1) $y = x^2 + ax - 4$

$\frac{dy}{dx} = 2x + a$

At point P:

$k = 3^2 + 3a - 4 = 5 + 3a$ — ①

$3 = 2 \times 3 + a = 6 + a$ — ②

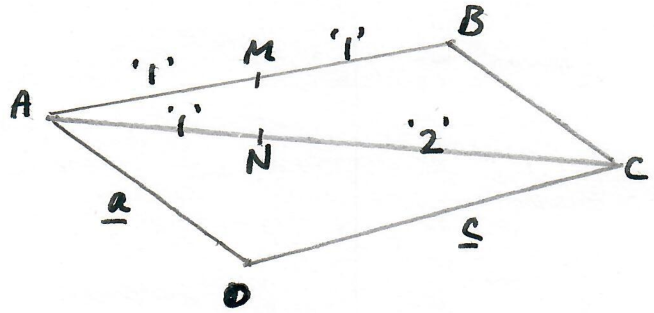
From ②: $3 = 6 + a$

$\underline{\underline{a = -3}}$

From ①: $k = 5 + 3a = 5 - 9 = -4$

$\underline{\underline{k = -4}}$

2)



$\vec{AC} = \vec{AO} + \vec{OC} = -\underline{a} + \underline{c} = \underline{\underline{\underline{c} - \underline{a}}}$

$\vec{ON} = \vec{OA} + \vec{AN} = \underline{a} + \frac{1}{3}\vec{AC} = \underline{a} + \frac{1}{3}\underline{c} - \frac{1}{3}\underline{a}$

$= \underline{\underline{\frac{2}{3}\underline{a} + \frac{1}{3}\underline{c}}}$

$\vec{NM} = \vec{NA} + \vec{AM}$

$= \frac{-1}{3}\vec{AC} + \frac{1}{2}\underline{c}$

$= \frac{-1}{3}(\underline{c} - \underline{a}) + \frac{1}{2}\underline{c}$

$= \underline{\underline{\frac{1}{3}\underline{a} + \frac{1}{6}\underline{c}}}$

O, M and N are collinear if one can be written as a linear combination of the other two. In this case that would mean showing $\overrightarrow{OM}$ is a multiple of $\overrightarrow{ON}$ or $\overrightarrow{NM}$.

$\overrightarrow{OM} = \underline{a} + \frac{1}{2}\underline{c}$

which is $3 \times \overrightarrow{NM}$, so the points are collinear.

(For good measure,

$\overrightarrow{ON} = \overrightarrow{OM} - \overrightarrow{NM} = \frac{2}{3}\underline{a} + \frac{1}{3}\underline{c}$,

which again is $2 \times \overrightarrow{NM}$)

3) $5x^2 + Ax + 7 = B(x-2)^2 + C$

we can do this by picking values for x:

$x = 0 \qquad 7 = 4B + C$

$x = 1 \qquad 12 + A = B + C$

$x = 2 \qquad 20 + 2A + 7 = C$

or we can expand the RHS and equate coeffs:

$5x^2 + Ax + 7 = Bx^2 - 4Bx + 4B + C$

$5 = B$
$A = -4B$
$7 = 4B + C$

$\underline{B = 5}$
$\underline{A = -20}$
$\underline{\underline{C = -13}}$

4) $x^2+y^2-6x+14y+33=0$

a) Using standard approach to a ⊙ $x^2+y^2+px+qy+r=0$,

Centre is $\underline{(3,-7)}$

Radius is $\sqrt{3^2+7^2-33} = \sqrt{9+49-33}$

$= \sqrt{25} = \underline{\underline{5}}$

We can also recast the equation as:

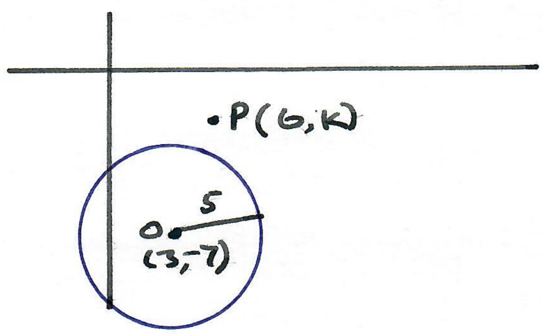
$(x-3)^2+(y+7)^2 = 25$

which confirms the result.

b) The centre of the circle has $y=-7$ and the radius is only 5, so the highest possible value is when $y=-2$ (and $x=3$)

– so the circle lies entirely below the x-axis.

c)



The length PO must be >5.

