1) a) Counting from a_1 = seats in row 1,

$$a_n = a_1 + (n-1)d$$

so $a_6 = a_1 + 5d = 23$ ——— 1

$a_{15} = a_1 + 14d = 50$ ——— 2

2 − 1 : $9d = 27$ so $d = 3$

in 1 : $a_1 + 15 = 23$

so $a_1 = 8$.

b) $a_1 = a_1$
$a_2 = a_1 + d$
$\vdots$
$a_{20} = a_1 + 19d$

so the total number of seats is

$$20a_1 + \frac{19(19+1)}{2}d$$

$= 20 \times 8 + 190 \times 3$

$= 160 + 570$

$= \underline{\underline{730}}$ seats

2) a) Integration by parts uses the identity

$$\frac{d}{dx}(uv) = u\frac{dv}{dx} + v\frac{du}{dx}$$

recast as $\int u\,dv = uv - \int v\,du$

Let $u = x$ $\quad du = 1$

$dv = \cos\frac{x}{2}$ $\quad v = 2\sin\frac{1}{2}x$

Then $\int x\cos\frac{x}{2}\,dx = 2x\sin\frac{x}{2} - 2\int \sin\frac{x}{2}\,dx$

$$= 2x\sin\frac{x}{2} + 4\cos\frac{x}{2} + C$$

b) Similarly in $\int x^2 \sin\frac{x}{2}\,dx$

Let $u = x^2$ $\quad du = 2x$

$dv = \sin\frac{x}{2}$ $\quad v = -2\cos\frac{x}{2}$

So $\int x^2 \sin\frac{x}{2}\,dx = -2x^2\cos\frac{x}{2} + 2\int 2x\cos\frac{x}{2}\,dx$

$$= -2x^2\cos\frac{x}{2} + 4\left(2x\sin\frac{x}{2} + 4\cos\frac{x}{2}\right) + c'$$

$$= -2x^2\cos\frac{x}{2} + 8x\sin\frac{x}{2} + 16\cos\frac{x}{2} + c'$$

Presumably you could generalise this for $\int x^n \sin px\,dx$, if you had a mind to.

3) This one seems remarkably easy...

a) $|AD|^2 = (9-18)^2 + (40-0)^2 = 9^2 + 40^2 = 1681$

which is 41^2

So length of AD = 41m.

b) In Δ AOD use the cosine rule to find $\angle ADO$:

OA = 41
AD = 41
OD = 18

$$\cos ADO = \frac{AD^2 + OD^2 - AD^2}{2\,AD.OD}$$

$$= \frac{41^2 + 18^2 - 41^2}{2.41.18} = \frac{324}{2.41.18} = \frac{9}{41}$$

So $\angle ADO = 1.3495$

BDC is also 1.3495,

so $\angle ADB = \pi - 2 \times 1.3495$

$= 0.4426$ rad to 4dp.

c) The two Δs have base = 18 and height 40

so the area of each Δ is $\frac{1}{2} \times 18 \times 40 = 360m^2$

Area of the sector $= \frac{0.4426 \times 41^2}{2} = 372m^2$

$\left(\frac{\theta r^2}{2}\right)$

So the total area is $2 \times 360 + 372 = \underline{\underline{1092m^2}}$

4) $I = \int x^5 \sqrt{1-x^3}\, dx$

Let $t = \sqrt{1-x^3} = (1-x^3)^{\frac{1}{2}}$

Then $x^3 = 1 - t^2$

and $\dfrac{dt}{dx} = \dfrac{1}{2}(1-x^3)^{-\frac{1}{2}}(-3x^2) = \dfrac{-3x^2}{2} \dfrac{1}{t}$

Substitute into I

$$= \int \frac{2t}{-3x^2} \cdot x^5 t \; dt$$

$$= \frac{-2}{3} \int x^3 t^2 \, dt$$

$$= \frac{-2}{3} \int (1-t^2) t^2 \, dt$$

$$= \frac{-2}{3} \int (t^2 - t^4) \, dt.$$

$$= \frac{-2}{3} \left(\frac{t^3}{3} - \frac{t^5}{5} \right) + C$$

$$= \frac{-2}{45} \left(5t^3 - 3t^5 \right) + C$$

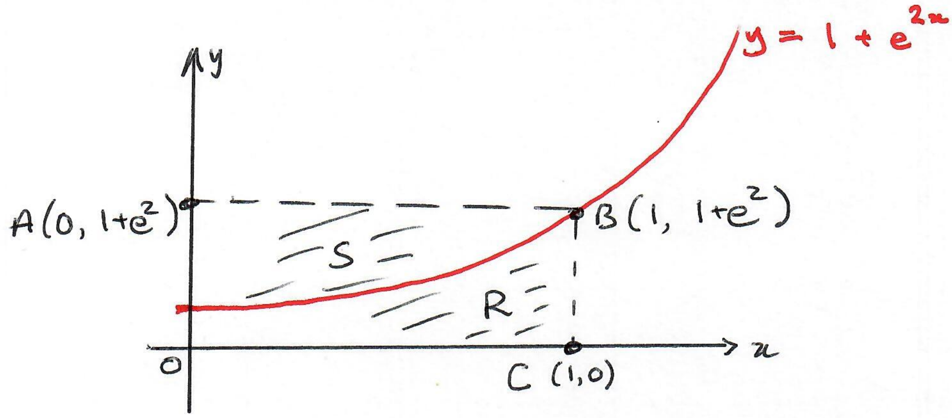
$$= \frac{-2}{45} t^3 (5 - 3t^2) + C$$

Subs. back for x:

$$= \frac{-2}{45} (1-x^3)^{\frac{3}{2}} \left(5 - 3(1-x^3) \right) + C$$

$$= \frac{-2}{45} (2 + 3x^3)(1-x^3)^{\frac{3}{2}} + C$$

5)



To show $R = S$,
It's tempting to think we should integrate across each, by x for R and by y for S.
But this would mean $\frac{1}{2}\int \ln(y-1)\,dy$ which isn't nice.

