1) Tackling this as a 'proof' (eg by induction) doesn't seem to work: but we can consider 3 cases and cover all the ground.

Consider

$m = 3n$ Then $m^2 = 9n^2$

equating $9n^2 = 3k+2$ would require 2 to have a factor of 3, which it doesn't.

$m = 3n+1$ $m^2 = 9n^2 + 6n + 1$

equating: $9n^2+6n+1 = 3k+2$

$9n^2+6n = 3k+1$

would again require 1 to have a factor of 3.

$m = 3n+2$ $m^2 = 9n^2 + 12n + 4$

equating: $9n^2+12n+4 = 3k+2$

$9n^2+12n = 3k-2$

would again require -2 to have a factor of 3

So for all possible integers m, it is impossible that $m^2 = 3k+2$.

2) $p = \log_6 25 \qquad q = \log_6 2$

i) $\log_6(200) = \log_6 25 + \log_6 8$

$= \log_6 25 + 3\log_6 2$

$= \underline{\underline{p + 3q}}$

ii) $\log_6(3.2) = \log_6\left(\frac{2^5}{2\times 5}\right)$

$= \log_6\left(\frac{2^4}{5}\right)$

$= \log_6(2^4) - \log_6(5)$

and $\log_6(5) = \frac{p}{2}$

So $\log_6(3.2) = \underline{\underline{4q - \frac{p}{2}}}$

iii) $\log_6(75) = \log_6(3) + \log_6(25)$

we also know $\log_6(6) = 1$,

So $\log_6(3) = \log_6(6) - \log_6(2)$

$= 1 - q$

So $\log_6(75) = \underline{\underline{1 + p - q}}$

3) $(2x+3)^2 - (4-x)^2 = 45$

A Just multiply it out:

$$4x^2 + 12x + 9 - 16 + 8x - x^2 = 45$$

$$3x^2 + 20x - 7 = 45$$

$$3x^2 + 20x - 52 = 0$$

$$(3x - 26)(x + 2) = 0$$

So $x = \frac{26}{3}$ or -2.

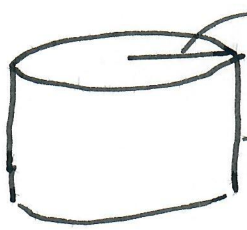
B Use 'difference of squares':

$$(2x+3+4-x)(2x+3-4+x) = 45$$

$$(x+7)(3x-1) = 45$$

but it doesn't make a difference.

4)



$r = \frac{1}{\sqrt{2}-1}$

$h = \sqrt{2}+1$

Area of the cross-section $= \pi r^2$

$$= \pi \left(\frac{1}{\sqrt{2}-1}\right)^2$$

Volume $= \pi r^2 h$

$$= \pi \left(\frac{1}{\sqrt{2}-1}\right)^2 (\sqrt{2}+1)$$

$$= \pi \frac{\sqrt{2}+1}{2+1-2\sqrt{2}}$$

$$= \pi \frac{\sqrt{2}+1}{3-2\sqrt{2}}$$

$$= \pi \frac{(\sqrt{2}+1)(3+2\sqrt{2})}{(3-2\sqrt{2})(3+2\sqrt{2})}$$

$$= \pi \frac{3\sqrt{2}+3+4+2\sqrt{2}}{9-8}$$

$$= \pi \frac{7+5\sqrt{2}}{1}$$

$= \underline{\pi(7+5\sqrt{2})}$ cm as required.

