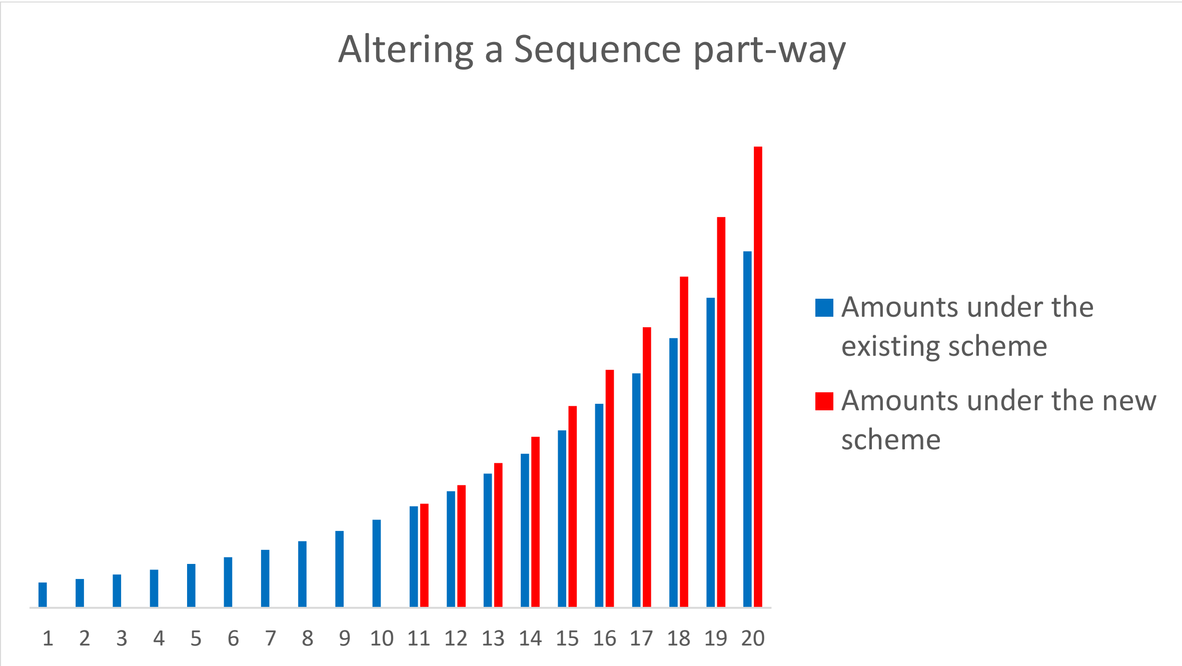
Altering a Geometric Series part-way (Gifts from Grandma)

This problem popped up as a **Gifts from Grandma** scheme, but it's relevant to many other situations - e.g. where you take out a loan and interest rates change, or where general inflation rates are changing.

It's presented here for two Geometric series, but you'd do it similarly if you were working with Arithmetic series or a mix of the two.



Under the existing scheme with interest rate r_1

$$u_n = u_1 r_1^{n-1}$$

$$S_n = \sum_{i=1}^{n} u_i = \frac{u_1(r_1^n - 1)}{r_1 - 1}$$

$$S_{10} = \frac{u_1(r_1^{10} - 1)}{r_1 - 1}$$

Expected Total S_{20} under the existing scheme

$$S_{20} = \frac{u_1(r_1^{20} - 1)}{r_1 - 1}$$

Total overall wanted by changing to a new scheme with a new interest rate r_2 is T_{20}

So the new set of payments u_{11}, u_{12} ... u_{20} have to sum to the difference $T_{20} - S_{10}$

$$\sum_{i=11}^{20} u_i = T_{20} - S_{20}$$

And (very importantly) we want the new $u_{11}$to be calculated under the new scheme, not the old one – i.e. with the new rate r_2 – from the existing value of u_{10}.

This means

$$u_{11} = u_{10} r_2 \text{ (note } r_2 \text{ not } r_1\text{)}$$

This means we can write out a new sum for the 11-20 part of the series:

$$\sum_{i=11}^{20} u_i = T_{20} - S_{20} = \frac{u_{11}(r_2^n - 1)}{r_2 - 1}$$

But since we only know u_{10} at this stage, this becomes:

$$T_{20} - S_{20} = \frac{(u_{10} r_2)(r_2^n - 1)}{\mathrm{r}_2 - 1} \text{(note the extra } r_2\text{)}$$

Happily we know everything here except r_2, so if we solve this equation using logarithms / calculator / computer we get the answer we want.

Summary

What we're doing is splitting the sequence into two parts:

- First 10 terms: ratio r_1
- Next 10 terms: ratio r_2, starting after u_{10}

Things to watch out for

- The sum for second series mustn't include u_{10} (we've already added that in with S_{10}) so we have to start with u_{11}.
- u_{11} must itself be calculated with the new ratio r_2. (Granny isn't allowed a year without an interest increase)

Finally

If you wanted to write out the overall amount it would be:

$$T_{20} = \frac{u_1(r_1^{10} - 1)}{r_1 - 1} + \frac{u_{10} r_2 (r_2^{10} - 1)}{r_2 - 1}$$

Altering Geometric Series part-way: Worksheet

Geometric sequences where the ratio changes part-way through

Section A – Foundations

1. Basic change of ratio

A geometric sequence has first term $u_1 = 3$ and ratio $r_1 = 2$ for the first 6 terms.
From term 7 onwards, the ratio changes to $r_2 = 3$.