Instead, since OABC is a rectangle, we need only show R is half of OABC, which is itself $1 \times (1+e^2) = 1+e^2$.

To find R:

$$R = \int_0^1 (1+e^{2x})\,dx$$

$$= x + \frac{1}{2}e^{2x}\Big|_0^1$$

$$= 1 + \frac{1}{2}e^2 - 0 - \frac{1}{2}e^0$$

$$= 1 + \frac{1}{2}e^2 - \frac{1}{2}$$

$$= \frac{1}{2}(1+e^2)$$

So R is half of OABC, S is also half, and $R = S$ as required.

6) We know for a geometric series $a_1, a_2, \ldots$

where $a_2 = ra_1$, $a_3 = r^2 a_1$

the sum to ∞ (if it exists) is given by

$$\frac{a_1}{1-r} \text{ with } |r| < 1.$$

So here: $a_1 = 1200$

$$\frac{a_1}{1-r} = 1600$$

$$1200 = 1600(1-r)$$

$$1200 = 1600 - 1600r$$

$$1600r = 400$$

$$r = \frac{1}{4}$$

a) We also know

Sum of n terms $= S_n = \dfrac{a_1(1-r^n)}{1-r}$

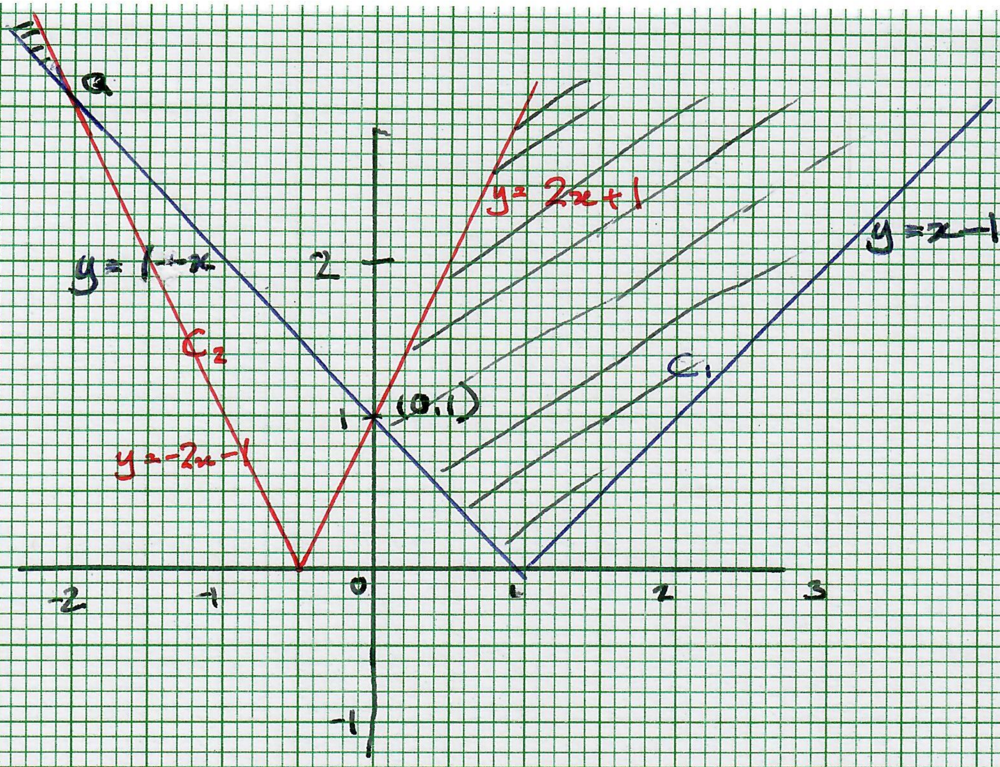
So $$S_5 = \frac{1200\left(1-\left(\frac{1}{4}\right)^5\right)}{1-\frac{1}{4}}$$

$$= \frac{1200 \times \left(1 - \frac{1}{1024}\right)}{3/4}$$

$$= 1200 \times \frac{1023}{1024} \times \frac{4}{3}$$

$$= 1598.44$$

7)



C_1: $y = |x-1|$ is two straight lines, grad 1 and −1, $y = x-1$ and $y = 1-x$, meeting at (1, 0)

C_2: $y = |2x+1|$ is two straight lines, grad 2 and −2, $y = 2x+1$ and $y = -2x-1$, meeting at $(-\frac{1}{2}, 0)$

C_1 intercepts the x-axis at (1, 0)

C_2 — — — — — at $(-\frac{1}{2}, 0)$

Both curves intercept at (0, 1)

b) For $|2x+1|$ to be $\geqslant |x-1|$, C_2 must lie above C_1.

This occurs in the shaded areas.

To find Q: $1 - x = -2x - 1$

$x = -2 \quad (y = 3)$

So the required regions are

$x \leqslant -2$ (by calculation)

$x \geqslant 0$ (by observation)

MMDAS omits the equalities - which is wrong.

8) $y = \frac{1}{4}e^{2x-3} - 4\ln\left(\frac{x}{2}\right)$ —— ①

$= \frac{1}{4}e^{2x-3} - 4\ln x + 4\ln 2$

$\frac{dy}{dx} = \frac{2}{4}e^{2x-3} - \frac{4}{x} = \frac{1}{2}e^{2x-3} - \frac{4}{x}$

when $x=2$, this is $\frac{1}{2}e^{1} - \frac{4}{2} = \frac{1}{2}(e-4)$

So the tangent at that point is

$y = \frac{1}{2}(e-4)x + c$ —— ②

To find c:

from ①, when $x=2$, $y = \frac{1}{4}e^{1} - 4\ln(1) = \frac{e}{4}$

Subs the point $(2, \frac{e}{4})$ into ②

$\frac{e}{4} = \frac{1}{2}(e-4)\times 2 + c$

$= e - 4 + c$

$c = 4 + \frac{e}{4} - e = 4 - \frac{3}{4}e$

So the tangent is

$y = \frac{1}{2}(e-4)x + 4 - \frac{3}{4}e$

Expressed in c intercept form this is

$\frac{1}{2}(e-4)x + y = 4 - \frac{3}{4}e$

And
$$\frac{\frac{1}{2}(e+4)\,x}{4-\frac{3}{4}e} + \frac{y}{4-\frac{3}{4}e} = 1$$

This intercepts the y-axis when $x = 0$, ie

at $y = 4 - \frac{3}{4}e$

. . . . x . . . . y . . .