5) $f(x) = 3(2^{-x}) - 1 \quad x \in \mathbb{R}, x > 0$

$g(x) = \log_2 x \quad x \in \mathbb{R}, x \geq 1$

when $x = 0$, $f(x) = 3(2^0) - 1 = 3 \times 1 - 1 = 2$

$x = 1$, $f(x) = 3 \times \frac{1}{2} - 1 = \frac{1}{2}$

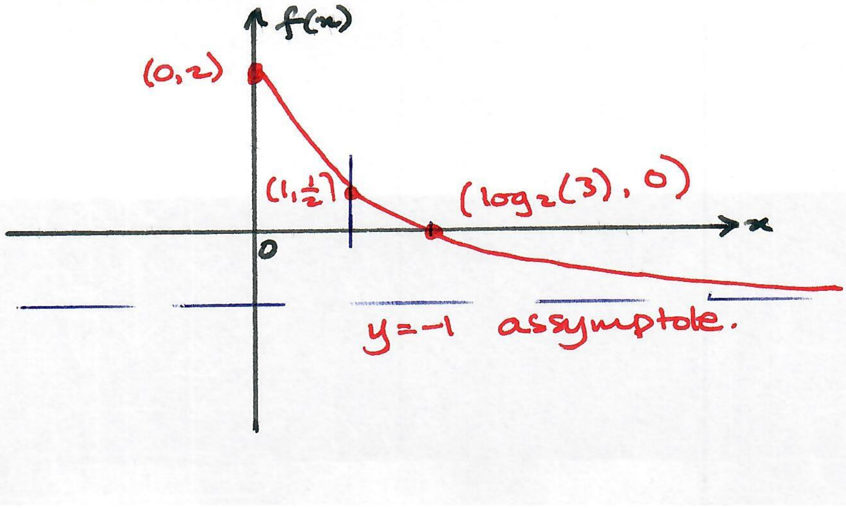
$y = 0$, $0 = 3(2^{-x}) - 1$

$2^{-x} = \frac{1}{3}$

$2^x = 3$

$x = \log_2(3)$

Also as $x \to \infty$, $f(x) \to -1$ from above.



b) The range of f (i.e. all possible values) runs from >-1 to 2: i.e. $(-1, 2]$

$$-1 < f(x) \leqslant 2$$

c) $f(g(x)) =$

$$3\left(2^{-\log_2 x}\right) - 1$$

$$= \frac{3}{2^{\log_2 x}} - 1$$

$$= \frac{3}{x} - 1$$

6) Geometric series $a_1 \; a_2 \; a_3 \; \ldots$

in general $a_i = a_1 r^{n-1}$

So $a_2 + a_4 = a_1(r + r^3) = 156$ —— ①

$a_3 + a_5 = a_1(r^2 + r^4) = 234$ —— ②

$= a_1 r(r + r^3) = 234$ —— ③

③ ÷ ①: $r = \frac{234}{156} = \frac{117}{78} = \frac{39}{26} = \frac{3}{2}$

And in ①: $\cancel{a_1\left(\frac{117}{78} + \left(\frac{117}{78}\right)^3\right) = 156}$

$$a_1\left(\frac{3}{2} + \frac{27}{8}\right) = 156$$

$$a_1\left(\frac{12 + 27}{8}\right) = 156$$

$$a_1 = \frac{156}{39} \times 8 = 32$$

7) $I = \int \frac{6x^2}{2x^{3/2}-1}\,dx$

Let $u = 2x^{3/2}-1$ (or otherwise... hah!)

Then $\frac{du}{dx} = 2.\frac{3}{2}x^{\frac{1}{2}} = 3x^{\frac{1}{2}}$

$du = 3x^{\frac{1}{2}}dx$

This can be used in the integral but we need to find an expression for $2x^{\frac{3}{2}}$.

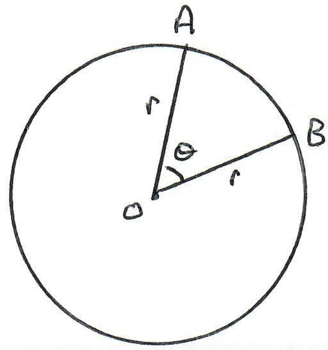
Lo ... it's $u+1$.

So $I = \int \frac{u+1}{u}\,du$

$= \int 1+\frac{1}{u}\,du$

$= u + \ln u + C$

$= 2x^{\frac{3}{2}}-1 + \ln\left(2x^{\frac{3}{2}}-1\right) + C$

8) 

Area AOB $= \frac{\theta r^2}{2} = 48$ — ①

Arc AB $= \theta r = 12$ — ②

① ÷ ② : $\frac{r}{2} = 4$ so $\underline{\underline{r = 8}}$ cm

in ② : $8\theta = 12$ so $\theta = \frac{3}{2}$ (radians)

9) Doing derivatives from first principles is hard work at the best of times, but partic. with an expression like this.

Let $f(x) = \dfrac{1}{x^2-2x}$ and h be a small increment in x.