(a) Find u_6
(b) Find u_7
(c) Find u_{10}
(d) Find the sum of the first 10 terms, S'_{10}

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2. Writing as two sums

A sequence has:

- $u_1 = 5$
- ratio $r_1 = 1.5$ for the first 8 terms
- ratio $r_2 = 0.5$ thereafter

(a) Write an expression for S'_{12}as the sum of two geometric series
(b) Evaluate S'_{12}

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3. Expressing the "join" term

A sequence starts with $u_1 = a$, ratio r_1 for 10 terms, then switches to ratio r_2.

(a) Show that $u_{10} = ar_1^9$
(b) Write an expression for u_{11}
(c) Hence write the first term of the second part in terms of a, r_1, r_2

Section B – Building the General Case

4. Deriving a closed form

A sequence has:

- first term u_1

- ratio r_1 for the first $m$terms
- ratio r_2 for the next $n$terms

(a) Write an expression for the sum of the first m terms
(b) Show that the first term of the second part is $u_1 r_1^{m-1} r_2$
(c) Write an expression for the sum of the next n terms
(d) Hence write a closed form for the total sum

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5. Special case check

Using your result from Question 4, show that if $r_1 = r_2$, your expression reduces to the usual geometric series formula.

Section C – Reverse Problems

6. Finding the second ratio

A sequence has:

- $u_1 = 2$
- ratio $r_1 = 2$ for the first 5 terms
- ratio r_2 thereafter

Given that $u_{10} = 2048$, find r_2.

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7. Matching totals

A sequence has:

- $u_1 = 4$
- ratio $r_1 = 3$ for the first 4 terms
- ratio r_2 thereafter

The total of the first 8 terms is $S_8' = 5000$. Find r_2.

Section D – Interpretation & Modelling

8. Growth and change

A population grows geometrically:

- initial population 100

- growth factor 1.2 per year for 5 years
- then changes to growth factor 0.9

(a) Find the population after 10 years
(b) Find the total population accumulated over the 10 years (sum of yearly populations)
(c) Explain what effect the change in ratio has on long-term growth

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9. Financial model

An investment pays:

- 5% compound growth annually for 6 years
- then 2% thereafter

Initial investment: $1000

(a) Find the value after 10 years
(b) Compare with constant 5% growth over 10 years
(c) Comment on the impact of the change

Section E – Challenge

10. Designing a sequence

Construct a sequence such that:

- $u_1 = 1$
- ratio $r_1 = 2$ for the first m terms
- ratio $r_2 = \frac{1}{2}$ thereafter
- total sum of the first $2m$ terms is exactly 100

Find m.

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11. Hidden structure

A sequence satisfies:

- first 6 terms form a geometric sequence with ratio r
- next 6 terms form a geometric sequence with ratio $\frac{1}{r}$
- the 6th and 7th terms connect in the usual way

Show that the total of the first 12 terms can be written in the form:

$$S = A(r^6 - 1) + B(1 - r^{-6})$$

and determine A and B.

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12. Extension (very hard)

A sequence changes ratio *twice*:

- ratio r_1 for first $m$terms
- ratio r_2 for next $n$terms
- ratio r_3 thereafter

(a) Write a general expression for the total of the first $m + n + p$ terms
(b) Simplify as far as possible
(c) Suggest a compact notation for multiple changes of ratio

Altering Geometric Series part-way: Real-life Examples

1. Grandma's Gift Scheme 🎁

Grandma gives you:

- $50 this year
- Each year for 5 years, she **doubles** the gift ($r_1 = 2$)
- After that, she decides it's getting expensive, so increases by only **10% per year** ($r_2 = 1.1$)

👉 The sequence changes behaviour at year 6:

- Fast growth → gentle growth
- A perfect example of an altered geometric series

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2. Savings Account Interest 💰

You invest $1000:

- For the first 3 years: **high interest (6%)** → $r_1 = 1.06$
- After that: rates drop to **2%** → $r_2 = 1.02$

👉 The balance each year follows a geometric pattern, but the ratio changes when the interest rate changes.

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3. Loan Repayment Structure 🏦

A loan balance evolves like this:

- Initially grows at **5% interest** per year (if unpaid)
- After restructuring, interest drops to **3%**

👉 The outstanding balance forms:

- one geometric phase (higher growth)
- followed by a second (lower growth)

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4. Population with Policy Change 🌱

A town's population:

- grows at **+4% annually** for 10 years ($r_1 = 1.04$)
- then declines slightly at **−1% annually** ($r_2 = 0.99$)

👉 This creates:

- an expanding geometric phase
- followed by a contracting one

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5. Business Growth then Saturation 📈

A startup's revenue:

- triples each year for 4 years ($r_1 = 3$)
- then stabilises to **20% growth** ($r_2 = 1.2$)

👉 Classic "early boom → mature growth" pattern

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6. Radioactive Decay with Environmental Change ☢️

A substance decays:

- initially with half-life behaviour ($r_1 = 0.5$)
- then conditions change (e.g. shielding removed), altering decay rate to $r_2 = 0.7$

👉 Still geometric — but with a different ratio after a certain time

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7. Subscription Growth then Market Saturation 📊

An app's users:

- double each month early on ($r_1 = 2$)
- then growth slows to **1.1× per month** ($r_2 = 1.1$)

👉 Another very natural "two-phase" geometric process