. . . . $x = \dfrac{4-\frac{3}{4}e}{\frac{1}{2}(e+4)}$

Simplify: this is when

$$x = \frac{(16-3e)}{2(e+4)}$$

$$y = \frac{1}{4}(16-3e)$$

So the Δ OAB has area

$$\frac{1}{2} \times \frac{(16-3e)}{} \times \frac{(16-3e)}{4}$$

$$= \frac{1}{16}\,\frac{(16-3e)^2}{(4-e)}$$ as required.

9) $y^3 - y^2 = e^x$ ——— ①

Differentiate:

$$3y^2 \frac{dy}{dx} - 2y\frac{dy}{dx} = e^x$$

$$\frac{dy}{dx}(3y^2 - 2y) = e^x$$

$$= \text{(we know) } y^3 - y^2$$

So $\frac{dy}{dx} = \frac{y^3 - y^2}{3y^2 - 2y}$ = (we know) $\frac{6}{5}$ at point P.

So at P: $5(y^3 - y^2) = 18y^2 - 12y$

Assume $y \neq 0$ (since $e^x = 0$ is impossible):

$$5y^3 - 5y^2 = 18y^2 - 12y$$

$$5y^2 - 5y = 18y - 12$$

$$5y^2 - 23y + 12 = 0$$

$$(5y - 3)(y - 4) = 0$$

So $y = \frac{3}{5}$ or 4.

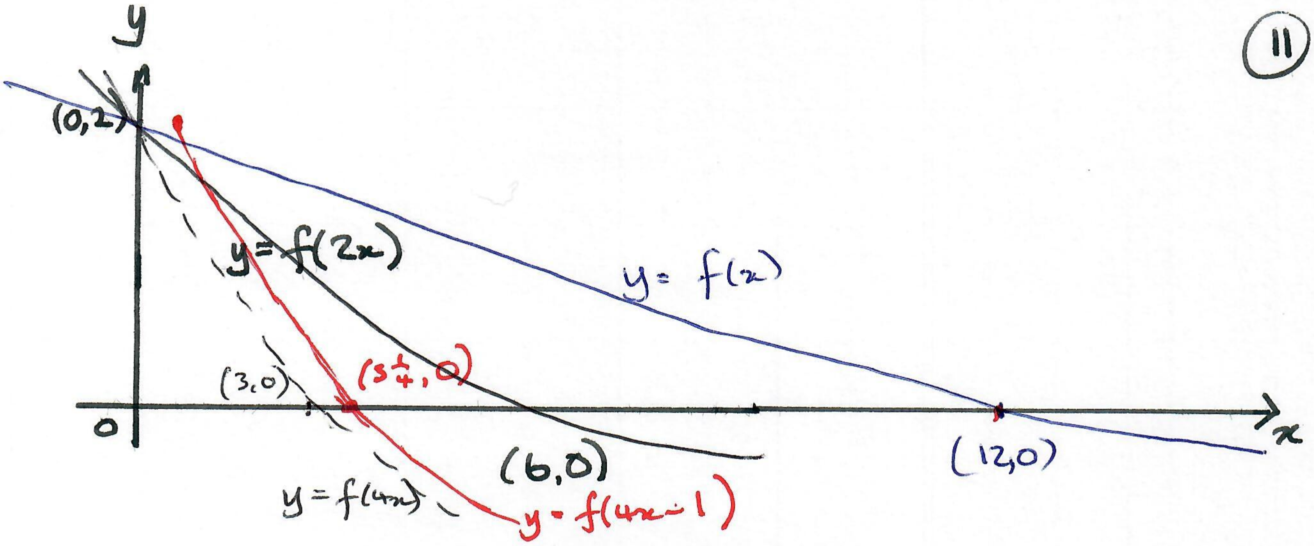
If $y = \frac{3}{5}$ then in ①: $\frac{3^3}{5^3} - \frac{3^2}{5^2} = e^x$

LHS $= \frac{27}{125} - \frac{9}{25} = \frac{27 - 45}{125}$ which is $-$ve, so discount this.

If $y = 4$ then in ①: $64 - 16 = e^x$

$\underline{x = \ln(48)}$ and this is the only solution.

10)



a) To get from $y = f(x)$ to $y = f(2x)$ means (eg) if $f(2 \times 6) = 0$ then $f(12) = 0$ – i.e. the curve has been shrunk by a factor of 2 in the x-direction.

It passes through (0,2) as before, but meets the x-axis at (12,0)

b) $f(4x)$ shrinks $f(x)$ horizontally by a factor of 4 : then going to $f(4x-1)$ moves it rightwards by $\frac{1}{4}$

This curve meets the x-axis at

$(3\frac{1}{4}, 0)$ $\quad \left(f\left(4 \times 3\tfrac{1}{4} - 1\right) = f(12) = 0 \right)$

We don't know where it meets the y-axis ... but we're not asked for that anyway.

