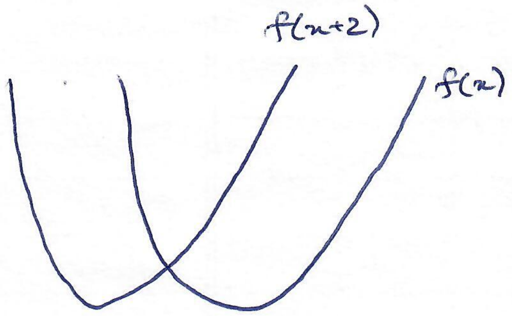


1) $f(x) = x^2 - 7x + 6$

$f(x+2) = (x+2)^2 - 7(x+2) + 6$

$= x^2 + 4x + 4 - 7x - 14 + 6$

$= x^2 - 3x - 4$

So if $f(x) = f(x+2)$,

$x^2 - 7x + 6 = x^2 - 3x - 4$

$-4x = -10$

$$x = \frac{5}{2}$$

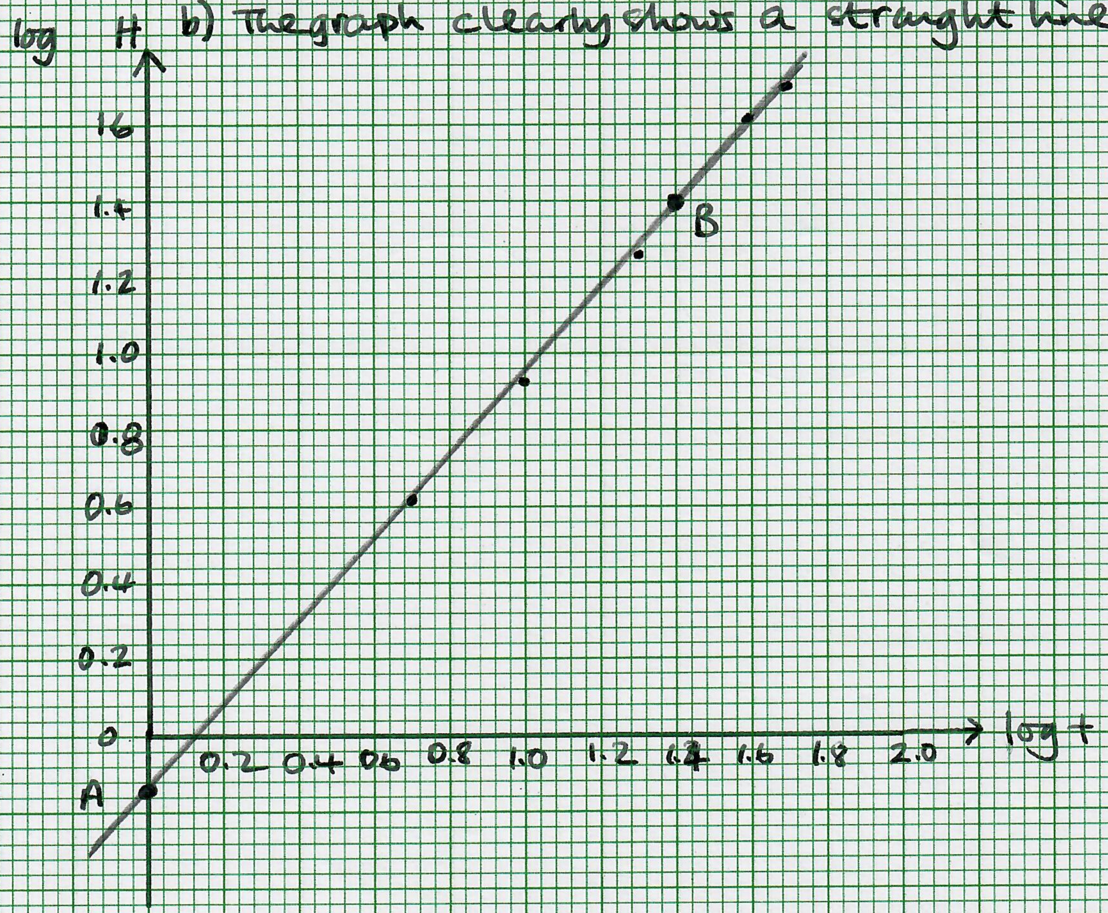
2) $H = kt^n$

Take logarithms: $\log H = \log k + n \log t$

This has the form of a standard straight line expressed as $y = c + mx$

t	5	10	20	40	50
H	4.1	8.5	18.0	42.0	50.0
$\log t$	0.70	1.00	1.30	1.60	1.70
$\log H$	0.61	0.93	1.26	1.62	1.70

b) The graph clearly shows a straight line relationship



A is $(0, -0.14)$

B is $(1.4, 1.4)$

So the gradient $n = \dfrac{1.4 + 0.14}{1.4} = \underline{\underline{1.1}}$

And the y intercept is -0.14

So $\log k = -0.14$

$\underline{\underline{k = 0.72}}$

So the equation for the experimental data is:

$$\underline{\underline{H = 0.72t^{1.1}}}$$

3) $y = x^2 + 8x + 12$

This factorises as $(x+6)(x+2)$, but that doesn't help identify transformations. Instead, look for a simple shift in x and y:

$$y = x^2 + 8x + 12 = (x+4)^2 - 4$$

In this form we can identify a shift <u>left</u> by 4 and then ~~up~~ down by 4, shown by:

$$y + 4 = (x+4)^2$$

This can be represented as $\begin{pmatrix} -4 \\ -4 \end{pmatrix}$.