so $(k-(-7))^2+(6-3)^2 > 25$

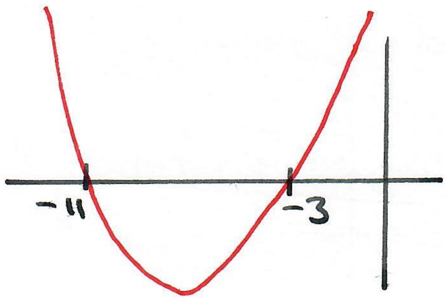
$k^2+14k+49+9 > 25$

$k^2+14k+33 > 0$

Roots where this expression $= 0$ are given by $(k+11)(k+3) = 0$

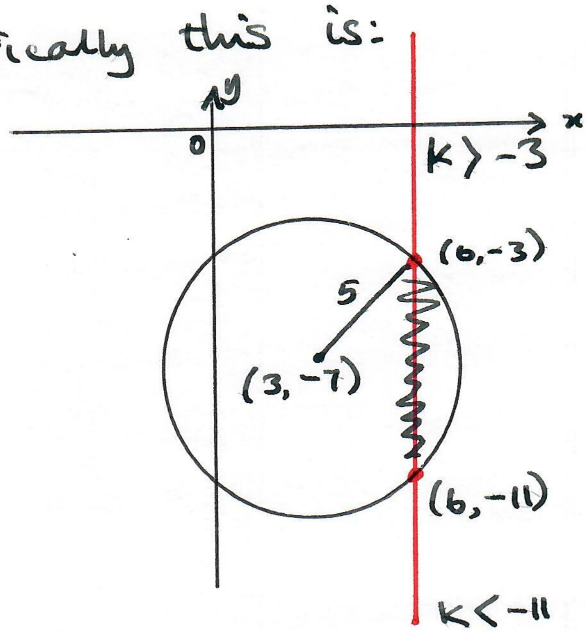
i.e. $k = -11$ or -3.

And this expression is an 'x^2' parabola:



So valid values of k are $k < -11$
or $k > -3$

Geometrically this is:



5) a) $f(\theta) = 2\cos\theta + 3\sin\theta$

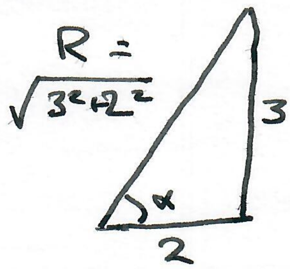
Express this as $R\cos\alpha\cos\theta + R\sin\alpha\sin\theta$

and use the identity

$\cos(\theta - \alpha) = \cos\theta\cos\alpha + \sin\theta\sin\alpha$

We have $2 = R\cos\alpha$

$3 = R\sin\alpha$



So $R = \sqrt{2^2+3^2} = \sqrt{13}$

and $\tan\alpha = \frac{3}{2}$ so $\alpha = 0.983^c$

So $f(x) = \sqrt{13}\cos(\theta - 0.983)$

b) This reaches a maximum value of $\underline{\underline{\sqrt{13}}}$

when $\cos(\theta - 0.983) = 1$

The smallest possible positive to satisfy this occurs when $\theta - 0.983 = 0$,

i.e. when $\underline{\underline{\theta = 0.983}}$

c) $T = 16 + 2\cos\left(\frac{\pi t}{12}\right) + 3\sin\left(\frac{\pi t}{12}\right)$