11) a) Length of one semi-⊙ part $= \pi r$

- - - - - one straight part $= x$

So total length of track $= 2x + 2\pi r = 400$ (given) ——— ①

Area of one semi⊙ part $= \dfrac{\pi r^2}{2}$

Area of part between straight parts $= 2rx$

So total area $= \pi r^2 + 2rx = A.$ ——— ②

From ①, $2x = 400 - 2\pi r$

$x = 200 - \pi r$

So in ②: $A = \pi r^2 + 2r(200 - \pi r)$

$= \pi r^2 + 400r - 2\pi r^2$

$= 400r - \pi r^2$ as required.

b) A is stationary when $\dfrac{dA}{dr} = 0$:

$$\frac{dA}{dr} = 400 - 2\pi r = 0 \quad \text{——— ③}$$

So $2\pi r = 400$

$r = \dfrac{400}{2\pi} = 63.66$ m

c) To show this is a maximum we need $\dfrac{d^2A}{dr^2} < 0$

from ③ $\dfrac{d^2A}{dr^2} = -2\pi$ which is certainly < 0.

So this value of r gives a [illegible] for A.

d) For this value of r. $\left(r = \frac{400}{2\pi}\right)$

$$A = 400r - \pi r^2$$

$$= 400 \times \frac{400}{2\pi} - \pi\left(\frac{400}{2\pi}\right)^2$$

$$= \frac{160000}{2\pi} - \frac{160000}{4\pi}$$

$$= \frac{80000}{\pi} - \frac{40000}{\pi}$$

$$= \frac{40000}{\pi}\,m^2 \text{ as required.}$$

e) Check what x is:

$$x = 200 - \pi r$$

$$= 200 - \pi \times 63.66\ldots$$

$$= 0.0062\ldots \text{ which is almost } 0$$

Working it out accurately would show $x = 0$.
So the track would be circular.

This probably isn't suitable.

12) As always, we know a circle

$$x^2 + y^2 + 2px + 2qy + r = 0$$

has centre $(-p, -q)$ and radius $\sqrt{p^2 + q^2 - r}$

which gives centre $(-10, 1)$ and radius

$$\sqrt{10^2 + 1^2 - 52} = \sqrt{49} = 7$$

for the circle given here.

But to do it by forming into squares:

$$x^2 + y^2 + 20x - 2y + 52 = 0$$

$$(x + 10)^2 - 100 + (y-1)^2 - 1 + 52 = 0$$

$$(x + 10)^2 + (y-1)^2 = 49 = 7^2$$

So the centre is $(-10, 1)$ and the radius is 7.

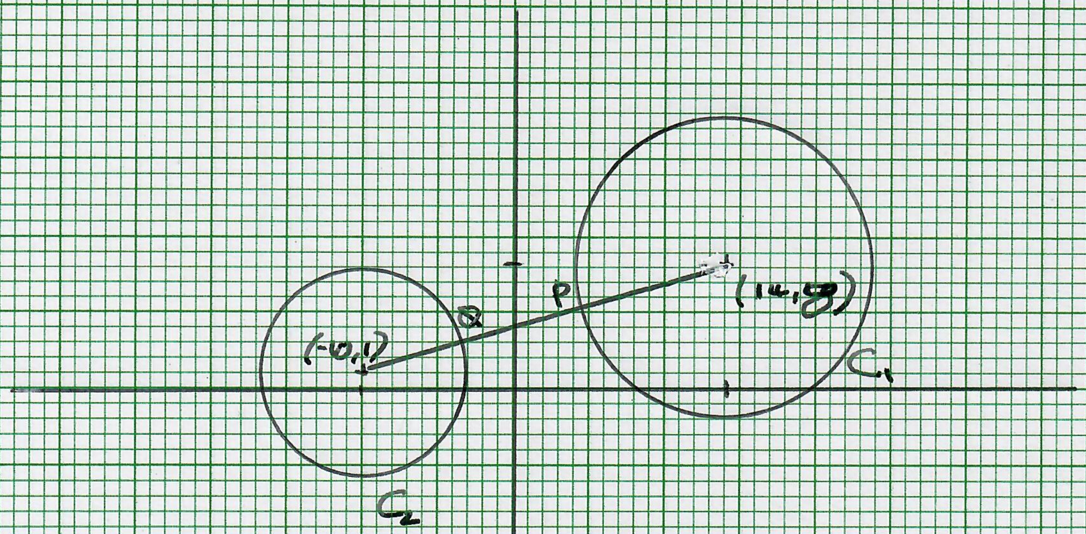
b) Circle C_2 has an equation

$$(x - 14)^2 + (y-8) = 100$$

i.e $x^2 - 28x + 196 + y^2 - 16y + 64 = 100$

or $x^2 - 28x + y^2 - 16y + 160 = 0$

(but this isn't going anywhere)



Assuming we're allowed to plot it all out and make a geometric argument...

the shortest distance PQ is on the line connecting the circles (you could prove this by considering other cases when they're <u>not</u> on the line)

<u>Distance between the centres:</u>

$$d^2 = (14+10)^2 + (8-1)^2 = 576 + 49$$
$$= 625$$

So $d = 25$.