Then $f(x+h) - f(x) =$

$$\frac{1}{(x+h)^2 - 2(x+h)} - \frac{1}{x^2-2x}$$

$$= \frac{1}{x^2+2xh+h^2-2x-2h} - \frac{1}{x^2-2x}$$

$$= \frac{(\cancel{x^2}-\cancel{2x}) - (\cancel{x^2}+2xh+h^2-\cancel{2x}-2h)}{(x^2+2xh+h^2-2x-2h)(x^2-2x)}$$

$$= \frac{2h-2xh-h^2}{(x^2+2xh+h^2-2x-2h)(x^2-2x)}$$

This looks rather full of h's, but we recall what we want is $\dfrac{f(x+h)-f(x)}{h}$ (actually $\lim\limits_{h\to 0}$)

So the expression becomes

$$\frac{2-2x-h}{(x^2+2xh+h^2-2x-2h)(x^2-2x)}$$

So $\lim_{h \to 0}$ of this is:

$$\frac{2-2x}{(x^2-2x)(x^2-2x)}$$

$$= \underline{\underline{\frac{2(1-x)}{(x^2-2x)^2}}}$$

Checking with the Quotient Rule:

$$\frac{d}{dx}\left(\frac{1}{x^2-2x}\right) = \frac{(x^2-2x).0-1(2x-2)}{(x^2-2x)^2}$$

$$= \frac{2-2x}{(x^2-2x)^2}$$

$$= \frac{2(1-x)}{(x^2-2x)^2} \quad \text{as expected.}$$

10) $\qquad \dfrac{3x^2}{y} - 5y = 2(x+8)$

To find a stationary point we look for $\dfrac{dy}{dx} = 0$

Differentiating:

$$\frac{y.6x - 3x^2y'}{y^2} - 5y' = 2$$

$$6xy - 3x^2y' - 5y'y^2 = 2y^2$$

$$y'(-3x^2 - 5y^2) = 2y^2 - 6xy$$

$$y' = -\frac{2y^2 - 6xy}{3x^2 + 5y^2}$$

The denominator is never 0 (we know $y \neq = 0$ is not on the curve)

So we can look for

$$0 = 2y^2 - 6xy$$

$$= 2y(y - 3x)$$

Again we discount $y = 0$, so $y = 3x$. ——— ①

Substitute into the original equation:

$$\frac{3x^2}{3x} - 15x = 2x + 16$$

$$x - 15x = 2x + 16$$

$$-16x = 16$$

$$x = -1$$

and from ①, $y = -3$

So the stationary point is $(-1, -3)$

MADAS SYN Paper B

11) a) $$\frac{dp}{dt} = kp\cos kt$$

Rearrange:

$$\frac{dp}{p} = k\cos kt\,dt$$

Integrate:

$$\ln p = \frac{k}{k}\sin kt + c$$

So $\ln p = \sin kt + c$

Substitute $(0, p_0)$:

$$\ln p_0 = \sin 0k + c$$

so $\underline{c = \ln p_0}$

So the solution required is:

$$\ln p = \sin kt + \ln p_0$$

$$\ln\left(\frac{p}{p_0}\right) = \ln p - \ln p_0 = \sin kt$$

$$\text{So } \frac{p}{p_0} = e^{\sin kt}$$

$$\underline{\underline{p = p_0 e^{\sin kt}}}$$

b) Assume $k=3$: $p = p_0 e^{\sin 3t}$

The population will reach p_0

when $p_0 = p_0 e^{\sin 3t}$

So $e^{\sin 3t} = 1$: $\sin 3t = 0$: $3t = \pi$: $t = \frac{\pi}{3} =$ 1.047 days = 1 day, 1 hour, 8 min.