$= 16 + \sqrt{13}\cos\left(\frac{\pi t}{12} - 0.983\right)$

So the maximum temperature is $16 + \sqrt{13}$

$= \underline{\underline{19.606\ ^\circ C}}$

and this occurs when

$$\frac{\pi t}{12} = 0.983$$

$$t = 3.755 \text{ hours}$$

$$= \underline{\underline{3 \text{ hours } 45 \text{ min.}}}$$

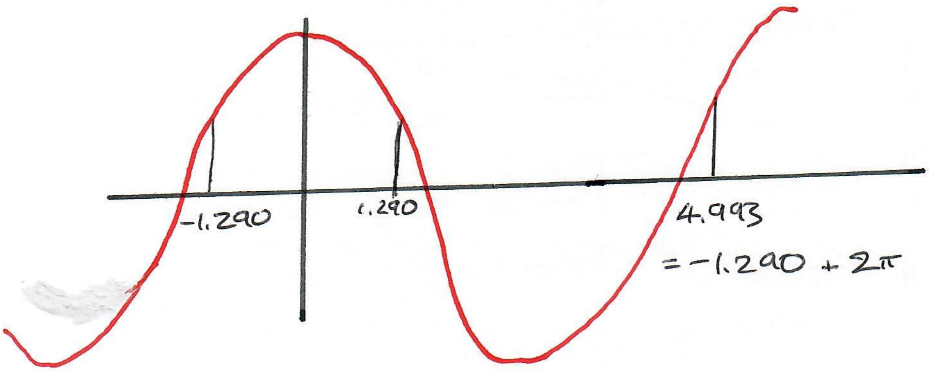
d) For the temperature to be 17°,

$$17 = 16 + \sqrt{13}\cos\left(\frac{\pi t}{12} - 0.983\right)$$

$$\frac{1}{\sqrt{13}} = \cos\left(\frac{\pi t}{12} - 0.983\right) = 0.277$$

$$\frac{\pi t}{12} - 0.983 = 1.290 \pm 2n\pi$$

and $4.993 \pm 2n\pi$



$\frac{\pi t}{12} - 0.983 = 1.290 \Rightarrow t = 8.68 = \underline{\underline{08:41}}$ (h:m)

$\frac{\pi t}{12} - 0.983 = 4.993 \Rightarrow t = 22.83 = \underline{\underline{22:50}}$ (h:m)

6) Using the trig. identity $\tan(A+B) = \dfrac{\tan A + \tan B}{1 - \tan A \tan B}$

we get $\tan 20^\circ = t = \dfrac{2\tan(10^\circ)}{1 - \tan^2(10^\circ)}$

So $t(1 - \tan^2 10^\circ) = 2\tan 10^\circ$

$\tan^2 10^\circ \, t + 2\tan 10^\circ - t = 0$

$\tan 10^\circ = \dfrac{-2 \pm \sqrt{4 + 4t^2}}{2t}$

$= \dfrac{-1 \pm \sqrt{1 + t^2}}{2t}$

So the first +ve value is

$\dfrac{-1 + \sqrt{1 + t^2}}{2}$

7) $a_1 - - -$ with $a_{n+1} = p + q a_n$

a) $a_1 = 250$

$a_2 = 220 = p + q \cdot 250$ $\therefore$ $p + 250q = 220$

$p = 220 - 250q$ — ①

$a_3 = 196 = p + q(p + q \cdot 250)$

$\therefore$ $250q^2 + pq + p = 196$ — ②

Subs ① into ②:

$$250q^2 + (220-250q)q + (220-250q) = 196$$

$$(250-250)q^2 + (220-250)q + 220 = 196$$

$$-30q = -24$$

$$q = \frac{24}{30} = \underline{\frac{4}{5}}$$

in ①: $p = 220 - 250q$

$$= 220 - 250 \times \frac{4}{5}$$

$$= \underline{\underline{20}}$$

and in general $a_{n+1} = 20 + \frac{4}{5}a_n$

b) Showing the limit (converged value) is easy enough if we <u>assume that one exists</u>:

$$a_\infty = 20 + \frac{4}{5}a_\infty$$

So $a_\infty \times \frac{1}{5} = 20$: $a_\infty = 100$ as required.

This doesn't actually prove it does converge, and on just one value... MADAS doesn't ask for that in the solution sheet.

8) a) $y = \sqrt{x^2+16}$

the new graph is $y = \sqrt{16x^2+16}$

So the transformation would be to <u>shrink</u> the graph in the x-direction by a factor of 4. In vector terms this is (a <u>stretch</u> of) $\begin{pmatrix} \frac{1}{4} \\ 1 \end{pmatrix}$

b) A translation $\begin{pmatrix} k \\ 0 \end{pmatrix}$ means the values of y are given by $f(x-k)$.

So $y = f(x-k) = \sqrt{(x-k^2)+16}$

And this curve passes through $(6,5)$:

$$5 = \sqrt{(6-k)^2+16}$$

$$25 = (6-k)^2+16$$

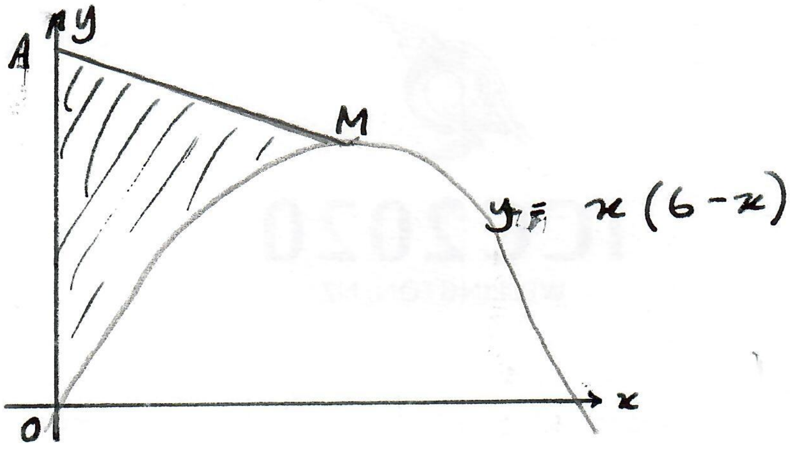
$$9 = (6-k)^2$$

$6-k = -3$: $\underline{\underline{k=9}}$

$6-k = +3$: $\underline{\underline{k=3}}$

So possible values of k are 3 and 9.

a)



First we find the point M. From the equation of the parabola we know this is where $x = 3$.

and $y = 3(6-3) = 9$.