Subtract the two radii 7 and 10 to give

$$PQ = 25 - 7 - 10 = \underline{\underline{8}}$$

13) C_1: $y = (x-7)(x^2+2x-3)$

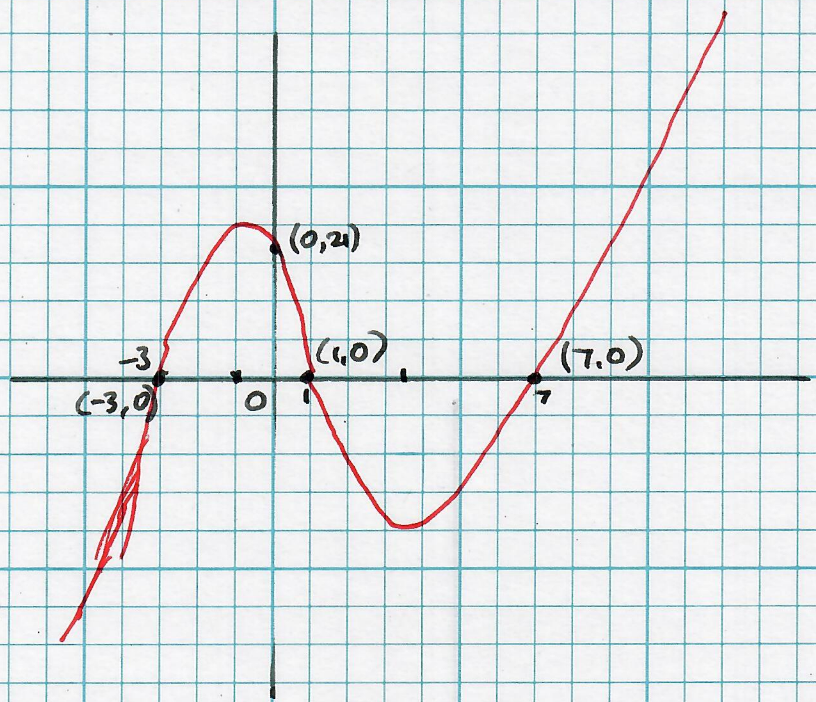
$= (x-7)(x+3)(x-1)$

So C_1 has zeroes at $x = -3, 1$ and 7.

when $x \to -\infty$, $y \to -\infty$

$x \to \infty$ $y \to \infty$

when $x = 0$, $y = -7 \times 3 \times -1 = 21$



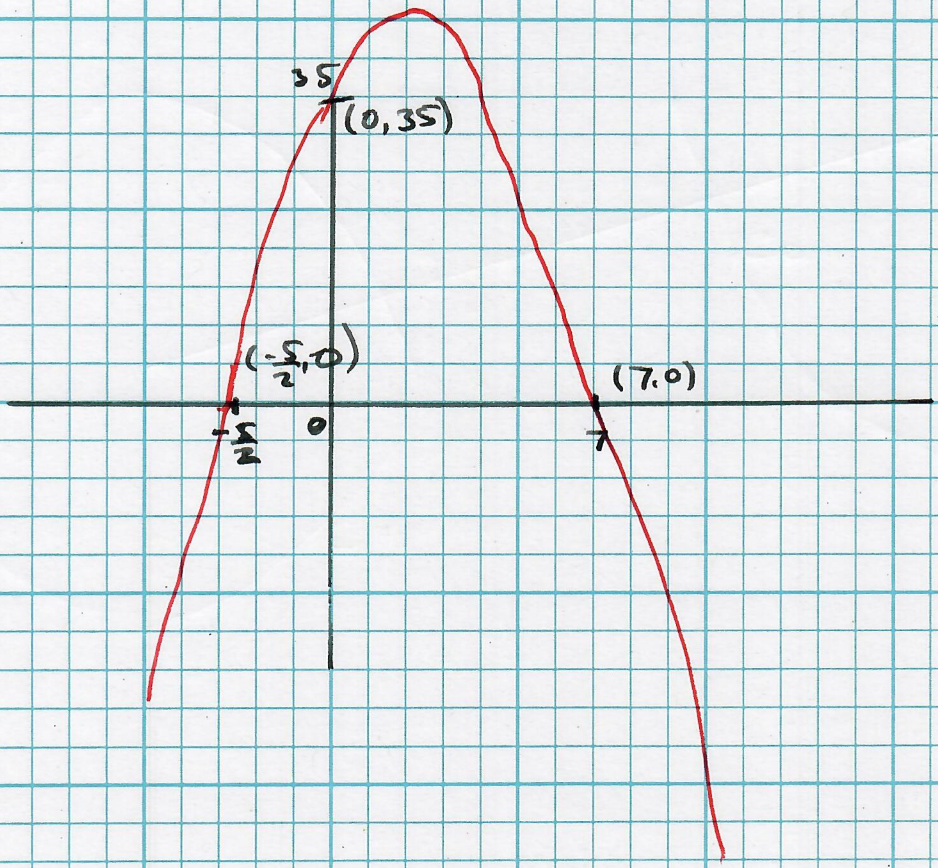
C_2: $y = (2x+5)(7-x)$

C_2 has zeroes at $x = \frac{-5}{2}$ and $x = 7$

when $x \to -\infty$ $y \to \infty$

$x \to \infty$ $y \to -\infty$

$x = 0$ $y = 5 \times 7 = 35$



b) This might be easier if we plotted both curves on the same diagram, but here goes.

$$(x-7)(x^2+2x-3) = (2x+5)(7-x)$$

one solution is clearly $x=7$ (both sides $=0$) and dividing throughout, we have

$$x^2+2x-3 = -2x-5 \quad \text{(note the } -1 \text{ factor)}$$

$$\therefore \; x^2+4x+2=0$$

$$x = \frac{-4 \pm \sqrt{16-8}}{2} = -2 \pm \sqrt{2}$$

So the 3 solutions are $x = -2-\sqrt{2}$
$-2+\sqrt{2}$
7

14) $y = 2 - 3e^x$

$y = 1 + e^{x+1} + e^{x-1}$

The obvious way to do this is to solve for x, but it's simpler just to work with e^x and go directly for y. So...

Let $t = e^x$

Then $y = 2 - 3t$ $\qquad t = \frac{2-y}{3}$ ①

$y = 1 + et + \frac{t}{e}$

or $ey = e + e^2 t + t$ ——— ②

Subs. ① into ②:

$$ey = e + \frac{e^2(2-y)}{3} + \frac{(2-y)}{3}$$

$$3ey = 3e + 2e^2 - e^2 y + 2 - y$$

$$y(3e + e^2 + 1) = 3e + 2e^2 + 2$$

So $y = \dfrac{2e^2 + 3e + 2}{e^2 + 3e + 1}$ as required.