12) a) $x = \dfrac{t+3}{t+1}$ $\quad y = \dfrac{2}{t+2}$ $\quad t \in \mathbb{R},\ t \neq -1,\ t \neq -2$

$$\frac{dx}{dt} = \frac{(t+1).1 - (t+3).1}{(t+1)^2} = \frac{t+1-t-3}{(t+1)^2}$$

$$= \frac{-2}{(t+1)^2}$$

$$\frac{dy}{dt} = \frac{(t+2).0 - 2.1}{(t+2)^2} = \frac{-2}{(t+2)^2}$$

So $\dfrac{dy}{dx} = \dfrac{-2}{(t+2)^2} \times \dfrac{(t+1)^2}{-2} = \left(\dfrac{t+1}{t+2}\right)^2$

as required.

b) You could do this by substituting the given expressions for x and y into the proposed equation ... but actually it's easy just to solve them:

from 1:
$$x(t+1) = t+3$$
$$xt - t = 3 - x$$
$$t(x-1) = 3 - x$$
$$t = \frac{3-x}{x-1}$$

so in 2:
$$y = \frac{2}{t+2} = \frac{2}{\frac{3-x}{x+1} + 2}$$

$$= \frac{2 \quad (x+1)}{3 - x + 2x - 2} = \frac{2(x-1)}{x+1}$$

as required.

13) $f(x) = 3\ln 2x \qquad x \in \mathbb{R} \quad x > 0$

$g(x) = 2x^2 + 1 \qquad x \in \mathbb{R}$

Using the chain rule (and noting $f(x) = 3(\ln x + \ln 2)$:

$$\frac{d}{dx} gf(x) = \frac{dg}{dx}(f(x)) \cdot \frac{df}{dx}$$

$$= 4(f(x)) \cdot \frac{3}{x}$$

$$= \frac{12}{x} \cdot 3(\ln x + \ln 2)$$

$$= \frac{36}{x}(\ln x + \ln 2)$$

So when $x = e$, this is

$\frac{36}{e}(1 + \ln 2)$ as required.

14) $I = \int_k^{2k} \frac{3x-5}{x(x-1)}\,dx$

Separate into fractions:

$$\frac{3x-5}{x(x-1)} = \frac{A}{x} + \frac{B}{x-1} = \frac{A(x-1)+Bx}{x(x-1)}$$

So $\left.\begin{aligned} A + B &= 3 \\ -A &= -5 \end{aligned}\right\}$ $A = 5$, $B = -2$ and the expression is

$$\frac{5}{x} - \frac{2}{x-1}.$$

So $I = \int_{k}^{2k} \left(\frac{5}{x} - \frac{2}{x-1} \right) dx$

$= \left[5\ln x - 2\ln(x-1) \right]_{k}^{2k}$

$= 5\ln 2k - 2\ln(2k-1) - 5\ln k + 2\ln(k-1)$

$= \ln \frac{(2k)^5}{(2k-1)^2} \frac{(k-1)^2}{k^5} = \ln 72$ (given)

So $\frac{(2k)^5}{k^5} \cdot \frac{(k-1)^2}{(2k-1)^2} = 72$

$32 \frac{(k-1)^2}{(2k-1)^2} = 72$

$4(k-1)^2 = 9(2k-1)^2$

$4k^2 - 8k + 4 = 36k^2 - 36k + 9$

$0 = 32k^2 - 28k + 5$

$= (8k - 5)(4k - 1)$

So $k = \frac{8}{5}$ or $\frac{1}{4}$

and because we are given $0 < k < 1$

this means $k = \frac{1}{4}$

15) A $(2,-1,4)$

B $(0,-5,10)$

C $(3,1,3)$

D $(6,7,-8)$

a) We're really looking for something like

$$C = \lambda A + (1-\lambda) B$$

but perhaps we just have to slog it all out:

$$\vec{AB} = (-2,-4,6) = -2(1,2,-3)$$

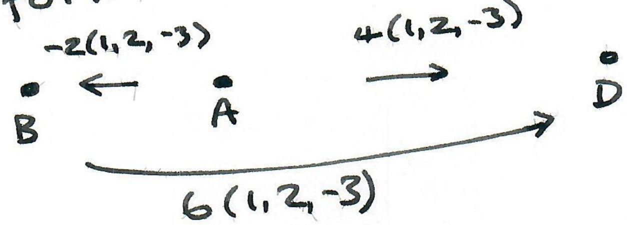
$$\vec{AC} = (1,2,-1)$$

$$\vec{AD} = (4,8,-12) = 4(1,2,-3)$$

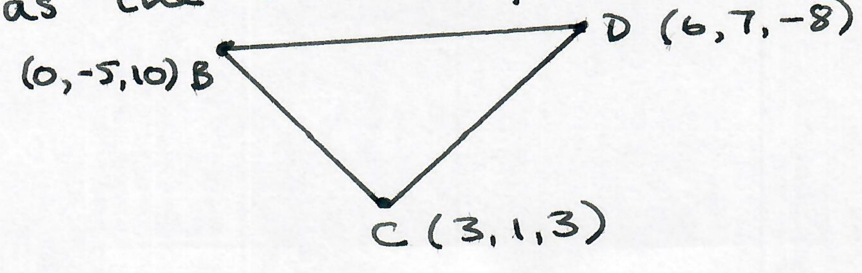
And

$$\vec{BD} = (6,12,-18) = 6(1,2,-3)$$

This tells us that A, B and D have a linear relationship and are collinear in a form:



b) It seems obvious to take B, D and C as the vertices of the Δ, so:



Lengths of the sides:

$$BD^2 = 6^2 + 12^2 + 18^2 = 36 + 144 + 324 = 504$$

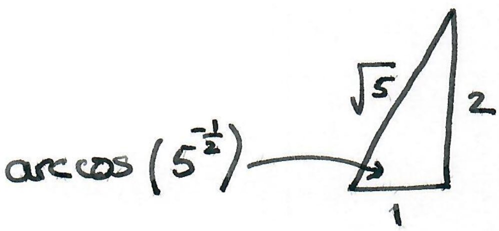
$$BC^2 = 3^2 + 6^2 + 7^2 = 9 + 36 + 49 = 94$$

$$DC^2 = 3^2 + 6^2 + 11^2 = 9 + 36 + 121 = 166$$

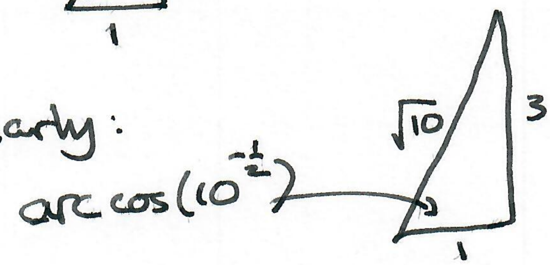
So BC is the shortest side and has length $\underline{\underline{\sqrt{94}}}$

16) This one looks a stinker but it drops out quickly once you get the right ideas.

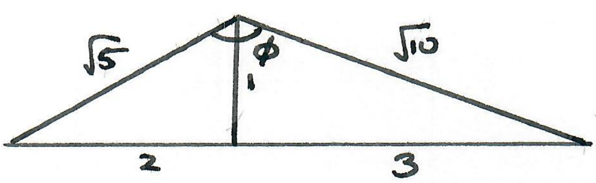
For $\cos^{-1}\left(\frac{1}{\sqrt{5}}\right)$ consider a right Δ with sides $1, 2, \sqrt{5}$:



And similarly:



Put these two Δs together:



We want to show the apex $\angle\phi$ is $\frac{3\pi}{4}$

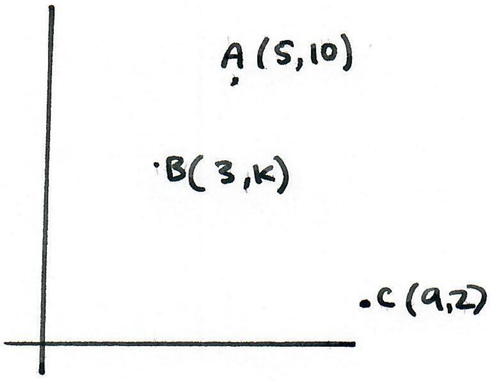
using the cosine formula:

$$\cos\phi = \frac{\sqrt{5}^2 + \sqrt{10}^2 - 5^2}{2\sqrt{5}\sqrt{10}} = \frac{5 + 10 - 25}{10\sqrt{2}} = \frac{-10}{10\sqrt{2}}$$

$$= \frac{-1}{\sqrt{2}} \text{ which means } \phi = \underline{\frac{3\pi}{4}}.$$

as required.