But to do it thoroughly:

$$\frac{dy}{dx} = 6 - 2x = 0 \quad \text{so} \quad x = 3 \checkmark$$

A is the point $(0,12)$

M —————— $(3,9)$

So gradient of $AM = \frac{12-9}{0-3} = -1$

Equation of AM is $y = -x + c$

subs. for $(0,12)$: $12 = c$

So AM is $y = -x + 12$

So the integral required is $\int_0^3 (-x+12) - x(6-x)\, dx$

$$= \int_0^3 +x^2 - x - 6x + 12\, dx$$

$$= \int_0^3 x^2 - 7x + 12\, dx = \left[\frac{x^3}{3} - \frac{7x^2}{2} + 12x\right]_0^3$$

$$= 9 - \frac{63}{2} + 36 = \underline{\underline{13.5}}$$

10) $(2-3x)^2(1+4x)^7$

$= (4-12x+9x^2)\left(1+\binom{7}{1}4x+\binom{7}{2}16x^2+\cdots\right)$

$= (4-12x+9x^2)(1+28x+21\times16x^2+\cdots)$

$= (4-12x+9x^2)(1+28x+336x^2+\cdots)$

$= \cdots + (4\times336-12\times28+9)x^2+\cdots$

$= (1344-336+9)x^2$

$= \underline{\underline{1017x^2}}$

11) we look for $7^x = 2\times5^x$

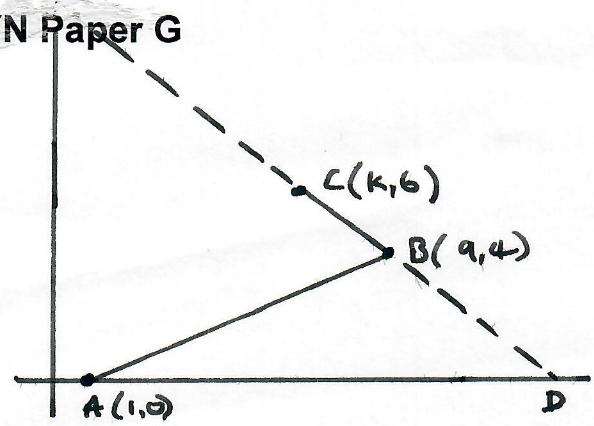
Take $\log_2$ of both sides:

$x\log_2 7 = \log_2 2 + x\log_2 5$

$x(\log_2 7-\log_2 5) = \log_2 2 = 1$

$$x = \frac{1}{\log_2 7 - \log_2 5}$$

12)



a) Gradient of AB $= \frac{4-0}{9-1} = \frac{4}{8} = \frac{1}{2}$

So gradient of CB (perpendicular) $= -2$

Equation of CB is $y = -2x + c$

At point $(9,4)$: $4 = -18 + c$

So $c = 22$ and CB is $y = -2x + 22$

Substituting for the point $(k,6)$:

$6 = -2k + 22$

$2k = 16$

$\underline{\underline{k = 8}}$

b) we have already found CB is

$y = -2x + 22$

so $2x + y = 22$.

In the terms of the question, this is

$ax + by = c$ with $a = 2$, $b = 1$, $c = 22$.

c) The point D is given by

$$0 = -2x + 22$$

so $x = 11$ and D is $(11, 0)$.

So $AD = 11 - 1 = 10$, and the altitude from B is 4.

$$\text{So } \triangle ABD = \frac{10 \times 4}{2} = \underline{\underline{20}}$$

d) $$\triangle ACD = \frac{10 \times 6}{2} = 30$$

$$\text{So } \triangle ABC = \triangle ACD - \triangle ABD = \underline{\underline{10}}$$