4) a) $x_{n+1} = \dfrac{k - 5x_n}{x_n}$ $\quad x_1 = 1 \quad k > 5$

$x_1 = 1$

$x_2 = \dfrac{k - 5 \times 1}{1} = k - 5$

$x_3 = \dfrac{k - 5(k-5)}{k-5} = \dfrac{25 - 4k}{k-5}$

b) we know $x_3 > 6$,

so $\dfrac{25-4k}{k-5} > 6$

$25 - 4k > 6k - 30$

$55 > 10k$

$5 < k < 5.5$

we know

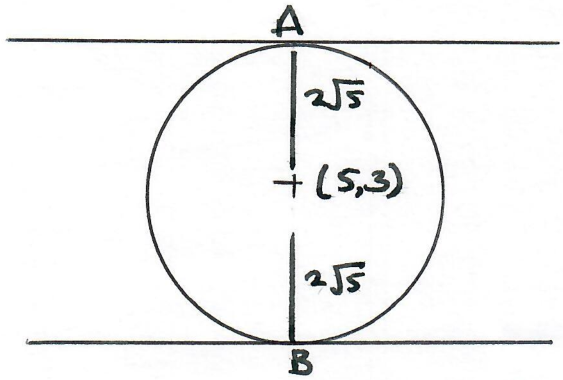
3) $x^2+y^2-10x-6y+14=0$

Rearrange:

$$(x-5)^2+(y-3)^2=20$$

So the centre is $(5,3)$ and the radius is

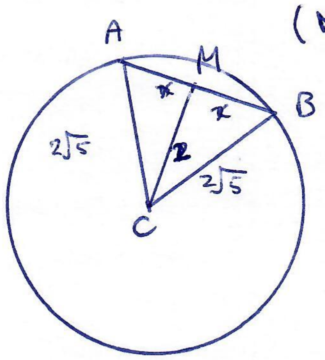
$\sqrt{20} = 2\sqrt{5}$.



a) $y=k$ is a horizontal line touching the circle at A or B;

so $\underline{\underline{k = 3+2\sqrt{5} \text{ or } 3-2\sqrt{5}.}}$

b) (Not well drawn)



Using Pythagoras, $x^2 = (2\sqrt{5})^2 - 2^2 = 20-4$
$= 16.$

$\underline{\underline{\text{So } AB = 8}}$

c) line: $x - 2y - 9 = 0$

$$x = 9 + 2y$$

Subs. into the circle equation:

$$(9+2y)^2 + y^2 - 10(9+2y) - 6y + 14 = 0$$

$$81 + 36y + 4y^2 + y^2 - 90 - 20y - 6y + 14 = 0$$

$$5y^2 + 10y + 5 = 0$$

$$y^2 + 2y + 1 = 0$$

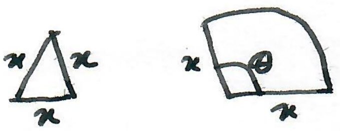
$$(y+1)^2 = 0 \quad \text{(a repeated root)}$$

So $y = -1$

and $x = 9 + 2 \times -1 = 7$

So the point D is $(7, -1)$

6)



Perimeter of $\Delta = 3x$

sector $= 2x + \theta x$

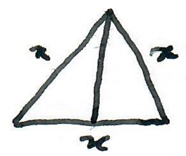
radial ↑ ↑ curve

and we know this is 60m

So $3x + 2x + \theta x = 60$

or $\underline{\underline{x\theta = 60 - 5x}}$ as required.

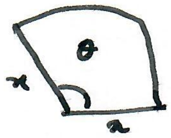
b)



Height of $\Delta = \frac{\sqrt{3}}{2}x$

Area of $\Delta = \frac{1}{2} x \frac{\sqrt{3}x}{2}$

$= \frac{\sqrt{3}}{4}x^2$



Area of sector $= \frac{\theta x^2}{2}$

and we know this $= (60-5x)\frac{x}{2}$

So the total area $A = \frac{\sqrt{3}}{4}x^2 + \frac{(60-5x)x}{2}$

$= \frac{\sqrt{3}}{4}x^2 - \frac{10}{4}x^2 + 30x$

$= \frac{1}{4}(\sqrt{3}-10)x^2 + 30x$

as required.

c) A is stationary when $\frac{dA}{dx} = 0$.

$\frac{dA}{dx} = \frac{2}{4}(\sqrt{3}-10)x + 30 = 0$

$\frac{1}{2}(10-\sqrt{3})x = 30$

$x = \frac{60}{10-\sqrt{3}} = \frac{60(10+\sqrt{3})}{(10-\sqrt{3})(10+\sqrt{3})}$

$= \frac{60(10+\sqrt{3})}{100-3}$

$= \frac{60}{97}(10+\sqrt{3})$

≈ 7.257

This is the only stationary point, and we observe $\frac{d^2A}{dx^2} = \frac{1}{2}(\sqrt{3} - 10)$.

This is negative, indicating a maximum

gradient is decreasing:

2nd deriv. is negative.

At this point $A = \frac{1}{4}(\sqrt{3}-10)(7.257)^2 + 30 \times 7.257$

$= 108.854$

$= 109 \text{ cm}^2$ to 3 sig. figs.

7) $x^2 + (k-1)x + k + 2 = 0$

Using the quadratic formula, this has repeated roots if the discriminant "$b^2 - 4ac$" > 0.