15.) $\sin\left(\arcsin\frac{1}{4} + \arccos x\right) = 1$

So $\arcsin\frac{1}{4} + \arccos x = \frac{\pi}{2} \pm 2n\pi$

Cast this in terms of x:

$$\arccos x = \frac{\pi}{2} + 2n\pi - \arcsin\frac{1}{4}$$

$$= \left(\frac{\pi}{2} - \arcsin\frac{1}{4}\right) + 2n\pi$$

Drop the $2n\pi$ term and only use principal value

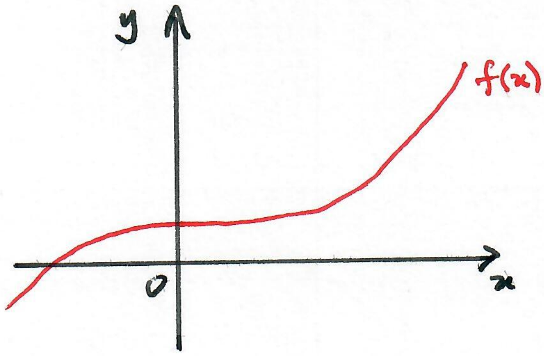
$$x = \cos\left(\frac{\pi}{2} - \arcsin\frac{1}{4}\right)$$

$$= \cos\left(\arccos\frac{1}{4}\right) \quad \left(\text{Because } \cos\left(\frac{\pi}{2}-\theta\right) = \sin\theta\right)$$

$$= \frac{1}{4}$$

(I'm not entirely convinced by this...)

16)



we're not given any sense of scaling (so we don't know where $x=2$) but we do know the function is continuous and smooth. This means it's differentiable everywhere, particularly at $x=2$

So at $x=2$ we have 4 equations

$$f(2) = 8a+2 \quad ①$$
$$f(2) = 4b-2 \quad ②$$

these are equal

$$\frac{df}{dx} = 3ax^2 = 12a \quad ③$$
$$\frac{df}{dx} = 2bx = 4b \quad ④$$

these are equal.

So $\quad 12a = 4b \qquad b = 3a$

so in ② and ① $\quad 8a+2 = 12a-2$

$$4 = 4a \qquad \underline{\underline{a=1.}}$$
$$\underline{\underline{b=3}}$$

as required.

17) $3\log_8(xy) = 4\log_2 x$ —— ①

$\log_2 y = 1 + \log_2 x = \log_2 2 + \log_2 x$

$= \log_2 2x$ —— ②

from ②: $y = 2x$ —— ③

①: $\log_8 (xy)^3 = 4\log_2 x$

and $\log_8 (xy)^3 = \dfrac{\log_2 (xy)^3}{\log_2 8}$

$= \dfrac{\log_2 (xy)^3}{3}$

$= \dfrac{3\log_2 (xy)}{3}$

$= \log_2 (xy)$

So ① gives

$\log_2 (xy) = 4\log_2 x = \log_2 x^4$

So $xy = x^4$

from ③: $2x^2 = x^4$

x can't be 0, so $x = \sqrt{2}$

$y = 2\sqrt{2}$

18) a) $$\frac{1}{t(t^2+1)} = \frac{At+B}{t^2+1} + \frac{C}{t}$$

Multiply throughout by $t(t^2+1)$:

$$1 = (At+B)t + C(t^2+1)$$

$$= At^2 + Bt + Ct^2 + C$$

$$= (A+C)t^2 + Bt + C$$

Equating coeffs of t:

$$A+C=0$$

$$B=0 \qquad \text{(looks wrong)}$$

$$C=1$$

So $A=-1$, $B=0$ and $C=1$

$$\frac{1}{t(t^2+1)} = \frac{-t}{t^2+1} + \frac{1}{t}$$

b) $$\frac{dm}{dt} = \frac{m}{t(t^2+1)}$$

So $$\int \frac{dm}{m} = \int \frac{\cancel{m}1}{t(t^2+1)}\,dt = \int \left(\frac{-t}{t^2+1} + \frac{1}{t}\right)dt$$

$$\int \frac{dm}{m} = \ln m.$$

$$\int \frac{dt}{t} = \ln t$$

for $\int \frac{-t}{t^2+1}\,dt$ $\cancel{t^2+1=u^2}$ $\cancel{2t\,dt=2u\,du}$

$=$ note $\frac{d}{dt}(t^2+1) = 2t$

So $\int \frac{2t}{1+t^2}\,dt = \ln(1+t^2)$

(check this by differentiating:

$\frac{d}{dt}(\ln(1+t^2)) = \frac{1}{1+t^2} \times 2t$ by the chain rule)

So $\int \frac{-t}{1+t^2}\,dt = -\frac{1}{2}\ln(1+t^2)$

Put all this together and include a constant c:

$$\ln m = \ln t - \frac{1}{2}\ln(1+t^2) + \ln C$$

$$= \ln \frac{Ct}{\sqrt{1+t^2}}$$

so $$m = \frac{Ct}{\sqrt{t^2+1}}$$

c) we know $10 = \frac{C.2}{\sqrt{2^2+1}} = \frac{2C}{\sqrt{5}}$

$$C = \frac{10\sqrt{5}}{2} = 5\sqrt{5}$$

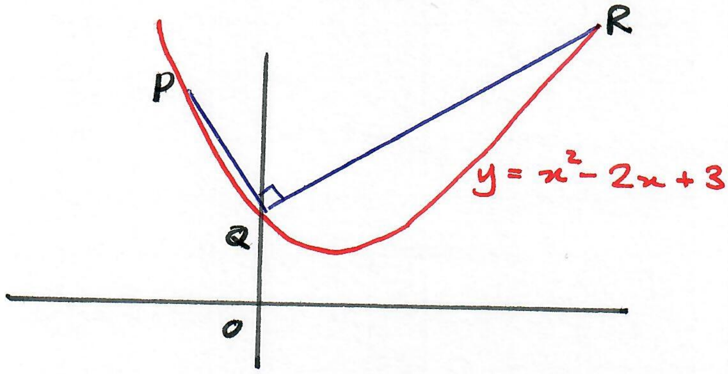
so $$m = \frac{5\sqrt{5}\,t}{\sqrt{t^2+1}}$$

when $t = 4$, $m = \frac{5\sqrt{5} \times 4}{\sqrt{16+1}} = \frac{20\sqrt{5}}{\sqrt{17}}$