13) $u_3 = x + 5$

$u_4 = 4x - 1$

$u_5 = 2x + 3$

In terms of the common ratio r; we know:

$$u_4 = 4x - 1 = (x+5)r = u_3 r$$

$$u_5 = 2x + 3 = (4x-1)r = u_4 r$$

$$\text{So } \frac{4x-1}{x+5} = \frac{2x+3}{4x-1} = r$$

$$\therefore \quad (4x-1)^2 = (2x+3)(x+5)$$

$$16x^2 - 8x + 1 = 2x^2 + 13x + 15$$

$$14x^2 - 21x - 14 = 0$$

$$2x^2 - 3x - 2 = 0$$

$$(2x+1)(x-2)$$

$$x = -\frac{1}{2} \text{ or } 2$$

if $x = \frac{-1}{2}$:

$r = \frac{2x+3}{4x-1} = \frac{2}{-3} = \frac{-2}{3}$ This is valid for a sum to ∞.

if $x = 2$:

$r = \frac{2x+3}{4x-1} = \frac{7}{7} = 1$ This is not valid for a sum to infinity, since the sum would not converge

So with $r = -\frac{2}{3}$ and $x = \frac{-1}{2}$

$$u_3 = x + 5 = \frac{15}{3} - \frac{2}{3} = \frac{13}{3}$$

$$u_2 = \frac{13}{3} \Big/ -\frac{2}{3} = -\frac{13 \times 3}{3 \times 2} = \frac{-13}{2}$$

$$u_1 = \frac{-13}{2} \Big/ -\frac{2}{3} = \frac{13 \times 3}{2 \times 2} = \frac{39}{4}$$

$$u_3 = x+5 = -\frac{1}{2}+5 = \frac{9}{2}$$

$$u_2 = \frac{9}{2}\Big/\frac{-2}{3} = -\frac{9\times 3}{2\times 2} = -\frac{27}{4}$$

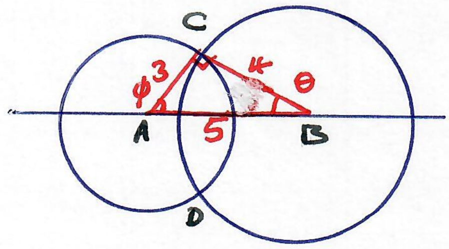
$$u_1 = -\frac{27}{4}\Big/\frac{-2}{3} = \frac{27\times 3}{4\times 2} = \frac{81}{8}$$

Using the formula $\sum_1^\infty u_n = \frac{u_1}{1-r}$, we get

$$\sum_1^\infty = \frac{81}{8}\times\frac{1}{1-\frac{-2}{3}} = \frac{81}{8}\Big/\frac{5}{3}$$

$$= \underline{\underline{\frac{243}{40}}}$$

14



Conveniently, ABC is a right-angled Δ with sides 3, 4 and 5.

So the outer arc from C to D (right hand $\odot$) has length

$$(2\pi - 2\theta)\times 4$$

And the outer arc from C to D on the left hand $\odot$

$$\left(2\pi - 2\left(\frac{\pi}{2}-\theta\right)\right)\times 3 \text{ or } (2\pi - 2\phi)\times 3$$
where $\theta + \phi = \frac{\pi}{2}$

And $\sin\theta = \frac{3}{5}$: $\theta = 0.643^c$

$\sin\phi = \frac{4}{5}$: $\phi = 0.927^c$

So the total perimeter is:

$4(2\pi - 2\theta) + 3(2\pi - 2\phi)$

$= 4 \times 4.997 + 3 \times 4.429$

$= 19.989 + 13.287$

$= \underline{\underline{33.276 \text{ cm}}}$

15) $I = \int_0^{\pi} \frac{1 + x\cos x}{x + e^{-\sin x}} \, dx$

when $x = 0$ $u = 1$

$x = \pi$ $u = 1 + \pi e^0$
$= 1 + \pi$

Let $u = 1 + xe^{\sin x}$

Then $\frac{du}{dx} = x\cos x \, e^{\sin x} + e^{\sin x}$

$= (x\cos x + 1)e^{\sin x}$

So $dx = \frac{du}{(x\cos x + 1)} e^{-\sin x}$

$I = \int_1^{1+\pi} \frac{\cancel{1 + x\cos x}}{x + e^{-\sin x}} \frac{e^{-\sin x}}{\cancel{x\cos x + 1}} \, du$

$= \int_1^{1+\pi} \frac{e^{-\sin x}}{x + e^{-\sin x}} \, du = \int_1^{1+\pi} \frac{1}{xe^{\sin x} + 1} \, du = \int_1^{1+\pi} \frac{1}{u} \, du$

$= \left[\ln u\right]_1^{1+\pi} = \ln(1+\pi) - \ln 1 = \underline{\underline{\ln(1+\pi)}}$

16) $C: \quad y = x(x-2)^3$

Using the chain rule,

$$\frac{dy}{dx} = x.3(x-2)^2 + (x-2)^3$$

$$= 3x(x^2-4x+4) + (x-2)(x^2-4x+4)$$

$$\begin{aligned} = \; & 3x^3 - 12x^2 + 12x \\ + \; & x^3 - 4x^2 + 4x \\ & \quad\; - 2x^2 + 8x - 8 \\ \hline = \; & 4x^3 - 18x^2 + 24x - 8 \end{aligned}$$

$= 10$ at the point given.