So $(k-1)^2 - 4(k+2) > 0$

Actually we're not asked to comment on k, so we can just go to

$$x = \frac{(1-k) \pm \sqrt{(k-1)^2 - 4(k+2)}}{2}$$

$$= \frac{(1-k) \pm \sqrt{k^2 - 2k + 1 - 4k - 8}}{2}$$

$$= \frac{(1-k) \pm \sqrt{k^2-6k-7}}{2}$$

$$= \frac{(1-k) \pm \sqrt{(k-7)(k+1)}}{2}$$

But we know the equation has repeated roots.

So the discriminant is 0: which means $k=-1$ or 7.

And the solution for x is $x = \frac{1-k}{2}$.

So if $k=-1$, $x = \frac{1-(-1)}{2} = 1$

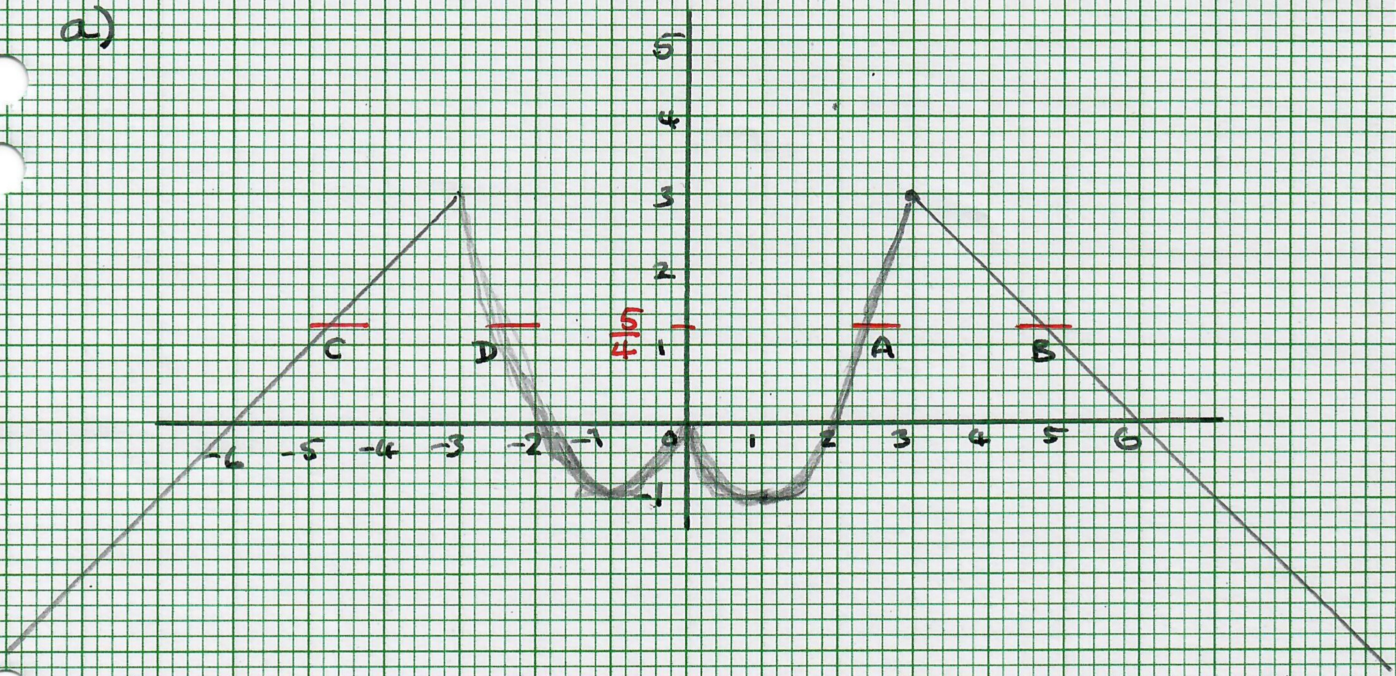
if $k=7$, $x = \frac{1-7}{2} = -3$.

8) Note the fact f is even: this means it's symmetrical about the y-axis. The rest of the question may appear to assume $x \geq 0$, but in fact we're expected to include $x<0$ too.

a) For $0 \leq x \leq 3$ the curve is an 'x^2' parabola with roots 0 and 2, and minimum at $(1,-1)$

For $x > 3$ $x-6$ is simply a st line with gradient -1.

a)



b) $f(x) = \frac{5}{4}$ at 4 points, symmetrically placed, and one on each part of the curve.

so: for point A, $x^2 - 2x = \frac{5}{4}$

$4x^2 - 8x = 5$

$4x^2 - 8x - 5 = 0$

$(2x+1)(2x-5) = 0$

so $x = -\frac{1}{2}$ (but not relevant)

or $x = \frac{5}{2} = 2\frac{1}{2}$

for point B: $6 - x = \frac{5}{4}$

$24 - 4x = 5$

$4x = 19$

$x = \frac{19}{4} = 4\frac{3}{4}$

Reflecting these in the y axis we get:

C: $x = -\frac{19}{4}$ B: $x = \frac{19}{4}$

D: $x = -\frac{5}{2}$ A: $x = \frac{5}{2}$

9) $\ln(2\sqrt{e}) - \frac{1}{3}\ln\left(\frac{8}{e^2}\right) - \ln\left(\frac{e}{3}\right)$

To get the answer in the required form, we gather all the 'e' terms together:

$$\ln 2 + \ln\sqrt{e} - \ln 8^{\frac{1}{3}} + \ln(e^2)^{\frac{1}{3}} - \ln e + \ln 3$$

$$= \ln \frac{\sqrt{e}\times(e^2)^{\frac{1}{3}}}{e} + \ln \frac{2}{8^{\frac{1}{3}}}\times 3$$

$$= \ln e^{\frac{1}{2}+\frac{2}{3}-1} + \ln \frac{2}{2}\times 3$$

$$= \ln e^{\frac{1}{6}} + \ln 3$$

$$= \frac{1}{6} + \ln 3$$

with $a=6$ and $b=3$ in the required form.