$= 10.85g$

d) As $t \to \infty$, $m \to \frac{5\sqrt{5}t}{\sqrt{t^2}} = 5\sqrt{5}g$. $(= 11.18g)$

19)



Gradient of PQ $= \dfrac{3-6}{0-(-1)} = \dfrac{-3}{1} = -3.$

So gradient of QR $= \dfrac{1}{3}$
(perpendicular)

So QR is of the form

$$y = \frac{1}{3}x + c$$

and it passes through Q $(0,3)$, so

$$3 = \frac{1}{3}0 + c$$

So $c = 3$

and QR is $y = \dfrac{x}{3} + 3.$

Subs. into the equation of the parabola:

$$\frac{x}{3} + 3 = x^2 - 2x + 3$$

$$x + 9 = 3x^2 - 6x + 9$$

$$0 = 3x^2 - 7x = x(3x - 7)$$ so $x = 0$ (Q) or $x = \frac{7}{3}$ (R)

and at R, $y = \frac{1}{3} \cdot \frac{7}{3} + 3 = \frac{34}{9}$ So R is $\left(\frac{7}{3}, \frac{34}{9}\right)$

20) $f(x) = 4^{ax+b}$

$f\left(\frac{2}{3}\right) = \frac{1}{4}\sqrt[3]{4} = 4^{\frac{2a}{3}+b}$ —— ①

$f\left(\frac{3}{2}\right) = 4^{\frac{3a}{2}+b} = \frac{1}{2}\sqrt{2}$ —— ②

explore ① : $4^{\frac{1}{3}-1} = 4^{\frac{2a}{3}+b}$

$4^{-\frac{2}{3}} = 4^{\frac{2a}{3}+b}$

So $\frac{2a}{3}+b = -\frac{2}{3}$: $b = -\frac{2}{3} - \frac{2}{3}a$ —— ③

explore ② : $2^{2\left(\frac{3a}{2}+b\right)} = 2^{\frac{1}{2}-1}$

So $3a+2b = -\frac{1}{2}$

from ③ : $3a + 2\left(-\frac{2}{3} - \frac{2}{3}a\right) = -\frac{1}{2}$

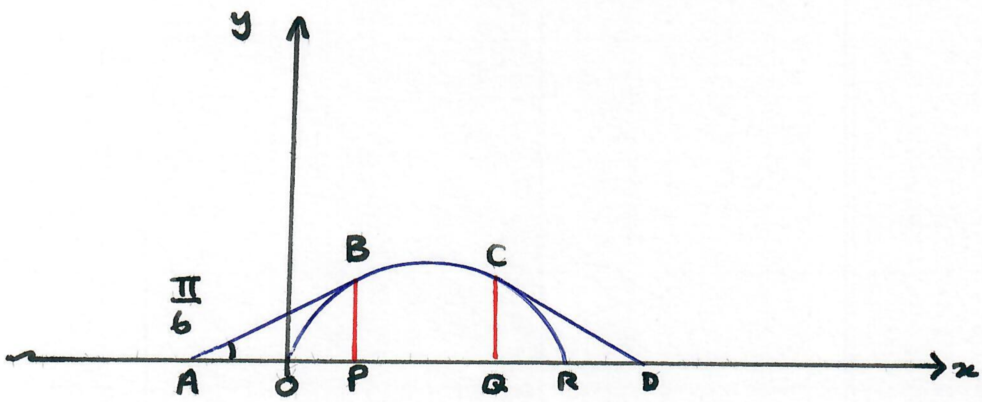
$3a - \frac{4}{3} - \frac{4}{3}a = -\frac{1}{2}$

$\frac{5}{3}a = -\frac{1}{2} + \frac{4}{3} = \frac{-3+5}{6} = \frac{2}{6} = \frac{1}{2}$

So $a = \frac{1}{2}$ $\cancel{a = \frac{5}{26} \times \frac{3}{8} = \frac{5}{16}}$

$2b = -\frac{1}{2} - 3a = -\frac{1}{2} - \frac{3}{2} = -2$ so $b = -1$

21)



For the circular part,

$$x = 6(2t - \sin 2t) = 12t - 6\sin 2t$$

$$y = 6(1 - \cos 2t) = 6 - 6\cos 2t$$

a) $$\frac{dx}{dt} = 12 - 6 \times 2\cos 2t = 12(1 - \cos 2t)$$

$$\frac{dy}{dt} = -6 \times 2 \times -\overset{\sin}{\cancel{\cos}}\, 2t = 12\sin 2t$$

Using standard expansions:

$$\cos 2t = \cos^2 t - \sin^2 t$$

$$\sin 2t = 2\sin t \cos t$$

So $$\frac{dx}{dt} = 12\left(1 - \cos^2 t + \sin^2 t\right) = 12\left(\sin^2 t + \sin^2 t\right)$$

$$= 24\sin^2 t.$$

$$\frac{dy}{dt} = 24\sin t\cos t$$

So dividing: $$\frac{dy}{dx} = \frac{dy}{dt}\cdot\frac{dt}{dx} = \frac{24\sin t\cos t}{24\sin^2 t} = \frac{\cos t}{\sin t}$$

$= \underline{\cot(t)}$ as required.

b) At O and R, $y = 0$

So $1 - \cos 2t = 0$

$\cos 2t = 1$

$2t = 0$ or 2π (or other $2n\pi$)

So at O $t = 0$: $x = 0$
at R $t = \pi$: $x = 6(2\pi - \sin 2\pi) = 12\pi$

So the distance OR is given by 12π.