So $\quad 4x^3 - 18x^2 + 24x - 18 = 0$

$$2x^3 - 9x^2 + 12x - 9 = 0$$

By inspection, this is 0 when $x = 3$:

$$(x-3)(2x^2-3x+3) = 0$$

So $x = 3$ is a solution but the quadratic term has no roots since its discriminant $3^2 - 4.2.3 = -15$

which is negative.

So there is only one point where the gradient is 10.

17)

$$\frac{dx}{dt} = k\frac{1}{x}$$

So $\quad x\,dx = k\,dt$

Integrating, $\quad \dfrac{x^2}{2} = kt + C$

or $\quad x^2 = At + B.$

Substituting for the two points:

$50^2 = A \times 0 + B \quad : \quad B = 2500$

$30^2 = 4A + B \quad : \quad 4A = 900 - 2500$

$= -1600$

$A = -400$

So the equation of motion is

$$x = \sqrt{-400t + 2500}$$

This is 0 when $400t = 2500$

$t = \dfrac{25}{4} = 6.25$ sec.

18) $$2x^2 + xy - y^2 - 4x - y + 20 = 0 \qquad (1)$$

Differentiate w.r.t x:

$$4x + x\frac{dy}{dx} + 1.y - 2y\frac{dy}{dx} - 4 - \frac{dy}{dx} = 0$$

Gathering terms:

$$\frac{dy}{dx}(x - 2y - 1) = -4x - y + 4$$

$$\frac{dy}{dx} = \frac{(-4x - y + 4)}{(x - 2y - 1)}$$

$$\frac{dy}{dx} = \frac{4x + y - 4}{2y - x + 1}$$

as required

This is zero when $4x + y - 4 = 0$, i.e.

$$y = -4x + 4 \qquad (3)$$

This is a straight line, and if we were simply plotting the given function it would suggest the stationary points are all along this line. But we know the function is 0 overall, so we solve for the exact crossing points.

Substitute for y in (1):

$$2x^2 + x(-4x + 4) - (-4x + 4)^2 - 4x - (-4x + 4) + 20 = 0$$

$$2x^2 - 4x^2 + 4x - 16x^2 + 32x - 16 - 4x + 4x - 4 + 20 = 0$$

$$-18x^2 + 36x = 0$$

$$-x^2 + 2x = 0$$

$$x(2 - x) = 0$$

So $x = 0$ or 2. From (3):

$x = 0$ gives $y = 4$, so the point is (0,4)

$x = 2$ gives $y = -4$, so the point is (2,- 4)

Consider the general equation in (2) and differentiate again:

$$4 + x\frac{d^2y}{dx^2} + \frac{dy}{dx} + \frac{dy}{dx} - 2y\frac{d^2y}{dx^2} - 2\left(\frac{dy}{dx}\right)^2 - \frac{d^2y}{dx^2} = 0$$

Regroup:

$$4 + 2\frac{dy}{dx} - 2\left(\frac{dy}{dx}\right)^2 + (x - 2y - 1)\frac{d^2y}{dx^2} = 0$$

as required

At the stationary points the gradient is 0, so this simplifies to:

$$4 + (x - 2y - 1)\frac{d^2y}{dx^2} = 0$$

$$\frac{d^2y}{dx^2} = \frac{4}{2y - x + 1}$$

At (0,4):

$$\frac{d^2y}{dx^2} = \frac{4}{8 - 0 + 1} = \frac{4}{9}$$

This is > 0 so (0,4) is a local minimum.

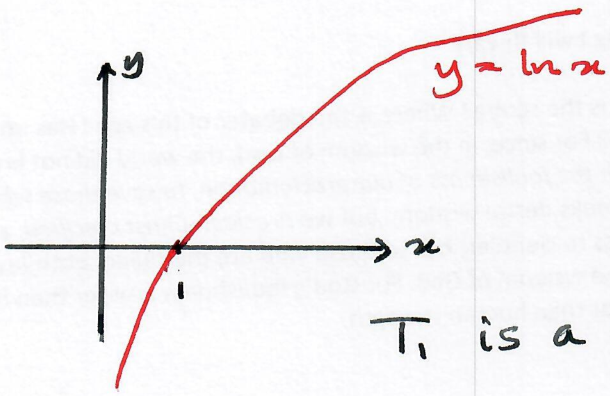
At (2,-4):

$$\frac{d^2y}{dx^2} = \frac{4}{-8 - 2 + 1} = -\frac{4}{9}$$

This is < 0 so (2,-4) is a local maximum.