10) $y = 3x^4 - 8x^3 - 6x^2 + 24x - 8 \qquad -2 \leqslant x \leqslant 3.$

This looks like a quartic of the form  or possibly just 

$x=1$ is not a root,

but $x=2$ gives $y = 48 - 64 - 24 + 48 - 8 = 0$

So $(x-2)$ is a factor.

Factorise: $y = (x-2)(3x^3 - 2x^2 - 10x + 4)$

By inspection we see there's a further factor $(x-2)$:

$$3x^3 - 2x^2 - 10x + 4$$

$$= (x-2)(3x^2 + 4x - 2)$$

So $(x-2)$ is a double root, and the others are given by

$$\frac{-4 \pm \sqrt{16 + 24}}{6} = \frac{-4 \pm \sqrt{40}}{6}$$

$$= -\frac{2}{3} \pm \frac{1}{3}\sqrt{10}$$

(which is very approximately

$$\frac{1}{3}(-2 - 3.1) = -1.7$$

$$\frac{1}{3}(-2 + 3.1) = 0.37\Big)$$

Now look at the derivative:

$$\frac{dy}{dx} = 12x^3 - 24x^2 - 12x + 24$$

$= 0$ for a stationary point.

$$\text{So } x^3 - 2x^2 - x + 2 = 0$$

$$= (x-2)(x^2 + 0x - 1)$$

$$= (x-2)(x-1)(x+1)$$

Find the nature of these points:

$$\frac{d^2y}{dx^2} = (12)(3x^2 - 4x - 1)$$

So when $x=-1$ $\frac{dy}{dx}=0$ and $\frac{d^2y}{dx^2} = 3+4-1 = 6$

$\therefore$ local minimum.

