theta$

$$\frac{dx}{d\theta} = \frac{-(2-\sin\theta)\sin\theta - \cos\theta(-\cos\theta)}{(2-\sin\theta)^2}$$

$$= \frac{-2\sin\theta + \sin^2\theta + \cos^2\theta}{(2-\sin\theta)^2} = \frac{+1 - 2\sin\theta}{(2-\sin\theta)^2}$$

$$\frac{dy}{d\theta} = 1 - 2\sin\theta.$$

So $$\frac{dy}{dx} = +\frac{(1-2\sin\theta)(2-\sin\theta)^2}{1-2\sin\theta}$$

$$= (2-\sin\theta)^2 \quad \text{as required.}$$

5) $y = x^2\cos 3x \qquad 0 \leq x \leq \frac{\pi}{6}$

$$\frac{dy}{dx} = x^2(-3\sin 3x) + 2x\cos 3x = 0 \text{ at } M.$$

So $-3a^2\sin 3a + 2a\cos 3a = 0$

$$2a\cos 3a = +3a^2\sin 3a$$

$$\frac{2}{3a} \cdot \frac{2a}{3a^2} = \frac{\sin 3a}{\cos 3a} = \tan 3a$$

So $3a = \tan^{-1}\left(\frac{2}{3a}\right)$

$$a = \frac{1}{3}\tan^{-1}\left(\frac{2}{3a}\right) \text{ as required.}$$

$$a = \frac{1}{3}\tan^{-1}\left(\frac{2}{3a}\right)$$

'Based on the equation', let's try

$$a_0 = \frac{2}{3}\frac{\pi}{6} = \frac{\pi}{9} = 0.3491$$

$$a_i = \frac{1}{3}\tan^{-1}\left(\frac{2}{3a_{i-1}}\right)$$

then $a_1 = \frac{1}{3}\tan^{-1}\left(\frac{2}{3 \times 0.3491}\right) = 0.3628$

$a_2 = 0.3575$

$a_3 = 0.3595$

$a_4 = 0.3587.$

These are oscillating and hence converging - so we can say to 2dp $\underline{\underline{a = 0.36.}}$

6)a) Let $E = 3\cos x + 2\cos(x-60°)$

unusually, this requires us to use two trig. identies – though they're basically the same:

$$\cos(A \pm B) = \cos A\cos B \mp \sin A \sin B.$$

So $\cos(x-60°) = \cos x \cos 60° + \sin x \sin 60°$
$= \frac{1}{2}\cos x + \frac{\sqrt{3}}{2}\sin x.$

$E = 3\cos x + \cos x + \sqrt{3}\sin x.$
$= 4\cos x + \sqrt{3}\sin x.$
$= R\cos A \cos x + R\sin A \sin x$ for some R, A.

And $\left.\begin{array}{l} R\cos A = 4 \\ R\sin A = \sqrt{3} \end{array}\right\}$ $\tan A = \frac{\sqrt{3}}{4}$: $A = 23.41°$

exactly: $R^2\cos^2 A + R^2\sin^2 A = R^2$
$= 4^2 + 3$ so $R = \sqrt{19}.$

$\left(R = \frac{4}{\cos 23.41°} = 4.3588 \right)$

So $E = 4.3588(\cos(23.41°)\cos x + \sin(23.41°)\sin x)$

$= 4.3588 \cos(x - 23.41°)$

b) So $3\cos 2\theta + 2\cos(2\theta - 60°) = 4.3588\cos(2\theta - 23.41°)$
$= 2.5$ (given)

$\cos(2\theta - 23.41°) = \frac{2.5}{4.3588} = 0.5736$

$2\theta - 23.41° = 55.00°$ or $-55.00° + 360° = 305°$
$\theta = 39.21°$ or $164.21°$

7) a) Let $u = \cos x$

then $\frac{du}{dx} = -\sin x$

$du = -\sin x \, dx$

Also $\sin 2x = 2\sin x \cos x$

and $e^{2\cos x} = e^{2u}$.

So $\int \sin 2x e^{2\cos x} dx = \int (2\cos x)e^{2\cos x}(-\sin x\, dx)$

$= \int -2ue^{2u} du$.