19) $f(x) = 6 - \ln(x+3)$

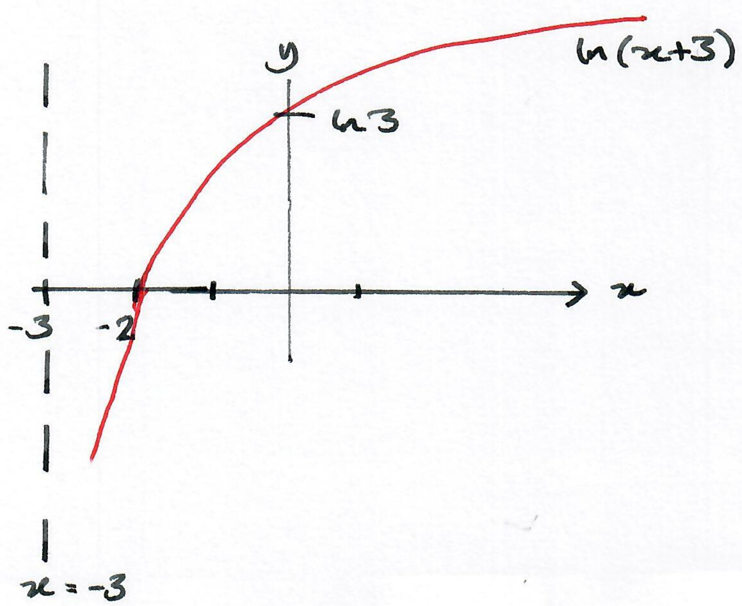
a)



T_1

$\ln(x) \rightarrow \ln(x+3)$

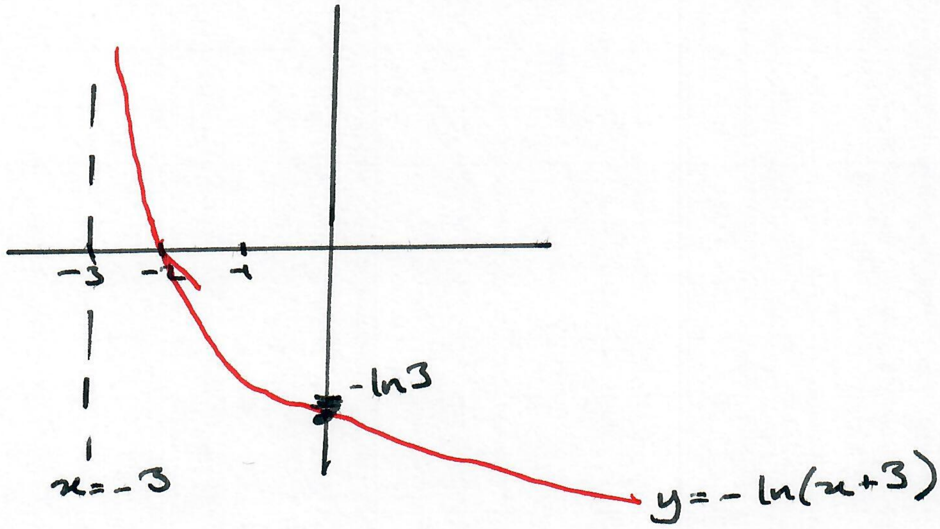
T_1 is a shift to the <u>left</u> by 3:



when $x = 0$,

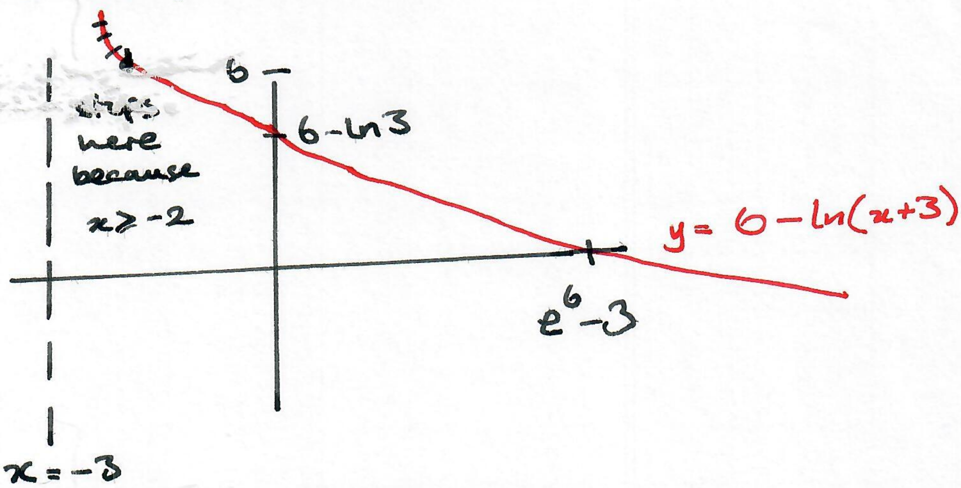
$\ln(x+3) = \ln 3$.