and $y = 3+8-6-24-8$
$= -27$

$x=1$ - - - - - - - $3-4-1 = -2$

$\therefore$ local maximum

and $y = 3-8-6+24-8$
$= 5$

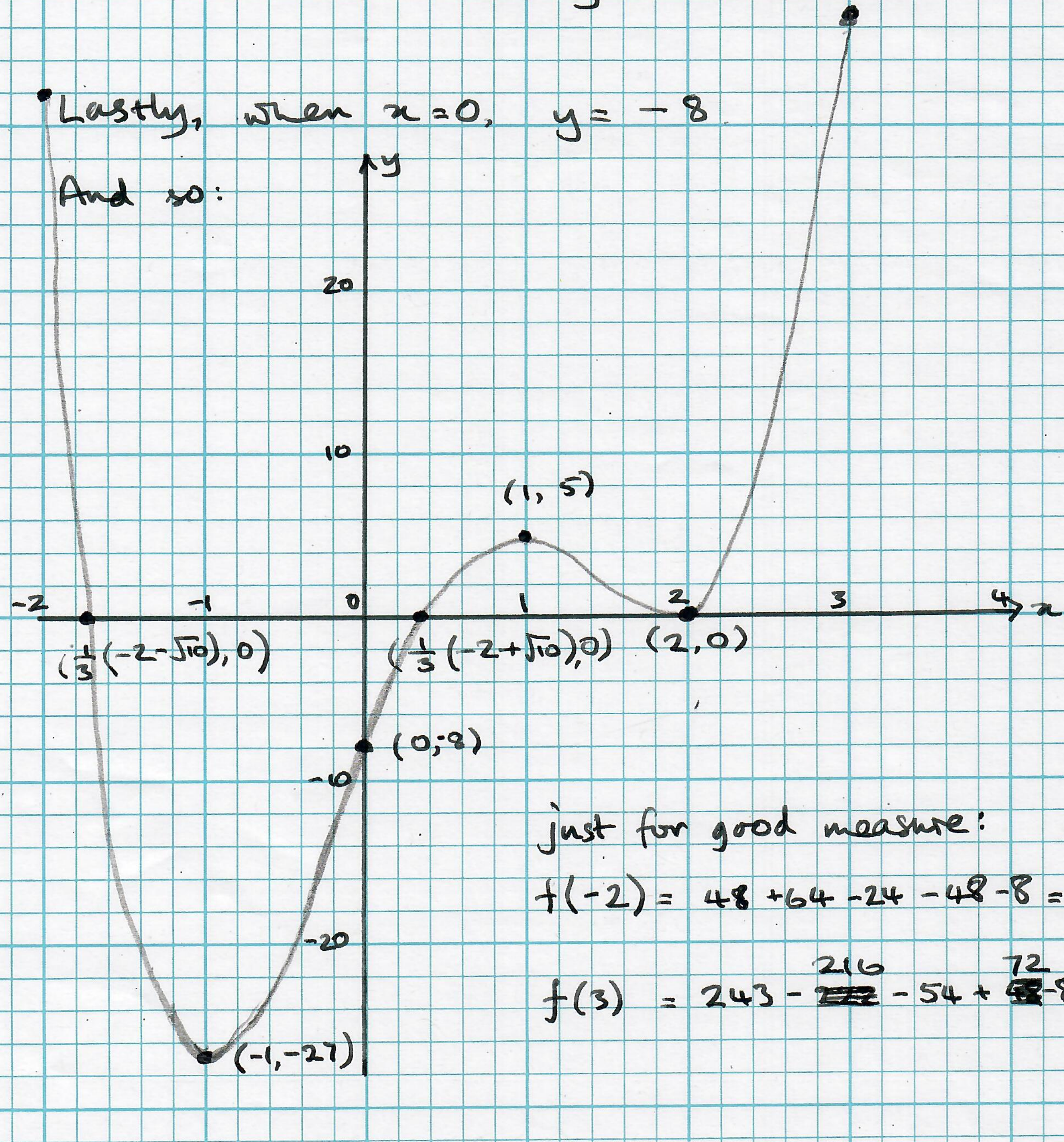
$x=2$ - - - - - - - $12-8-1 = 3$

$\therefore$ local minimum.

and $y=0$ (we know).

Lastly, when $x=0$, $y=-8$

And so:



just for good measure:

$f(-2) = 48+64-24-48-8 = 32$

$f(3) = 243-216-54+72-8 = 37$

11) $f(x) = \dfrac{3x+3}{x-2}$

a) There's clearly a vertical assymptote at

$x=2$

and as $x \to \infty$, $f(x) \to 3$ from above

as $x \to -\infty$, $f(x) \to 3$ from below.

Also if we separate into fractions:

$$f(x) = \frac{3(x-2)+9}{x-2} = 3 + \frac{9}{x-2}$$

Switching to y: $y - 3 = \dfrac{9}{x-2}$

This is a hyperbola with axes $y=3$ and $x=2$, lying in the upper right and lower left quadrants.

Also when $x=0$, $f(x) = \dfrac{3}{-2} = -\dfrac{3}{2}$

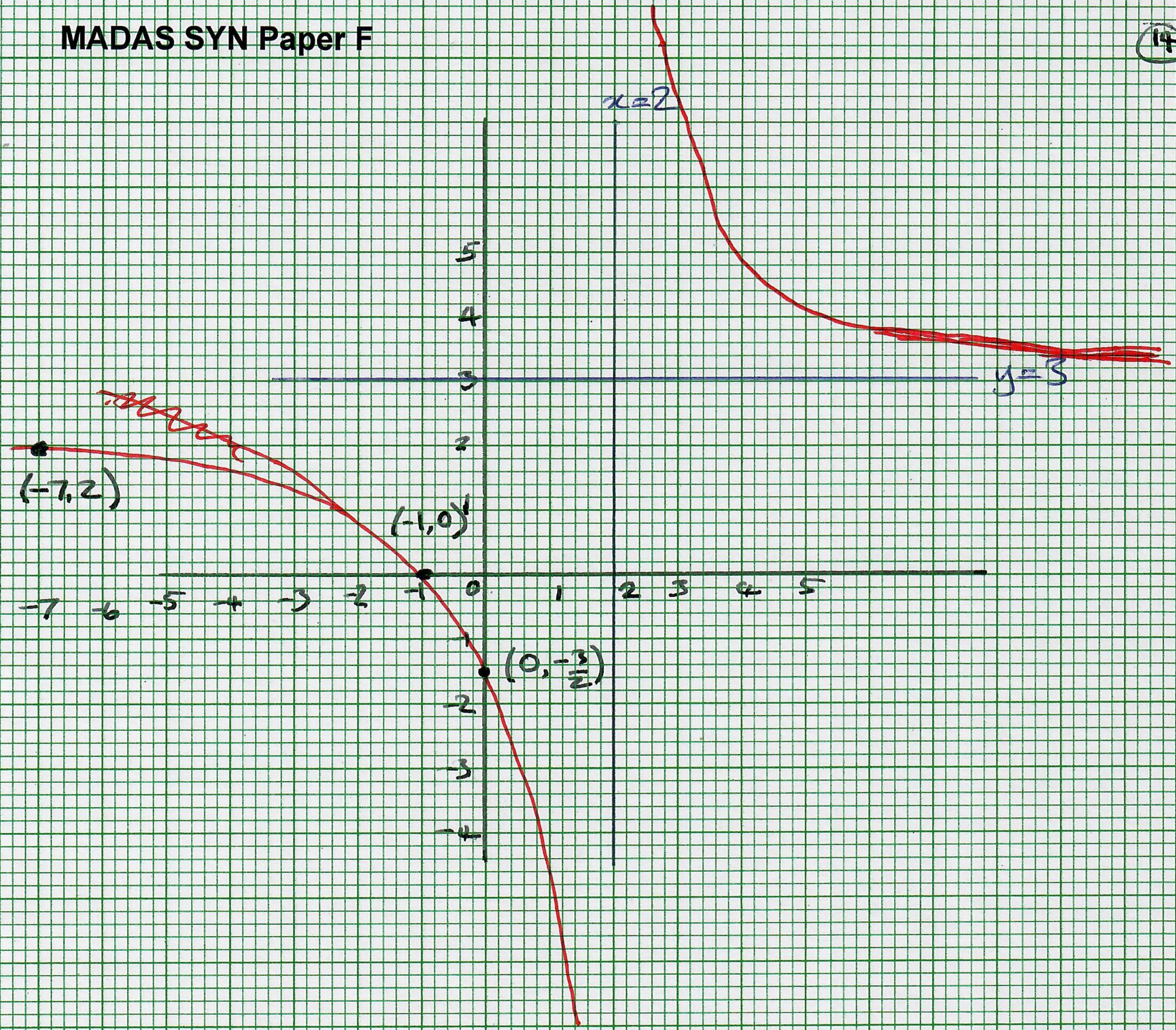
when $y=0$, $3x+3=0$

So $x=-1$

b) when $f(x)=2$, $(x-2)2 = 3x+3$

$2x-4 = 3x+3$

$\underline{\underline{-7 = x}}$



c) for $f(x) \geqslant 2$ we require

$$\underline{\underline{x \leq -7}}$$

or $\underline{\underline{x > 2}}$.