For the limits: when $x = 0$, $\cos 0 = 1$

$x = \pi$, $\cos \pi = -1$

So the integral is $-\int_1^{-1} 2ue^{2u} du$

$= \int_{-1}^{1} 2ue^{2u} du$ as required.

b) Use integration by parts / tabular integration:

$$\begin{array}{ccc} + & u & e^{2u} \\ - & 1 & \frac{1}{2}e^{2u} \end{array}$$

...... need to integrate this last part

$$2\int_{-1}^{1} ue^{2u}\,du = \left[2\left(\frac{1}{2}ue^{2u} - \int\frac{1}{2}e^{2u}\right)\right]_{-1}^{1}$$

$$= \left[\left(u-\frac{1}{2}\right)e^{2u}\right]_{-1}^{1}$$

$$= \frac{e^2}{2} - \left(-\frac{3}{2}\right)e^{-2}$$

$$= \underline{\underline{\frac{1}{2}e^2 + \frac{3}{2}e^{-2}}}$$

8) $$\frac{dy}{dx} = \frac{y^2+4}{x(y+4)}$$

Separating variables:

$$\frac{(y+4)}{(y^2+4)}\,dy = \frac{dx}{x}$$

$$\left(\frac{y}{y^2+4} + \frac{4}{y^2+4}\right)dy = \frac{dx}{x}$$

To Integrate $\frac{y}{y^2+4}\,dy$: Let $y^2+4=z$: $2y\,dy = dz$

So this is $\frac{1}{2z}\,dz = \frac{1}{2}\ln z = \frac{1}{2}\ln(y^2+4)$

To Integrate $\frac{4}{y^2+4}\,dy$: standard formula $4.\frac{1}{2}\tan^{-1}\left(\frac{y}{2}\right)$

To Integrate $\frac{dx}{x}$: $\ln x$.

Putting these together:

$$\frac{1}{2}\ln(y^2+4) + 2\tan^{-1}\left(\frac{y}{2}\right) = \ln x + c$$

Substitute for $(4, 2\sqrt{3})$:

$$\frac{1}{2}\ln(16) + 2\tan^{-1}(\sqrt{3}) = \ln(4) + c$$

$$\cancel{\ln 4} + 2\tan^{-1}(\sqrt{3}) = \cancel{\ln 4} + c$$

$$c = 2\tan^{-1}(\sqrt{3}) = \frac{2\pi}{3} = \frac{2}{3}\pi.$$

So $\ln\sqrt{y^2+4} + 2\tan^{-1}\left(\frac{y}{2}\right) = \ln x + \frac{2\pi}{3}$

So when $y = 2$:

$$\ln\sqrt{8} + \underbrace{2\tan^{-1}(1)}_{\substack{= 2\frac{\pi}{4} \\ = \frac{\pi}{2}}} = \ln x + \frac{2\pi}{3}$$

$$\ln x = \ln\sqrt{8} + \frac{\pi}{2} - \frac{2}{3}\pi$$

$$= \ln\sqrt{8} - \frac{\pi}{6}$$

$$x = e^{\ln\sqrt{8} - \frac{\pi}{6}}$$

$$= \frac{e^{\ln\sqrt{8}}}{e^{\frac{\pi}{6}}}$$

$$= \frac{\sqrt{8}}{e^{\pi/6}} \qquad (= 1.6755)$$

a) $l: \ r = ai + 3j + bk + \lambda(ci - 2j + 4k)$

$$= \begin{pmatrix} a \\ 3 \\ b \end{pmatrix} + \lambda \begin{pmatrix} c \\ -2 \\ 4 \end{pmatrix}$$

$$m: \ r = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} + \mu \begin{pmatrix} 2 \\ -3 \\ 1 \end{pmatrix}$$

$l \perp m$, so the dot product of direction vectors is 0:

$$\begin{pmatrix} c \\ -2 \\ 4 \end{pmatrix} \cdot \begin{pmatrix} 2 \\ -3 \\ 1 \end{pmatrix} = 0 = 2c + 6 + 4$$

$$2c = -10 : \quad \underline{\underline{c = -5}}$$

P is $\begin{pmatrix} 4 \\ 7 \\ -2 \end{pmatrix}$ and $P \in l$:

so for some λ, $\begin{pmatrix} 4 \\ 7 \\ -2 \end{pmatrix} = \begin{pmatrix} a \\ 3 \\ b \end{pmatrix} + \lambda \begin{pmatrix} -5 \\ -2 \\ 4 \end{pmatrix}$