T_2 is a reflection in the x-axis:



T_3 is a translation (shift) upwards by 6:

(The equation could be written as $y-6=-\ln(x+3)$ ... which explains why it's a shift upwards)



when $x=0$, $y=6-\ln(3)$

$y=0$ $\quad 0=6-\ln(x+3)$

$x+3=e^6$

$x=e^6-3$

b) Let $y = 6 - \ln(x+3) = f(x)$

$$\ln(x+3) = 6-y$$

$$x+3 = e^{6-y} = \frac{e^6}{e^y}$$

$$\underline{\underline{x = e^{6-y} - 3}}$$

So for $f^{-1}(x)$:

Domain $y \geq -2$

Range $x \leq 6$

c) $g : x \rightarrow e^{x^2} - 3$

$$fg(x) = f(e^{x^2}-3)$$

$$= 6 - \ln((e^{x^2}-3)+3)$$

$$= 6 - \ln e^{x^2}$$

$$= \underline{\underline{6 - x^2}}$$

20) $x = 2t + 4$ ①

$y = t^3 - 4t + 1$

$\frac{dx}{dt} = 2$ $\frac{dy}{dt} = 3t^2 - 4$

So $\frac{dy}{dx} = \frac{3t^2-4}{2}$

At $A(2,4)$, $x = 2 = 2t+4$ so $t = -1$

$y = t^3 - 4t + 1 = -1 + 4 + 1 = 4$ ✓

So $\frac{dy}{dx} = \frac{3-4}{2} = -\frac{1}{2}$.

So the gradient of the tangent is $-\frac{1}{2}$

and its equation is $y = -\frac{1}{2}x + c$

Subs. for point $A(2,4)$:

$4 = -\frac{1}{2} \cdot 2 + c$ $c = 5$

So the tangent is $y = -\frac{1}{2}x + 5$

or $2y + x = 10$ ② as required.

At the point B, the values x and y

satisfy both ① and ②, so

$2(t^3 - 4t + 1) + (2t+4) = 10$

$2t^3 - 8t + 2 + 2t + 4 = 10$

$2t^3 - 6t - 4 = 0$

$$t^3 - 3t - 2 = 0$$

we know this is satisfied at A where $t = -1$,

so $(t+1)$ is a factor:

$$(t+1)(t^2 - t - 2) = 0$$

$$(t+1)(t+1)(t-2) = 0$$

So $t = -1$ is a double root (expected) and $t = 2$ is the value at B.

B is the point $x = 2t + 4 = 8$

$$y = t^3 - 4t + 1 = 8 - 8 + 1$$

$$= 1$$

So B is $(8, 1)$

21) This is a problem with 3 linked variables V, h and t. It also mixes units.

Initially: $t = 0$ $V = 8.1\,l = 8100\ \text{cm}^3$ of liquid.

At any point (any t): $V = 36h^2$ — ①

$$\frac{dV}{dt} = 2t \quad \text{—— ②}$$

we want to know $\frac{dh}{dt}$ when $t = 2\text{min}$ (120 sec)

From ②, $V = \int 2t\, dt = t^2 + C$

when $t = 0$, $V = 8100 = 0 + C$

So in general $V = t^2 + 8100$

and after 2 minutes, $V = 14400 + 8100$

$= 22500\ \text{cm}^3$

We also know:

$$\frac{dh}{dt} = \frac{dh}{dV} \times \frac{dV}{dt}$$

and $\dfrac{dV}{dh} = 72h$

$$\frac{dV}{dt} = 240$$

So $\dfrac{dh}{dt} = \dfrac{240}{72h} = \dfrac{10}{3h}$

and $h = \sqrt{\dfrac{V}{36}} = \sqrt{\dfrac{22500}{36}} = \dfrac{150}{6} = 25$

So $\dfrac{dh}{dt} = \dfrac{10}{3 \times 25} = 0.133$ cm/s.