12) $$y = x - 2\ln(x^2+4)$$

$$\frac{dy}{dx} = 1 - 2\frac{2x}{x^2+4} = 1 - \frac{4x}{x^2+4}$$

a) $$\frac{d^2y}{dx^2} = -\frac{(x^2+4)4 - 4x(2x)}{(x^2+4)^2}$$

$$= -\frac{4x^2+16-8x^2}{(x^2+4)^2}$$

$$= \frac{4x^2-16}{(x^2+4)^2}$$

$$= \frac{4(x^2-4)}{(x^2+4)^2}$$ as required.

b) For a stationary point $\frac{dy}{dx} = 0$

so $$1 = \frac{4x}{x^2+4}$$

$$x^2 - 4x + 4 = 0$$

$$(x-2)^2 = 0$$

$\underline{x = 2}$ (repeated)

At $x = 2$, $\frac{d^2y}{dx^2} = \frac{4(0)}{64} = 0$ so the stationary point is a point of inflexion.

And at $x = 2$, $y = 2 - 2\ln(2^2+4)$

$$= 2 - 2\ln 8$$

$$= \underline{2 - 6\ln 2}$$

13) a) $\sum_{i=1}^{n} u_i = n u_1 + \frac{n(n-1)d}{2}$

So $\sum_{i=1}^{25} u_i = 1050 = 25u_1 + \frac{24.25d}{2}$

$u_{25} = 72 = u_1 + 24d$

So $u_1 + 24d = 72$

$25u_1 + 600d = 1800$

$25u_1 + 300d = 1050$

$300d = 750$

$\underline{\underline{d = 2.5}}$

$u_1 + 60 = 72 : \underline{\underline{u_1 = 12}}$

b) For convenience, let $\sum_1^k u_n = T.$

Then we have:

$117(1050 - T) = 233T$

$122850 = 350T$

$T = 351$

So $351 = \sum_{i=1}^{k} u_i = k.12 + \frac{k(k-1)}{2}2.5$

$702 = 24k + 2.5k^2 - 2.5k$

$1404 = 48k + 5k^2 - 5k$

$5k^2 + 43k - 1404 = 0$

$k = \frac{-43 \pm \sqrt{43^2 + 4.5.1404}}{2}$

$= (-216) \text{ or } \underline{\underline{13}}$

14) $\sqrt{2}x + \sqrt{3}y = 5$ ——— ①

$(5\sqrt{3} - \sqrt{2})x + (5\sqrt{2} - \sqrt{3})y = 10\sqrt{6}$ ——— ②

For a start, we can simplify:

①+②: $5\sqrt{3}x + 5\sqrt{2}y = 5 + 10\sqrt{6}$

$\sqrt{3}x + \sqrt{2}y = 1 + 2\sqrt{6}$ ——— ③

$\sqrt{2}x + \sqrt{3}y = 5$ ——— ① again.

We can do the usual multiply and subtract...
but we can also do it by matrices:

$$\begin{pmatrix} \sqrt{3} & \sqrt{2} \\ \sqrt{2} & \sqrt{3} \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 + 2\sqrt{6} \\ 5 \end{pmatrix}$$

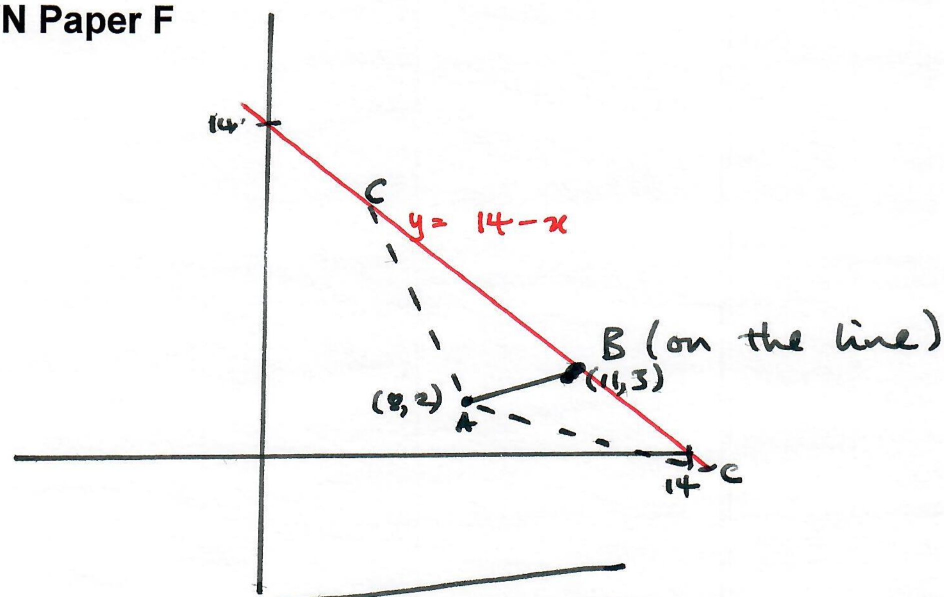
pre-multiply by the inverse: $1.\begin{pmatrix} \sqrt{3} & -\sqrt{2} \\ -\sqrt{2} & \sqrt{3} \end{pmatrix}$