Equating coefficients:

$4 = a - 5\lambda \ : \quad 4 = a + 10 \quad a = -6$

$7 = 3 - 2\lambda \ : \quad \lambda = -2$

$-2 = b + 4\lambda \ : \quad -2 = b - 8 \quad b = 6.$

So $l: \ r = \begin{pmatrix} -6 \\ 3 \\ 6 \end{pmatrix} + \lambda \begin{pmatrix} -5 \\ -2 \\ 4 \end{pmatrix}$

$m: \ r = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} + \mu \begin{pmatrix} 2 \\ -3 \\ 1 \end{pmatrix}$

For a general point Q on m:

$$\overrightarrow{PQ} = \begin{pmatrix}1\\2\\3\end{pmatrix} + \mu\begin{pmatrix}2\\-3\\1\end{pmatrix} - \begin{pmatrix}4\\7\\-2\end{pmatrix}$$

$$= \begin{pmatrix}-3+2\mu\\-5-3\mu\\5+\mu\end{pmatrix}$$

And we know (actually we don't, because the wording is careless: $\overrightarrow{PQ} \perp \begin{pmatrix}2\\-3\\1\end{pmatrix}$

So $\quad 2(-3+2\mu) - 3(-5-3\mu) + 1(5+\mu) = 0$ (dot product)

$$-6+15+5 + (4+9+1)\mu = 0$$

$$14 + 14\mu = 0 \qquad \underline{\mu = -1.}$$

So Q is $\begin{pmatrix}1\\2\\3\end{pmatrix} - \begin{pmatrix}2\\-3\\1\end{pmatrix} = \begin{pmatrix}1-2\\2+3\\3-1\end{pmatrix} = \begin{pmatrix}-1\\5\\2\end{pmatrix}$

So the equation of line PQ is

$$r = \begin{pmatrix}4\\7\\-2\end{pmatrix} + \sigma\begin{pmatrix}-1-4\\5-7\\2--2\end{pmatrix} = \begin{pmatrix}4\\7\\-2\end{pmatrix} + \sigma\begin{pmatrix}-5\\-2\\4\end{pmatrix}$$

And at Q, $\sigma = 1$.

So at R, $\sigma = \frac{5}{2}$: $\quad R = \begin{pmatrix}4\\7\\-2\end{pmatrix} + \frac{5}{2}\begin{pmatrix}-5\\-2\\4\end{pmatrix} = \begin{pmatrix}4-\frac{25}{2}\\7-5\\-2+10\end{pmatrix}$

$$= \begin{pmatrix}-\frac{17}{2}\\2\\8\end{pmatrix}$$

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10) $f(x) = \dfrac{21 - 8x - 2x^2}{(1+2x)(3-x)^2}$

Express this as $\dfrac{A}{1+2x} + \dfrac{B}{3-x} + \dfrac{C}{(3-x)^2}$

$$= \frac{A(3-x)^2 + B(1+2x)(3-x) + C(1+2x)}{(1+2x)(3-x)^2}$$

$$= \frac{A(9-6x+x^2) + B(3+5x-2x^2) + C(1+2x)}{(\quad)(\quad)^2}$$

$$= \frac{x^2(A-2B) + x(-6A+5B+2C) + 1(9A+3B+C)}{(\quad)(\quad)^2}$$

So equating coefficients:

x^2: $A - 2B = -2$

x: $-6A + 5B + 2C = -8$

1: $9A + 3B + C = 21$

$\downarrow$

$18A + 6B + 2C = 42$

$\rightarrow$ $24A + B = 50$

$48A + 2B = 100$

$A - 2B = -2$

$49A = 98$

$\underline{A = 2}$

$2 - 2B = -2$

$-2B = -4$

$\underline{B = 2.}$

$9 \times 2 + 3 \times 2 + C = 21$

$18 + 6 + C = 21$ $\underline{C = -3.}$

So $f(x) = \dfrac{2}{2x+1} + \dfrac{2}{3-x} - \dfrac{3}{(3-x)^2}$

Check: $\dfrac{2(3-x)^2 + 2(2x+1)(3-x) - 3(2x+1)}{(1+2x)(3-x)^2}$

$= \dfrac{2(9-6x+x^2) + 2(3+5x-2x^2) - 3(2x+1)}{(\quad x \quad)^2}$

$= \dfrac{18 - 12x + 2x^2 + 6 + 10x - 4x^2 - 6x - 3}{(\quad x \quad)^2}$

$= \dfrac{21 - 8x - 2x^2}{(\quad x \quad)^2}$ ✓

b) $\dfrac{2}{2x+1} = 2(1+2x)^{-1} = 2\left(1 + (-1)2x + \dfrac{(-1)(-2)}{2}(2x)^2 + \dots\right)$

$= 2 - 4x + 8x^2 + \dots$

$\dfrac{2}{3-x} = \dfrac{2}{3\left(1-\frac{x}{3}\right)} = \dfrac{2}{3}\left(1-\dfrac{x}{3}\right)^{-1}$

$= \dfrac{2}{3}\left(1 + (-1)\left(-\dfrac{x}{3}\right) + \dfrac{(-1)(-2)}{2}\left(\dfrac{-x}{3}\right)^2 + \dots\right)$

$= \dfrac{2}{3}\left(1 + \dfrac{x}{3} + \dfrac{1}{9}x^2 + \dots\right)$

$= \dfrac{2}{3} + \dfrac{2}{9}x + \dfrac{2}{27}x^2 + \dots$

$\dfrac{-3}{(3-x)^2} = \dfrac{-3}{9\left(1-\frac{x}{3}\right)^2} = -\dfrac{1}{3}\left(1-\dfrac{x}{3}\right)^{-2}$

$= -\dfrac{1}{3}\left(1 + (-2)\left(\dfrac{-x}{3}\right) + \dfrac{(-2)(-3)}{2}\left(\dfrac{-x}{3}\right)^2 + \dots\right)$

$$= -\frac{1}{3} - \frac{2}{9}x - \frac{1}{9}x^2 + \cdots$$

So the sum is:

$$\left(2+\frac{2}{3}-\frac{1}{3}\right) + \left(-4+\frac{2}{9}-\frac{2}{9}\right)x + \left(\frac{2}{27}+8-\frac{1}{9}\right)x^2 + \cdots$$

$$= 2\tfrac{1}{3} - 4x + \frac{2+216-3}{27}x^2 + \cdots$$

$$= 2\tfrac{1}{3} - 4x + 7\tfrac{26}{27}x^2 + \cdots$$

11) $z = \dfrac{5a-2i}{3+ai}$ $\quad \arg(z) = -\dfrac{\pi}{4}$ $\quad$ so $\mathrm{Re}(z) = -\mathrm{Im}(z)$

$a \in \mathbb{Z}$ $\quad$ and $\mathrm{Re}(z) > 0$

$$z = \frac{(5a-2i)(3-ai)}{(3+ai)(3-ai)} = \frac{(15a - 2a - 6i - 5a^2 i)}{(3+a^2)}$$

$$= \frac{13a + i(-6-5a^2)}{(9+a^2)} \quad \text{——— } ①$$

So $13a = -(-6-5a^2) = 5a^2+6$

$$5a^2 - 13a + 6 = 0$$

$$(5a-3)(a-2) = 0$$

$a = 2$ (discard $a = \frac{3}{5}$ because $a \in \mathbb{Z}$)

so from ① $z = \dfrac{13.2 + i(-6 - 5.4)}{9+4}$

$= \underline{\underline{2 - 2i}}$ as required.

b) $z = 2 - 2i = 2(1-i) = 2\sqrt{2}\,e^{-\frac{\pi}{4}i}$

so $z^3 = (2\sqrt{2})^3 e^{-\frac{3\pi}{4}i}$

$= 16\sqrt{2}\,e^{-\frac{3\pi}{4}i}$

which is in the required form with $r = 16\sqrt{2}$

and $\theta = -\dfrac{3\pi}{4}$