17)



You could use Pythagoras on ABC but in fact it's neater just to work with the gradients : namely grad AB = $\frac{-1}{\text{grd}BC}$

So $\frac{5-3}{10-k} = -\frac{2-k}{9-3}$

$\frac{2}{10-k} = -\frac{2-k}{6}$

$12 = -(2-k)(10-k)$

$\underline{\underline{(10-k)(2-k)+12=0}}$ as required.

Doing this the Pythagoras way would lead more directly to the quadratic

$(k-4)(k-8)=0$

and hence $k=4$ or 8.

It's not clear how to pick the right value for k, but we'll start with 4:

$|AB|^2 = (5-3)^2 + (10-6)^2 = 2^2+4^2 = 20$

$|BC|^2 = (9-3)^2 + (2-4)^2 = 4^2+2^2 = 20$

So we made the right choice and the square's area is 20.

c) $\overrightarrow{BA} = (5,10) - (3,4) = (2,6)$

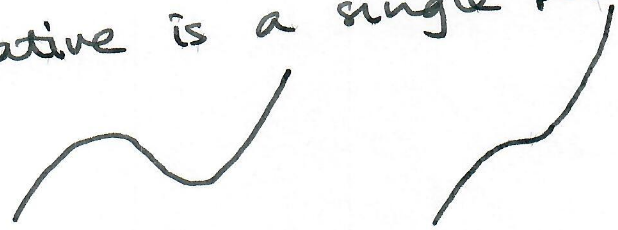
so $\overrightarrow{CD} = (9,2) + (2,6) = (11,8)$

i.e. D is point $(11,8)$

18) $y = ax^3 + bx^2 + cx + d$

has one local max and one local min.

(we note most cubics are like this ... the alternative is a single point of inflexion)



This implies the two stationary points are separate - i.e. there is not a double root.

$\frac{dy}{dx} = 3ax^2 + 2bx + c$

and this must be 0 in two diffent places.

Using the quadratic formula, this means the discriminant is >0. ie $(2b)^2 - 4(3a)c > 0$

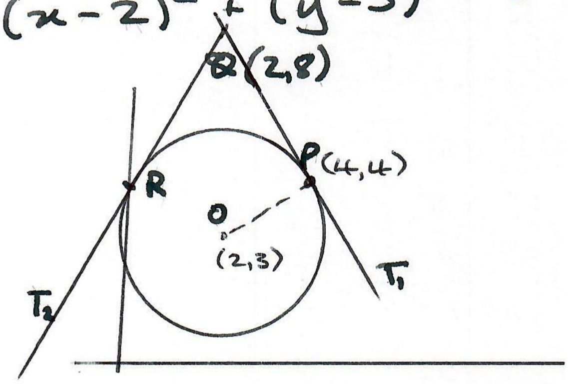
So $4b^2 - 12ac > 0$: $\underline{\underline{b^2 > 3ac}}$ as required.

19) $x^2+y^2-4x-6y+8=0$

This is a ⊙ centre $(2,3)$, radius $\sqrt{2^2+3^2-8} = \sqrt{5}$.

It can be written as

$(x-2)^2+(y-3)^2 = 5$



Gradient of OP $= \dfrac{4-3}{4-2} = \dfrac{1}{2}$

So gradient of $T_1 = -2$

T_1 is $y = -2x + c$

Subs. for P: $4 = -2\times 4 + c = -8 + c$

So $c = 12$ and T_1 is $\underline{\underline{y = -2x + 12}}$

b) The point Q is directly above the circle centre,

So OQ is an axis of symmetry.

Reflecting OP in OQ we can therefore say

R is $\underline{(0,4)}$.

By a Similar argument, gradient $T_2 = 2$

So T_2 is $y = 2x + c$

Subs for $(0,4)$: $4 = c$

So T_2 is $\underline{\underline{y = 2x + 4}}$

20) a) I'd have said this <u>is</u> one of the trig. identities but here goes. $\sin(A \pm B)$ is given by:

$$\sin(A+B) = \sin A\cos B + \cancel{\cos A \sin B}$$
$$\sin(A-B) = \sin A\cos B - \cancel{\cos A \sin B}$$