$$\begin{pmatrix} x \\ y \end{pmatrix} = 1.\begin{pmatrix} \sqrt{3} & -\sqrt{2} \\ -\sqrt{2} & \sqrt{3} \end{pmatrix}\begin{pmatrix} 1 + 2\sqrt{6} \\ 5 \end{pmatrix}$$

$$x = \sqrt{3} + 6\sqrt{2} - 5\sqrt{2} = \underline{\underline{\sqrt{3} + \sqrt{2}}}$$

$$y = -\sqrt{2} - 4\sqrt{3} + 5\sqrt{3} = \underline{\underline{-\sqrt{2} + \sqrt{3}}}$$

15)



$$\text{Distance } AB = \sqrt{(11-8)^2 + (3-2)^2}$$

$$= \sqrt{3^2 + 1^2}$$

$$= \sqrt{10}$$

Let C be the point(s) $(h, 14-h)$

Then Distance $AC = 2\sqrt{10}$

$$= \sqrt{(h-8)^2 + (14-h-2)^2}$$

So $40 = (h-8)^2 + \underbrace{(14-h-2)^2}_{=12-h}$

$$= h^2 - 16h + 64 + 144 - 24h + h^2$$

$$= 2h^2 - 40h + 208$$

$$2h^2 - 40h + 168 = 0$$

$$(h-14)(2h-12) = 0$$

$h = 6$: C is the point $\underline{\underline{(6,8)}}$

$h = 14$: C is the point $\underline{\underline{(14,0)}}$

16) a) $\sin 2x \equiv \dfrac{2\tan x}{1+\tan^2 x}$

'Proving' this rather depends on what we're allowed as 'given'. We'll assume it's OK to use

$$\cos^2 x + \sin^2 x = 1 \quad \text{—— ①}$$
$$\therefore \; 1 + \tan^2 x = \sec^2 x$$
$$\text{and } \sin 2x = 2\sin x \cos x \quad \text{—— ②}$$

So $\dfrac{2\tan x}{1+\tan^2 x} = \dfrac{2\tan x}{\sec^2 x}$

$$= \frac{2\sin x}{\cancel{\cos x}} \cos^{\cancel{2}} x$$

$$= 2\sin x \cos x$$

which is indeed $\sin 2x$ as required.

b) Let $\dfrac{8}{(3t+1)(t+3)} = \dfrac{A}{3t+1} + \dfrac{B}{t+3}$

$$= \frac{A(t+3)}{(3t+1)(t+3)} + \frac{B(3t+1)}{(t+3)(3t+1)}$$

$$= \frac{At + 3A + 3Bt + B}{\sim}$$

Equating coeffs: of 1: $3A + B = 8$

of t: $A + 3B = 0$

$3A + 9B = 0$

$8B = -8$ $\quad B = -1$

$A = +3$

To find $\displaystyle\int_0^{\frac{\pi}{4}} \frac{8}{3+5\sin 2x}\,dx$:

First let $t = \tan x$ and substitute for $\sin 2x$:

$$\frac{8}{3+5\left(\frac{2t}{1+t^2}\right)} = \frac{8(1+t^2)}{3(1+t^2)+10t} = \frac{8(1+t^2)}{3t^2+10t+3}$$

Also $\dfrac{dt}{dx} = \sec^2 x = 1+t^2$

So $dx = \dfrac{dt}{1+t^2}$.

So the integral is $\displaystyle\int \frac{8\cancel{(1+t^2)}}{3t^2+10t+3}\,\frac{dt}{\cancel{(1+t^2)}}$

$$= \int \frac{8}{3t^2+10t+3}\,dt$$

$$= 8\int \frac{1}{(3t+1)(t+3)}\,dt$$

with limits
$x=0$: $t=0$
$x=\frac{\pi}{4}$: $t=1$

$$= \cancel{8}\int_0^1 \frac{3}{3t+1} - \frac{1}{t+3}\,dt$$

$$= \cancel{8}\Big[\ln(3t+1) - \ln(t+3)\Big]_0^1 = \cancel{8}\Big[\ln 4 - \ln 4 - \ln 1 + \ln 3\Big]$$

$= \ln 3$ as required

17) $x = 3at$ $\quad t = \frac{x}{3a}$

$y = at^3$

$\frac{dy}{dt} = 3at^2 \qquad \frac{dt}{dx} = \frac{1}{3a}$

So $\frac{dy}{dx} = \frac{dy}{dt} \times \frac{dt}{dx} = \frac{3at^2}{3a} = t^2$ at the point $(3at, at^3)$

At this point the normal has gradient

$$\frac{-1}{t^2}$$

and has the form $y = \frac{-1}{t^2}x + c$

Substituting for t:

$$at^3 = \frac{-1}{t^2}3at + c$$

So $c = at^3 + \frac{3a}{t}$

So the normal is

$$y = \frac{-x}{t^2} + at^3 + \frac{3a}{t}$$

or $yt^2 + x = 3at + t^5$

We know that at the point P

$\frac{dy}{dx} = t^2$ and the normal has gradient $\frac{-1}{t^2}$

So use the two known points:

$$\text{gradient} = \frac{5-3}{-1-7} = -\frac{1}{4}$$

So $t^2 = 4$: $t = \pm 2$

so when $t = -2$, $x = -6a$
$y = -8a$

or when $t = 2$, $x = 6a$
$y = 8a$

MADAS takes this further and finds a value of a: but this assumes $(7,3)$ is on C, and it isn't. So I think the closest answer we can get is $(-6a, -8a)$ or $(6a, 8a)$