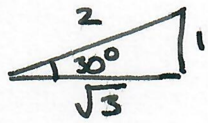
c) By symmetry maximum height occurs when $t = \frac{\pi}{2}$:

$$y = 6\left(1 - \cos \frac{2\pi}{2}\right) = 6(1-(-1))$$
$$= 12.$$

d) Gradient at B is equal to the gradient of AB. which is $\tan\left(\frac{\pi}{6}\right)$

ie $\tan(30°)$

$= \frac{1}{\sqrt{3}}$



So at B, $\frac{dy}{dx} = \cot t = \frac{1}{\sqrt{3}}$

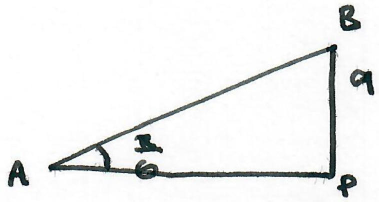
So $\tan t = \sqrt{3}$

so $t = (60°$ or$)$ $\frac{\pi}{3}$.

e) At the line BP $t = \frac{\pi}{3}$ so we know:

$$y = 6\left(1 - \cos \frac{2\pi}{3}\right) = 6\left(1 + \cos \frac{\pi}{3}\right)$$
$$= 6\left(1 + \frac{1}{2}\right)$$
$$= \underline{\underline{9}}$$

So in the Δ ABP we have



$$\frac{AP}{9} = \tan\left(\frac{\pi}{3}\right) = \sqrt{3}$$

So $\underline{\underline{AP = 9\sqrt{3}.}}$

So $AO = 9\sqrt{3} - OP$

At P, we know $t = \frac{\pi}{3}$

So $x = 6\left(\frac{2\pi}{3} - \sin\left(\frac{2\pi}{3}\right)\right) = OP = 6\left(\frac{2\pi}{3} - \frac{\sqrt{3}}{2}\right)$

$= 4\pi - 3\sqrt{3}$

So $AO = AP - OP = 9\sqrt{3} - (4\pi - 3\sqrt{3})$

$= 12\sqrt{3} - 4\pi.$

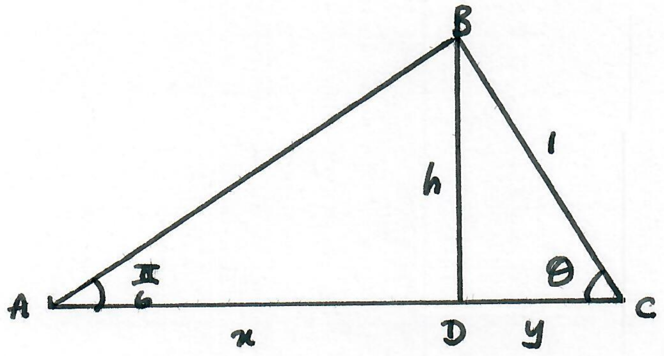
We know $OR = 12\pi$, so the distance AD

is $2(AO) + OR$

$= 24\sqrt{3} - 8\pi + 12\pi$

$= \underline{\underline{4\pi + 24\sqrt{3}}}$

22)



a) $\frac{h}{1} = \sin\theta$ so $h = \sin\theta$

$$\frac{x}{h} = \cot\left(\frac{\pi}{6}\right) = \tan\left(\frac{\pi}{3}\right) = \sqrt{3}$$

So $x = h\sqrt{3} = \sqrt{3}\sin\theta$ —— ①

Also $\frac{y}{1} = \cos\theta$ so $y = \cos\theta$ —— ②

① + ②: $x+y = \sqrt{3}\sin\theta + \cos\theta$

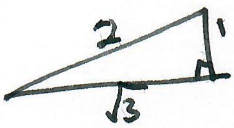
Area of $\triangle$ ABC

$$= \frac{h}{2}(x+y)$$

$$= \frac{\sin\theta}{2}\left(\sqrt{3}\sin\theta + \cos\theta\right)$$

$$= \sin\theta\left(\frac{\sqrt{3}}{2}\sin\theta + \frac{1}{2}\cos\theta\right)$$

Now $\sin\frac{\pi}{6} = \frac{1}{2}$ and $\cos\frac{\pi}{6} = \frac{\sqrt{3}}{2}$



So the area is $\sin\theta\left(\cos\frac{\pi}{6}\sin\theta + \sin\frac{\pi}{6}\cos\theta\right)$

which by standard identities

$= \underline{\underline{\sin\theta \, . \, \sin\left(\theta + \frac{\pi}{6}\right)}}$ as required.

b) $$\cos\left(\theta+\left(\theta+\frac{\pi}{6}\right)\right) = \cos\theta\cos\left(\theta+\frac{\pi}{6}\right) - \sin\theta\sin\left(\theta+\frac{\pi}{6}\right) \quad ①$$

$$\cos\left(\theta-\left(\theta+\frac{\pi}{6}\right)\right) = \cos\theta\cos\left(\theta+\frac{\pi}{6}\right) + \sin\theta\sin\left(\theta+\frac{\pi}{6}\right) \quad ②$$

On the LHS, observe

$$\cos\left(\theta+\left(\theta+\frac{\pi}{6}\right)\right) = \cos\left(2\theta+\frac{\pi}{6}\right)$$

$$\cos\left(\theta-\left(\theta+\frac{\pi}{6}\right)\right) = \cos\left(-\frac{\pi}{6}\right) = \frac{\sqrt{3}}{2}.$$

Subtract ① from ②, using these expressions on the LHS

$$\frac{\sqrt{3}}{2} - \cos\left(2\theta+\frac{\pi}{16}\right) = 2\sin\theta\sin\left(\theta+\frac{\pi}{6}\right)$$

So $$\Delta = \frac{1}{2}\left(\frac{\sqrt{3}}{2} - \cos\left(2\theta+\frac{\pi}{6}\right)\right)$$

$$= \frac{1}{4}\left(\sqrt{3} - 2\cos\left(2\theta+\frac{\pi}{6}\right)\right)$$

(which is as good as it gets)

c) We could do this by differentiating, but in fact it's simpler: the maximum is when $\cos\left(2\theta+\frac{\pi}{6}\right) = -1$, and $\Delta = \frac{1}{4}(\sqrt{3}+2)$

so $$2\theta+\frac{\pi}{6} = \pi$$

$$2\theta = \frac{5}{6}\pi$$

$$\theta = \frac{5}{12}\pi$$