18) $\tan(\arctan 3x - \arctan 2)$

$+ \tan(\arctan 3 - \arctan 2x) = \frac{3}{8}$

Pearson doesn't give a formula for $\tan(A+B)$

but MADAS assumes we know

$$\tan(A+B) = \frac{\tan A + \tan B}{1 - \tan A \tan B}$$

$$\text{—— } A-B = \frac{-}{+}$$

So the expression is:

$$\frac{\tan(\arctan 3x) - \tan(\arctan 2)}{1 + \tan(\arctan 3x)\tan(\arctan 2)}$$

$$+ \frac{\tan(\arctan 3) - \tan(\arctan 2x)}{1 + \tan(\arctan 3)\tan(\arctan 2x)}$$

which collapses nicely to:

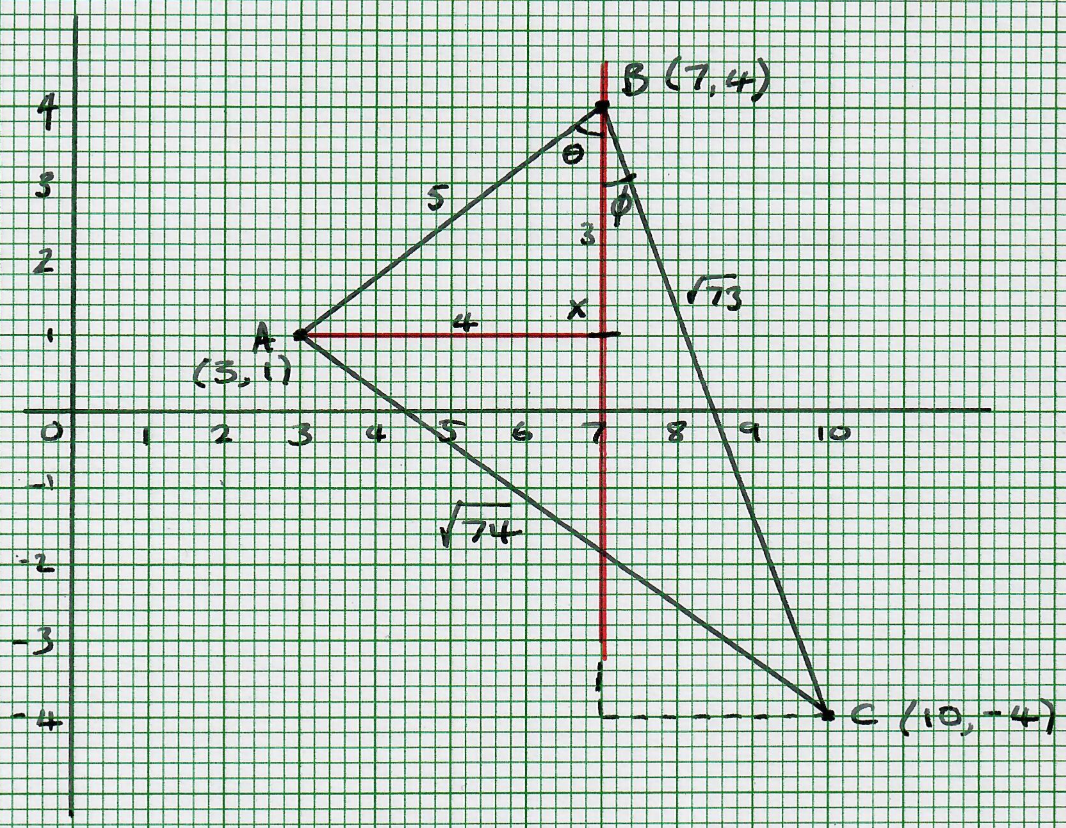
$$\frac{3x-2}{1+6x} + \frac{3-2x}{1+6x} = \frac{x+1}{6x+1} = \frac{3}{8}$$

So $8x+8 = 18x+3$

$5 = 10x$

$\underline{\underline{x = \frac{1}{2}}}$

19)



Start by calculating the side lengths:

$AB = \sqrt{3^2 + 4^2} = 5$

$AC = \sqrt{7^2 + 5^2} = \sqrt{74}$

$BC = \sqrt{3^2 + 8^2} = \sqrt{73}$

actually, not needed.

Marking the point $X(7,1)$ we can see $\triangle BAX$ is Pythagorean, so $\tan\theta = \frac{4}{3}$

We can also say $\tan\phi = \frac{3}{8}$

So if we can express $\tan(A+B)$ in terms of $\tan A$ and $\tan B$, we can find $\tan(\angle ABC)$.

19) Happily $-$ $\tan(A+B) = \dfrac{\tan A + \tan B}{1 - \tan A \tan B}$

So $\tan(\angle ABC) = \tan(\theta + \phi)$

$$= \frac{\tan\theta + \tan\phi}{1 - \tan\theta\tan\phi}$$

$$= \frac{\frac{4}{3} + \frac{3}{8}}{1 - \frac{4}{3}\times\frac{3}{8}} = \frac{\frac{32+9}{24}}{\frac{1}{2}}$$

$$= \frac{41}{12}$$ as required.