Adding: $\sin(A+B) + \sin(A-B) = 2\sin A\cos B$ —— ①

Let $P = A+B$, $Q = A-B$. Then $A = \dfrac{P+Q}{2}$, $B = \dfrac{P-Q}{2}$

Subs. into ①:

$$\sin P + \sin Q = 2\sin\left(\frac{P+Q}{2}\right)\cos\left(\frac{P-Q}{2}\right)$$

as required.

b) Applying this here:

$$\sin 7x + \sin x = 2\sin 4x\cos 3x = 0$$

So $\sin 4x = 0$: $4x = 0, \pi, 2\pi, 3\pi \ldots$

$$\underline{\underline{x = 0, \frac{\pi}{4}, \frac{\pi}{2}, \frac{3\pi}{4}}}$$

or $\cos 3x = 0$: $3x = \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}$

$$\underline{\underline{x = \frac{\pi}{6}, \frac{\pi}{2}, \frac{5\pi}{6}}}$$

21) a) Expansion is:

$$(1+bx)^n = 1 + \binom{n}{1}bx + \binom{n}{2}b^2x^2 + \binom{n}{3}(bx)^3 + \dots$$

$$= 1 + nbx + \frac{n(n-1)}{2}b^2x^2 + \dots$$

we know $= 1 - 6x + 27x^2 + \dots$

So $nb = -6$: $n = \frac{-6}{b}$, $b = \frac{-6}{n}$

and $\frac{n(n-1)}{2}b^2 = 27$

so $\frac{n(n-1)\,6^2}{2n^2} = 27$

$$36n^2 - 36n = 54n^2$$

$$18n^2 + 36n = 0$$

$$n(n+2) = 0$$

So $\underline{\underline{n \text{ is } -2}}$ and b is $\frac{-6}{-2} = \underline{\underline{3}}$ as required.

b) Coeff of $x^3 = \binom{n}{3}b^3$

$$= \frac{-2 \times -3 \times -4}{6} \times 3^3 = -4 \times 27 = \underline{\underline{-108}}$$

c) Valid for $|bx| < 1$ i.e $|3x| < 1$

i.e. $\underline{\underline{-\frac{1}{3} < x < \frac{1}{3}}}$

22) Everything here seems to say 'recast it to integrate wrt y, not x.

The bounds for y are 1 (upper)

and when $x=0$, $y=\sqrt[3]{-27} = -3$ (lower)

And the curve is

$$y = \sqrt[3]{8x-27}$$

$$y^3 = 8x - 27$$

$$8x = y^3 + 27$$

$$x = \frac{1}{8}(y^3+27)$$

So the area is given by

$$\int_{-3}^{1} \frac{1}{8}(y^3+27)\,dy$$

$$= \frac{1}{8}\left[\frac{y^4}{4} + 27y\right]_{-3}^{1}$$

$$= \frac{1}{8}\left[\frac{1}{4} + 27 - \frac{81}{4} + 81\right]$$

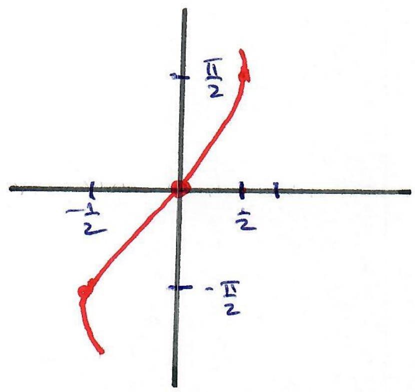
$$= \frac{1}{8}\left[27 - \frac{80}{4} + 81\right]$$

$$= \frac{1}{8}(27 - 20 + 81)$$

$$= \frac{1}{8}\,88$$

$= 11\text{m}^2$ as required.

23) $y = \arcsin 2x$



Let $2x = w$

then $\frac{dw}{dx} = 2$

Tempting again to turn this on its side, but we have a given formula that says

$$\frac{dy}{dx} = \frac{dy}{dw}\frac{dw}{dx} = \cancel{\frac{1}{\sqrt{1-\sin^2}}}$$

$$= \frac{1}{\sqrt{1-(2x)^2}} \cdot 2$$

$$= \frac{2}{\sqrt{1-4x^2}}$$ as required.

(This doesn't use a trig identity, though)

The 'right' way to do this is:

$$\sin y = 2x$$

Differentiate:

$$\cos y \frac{dy}{dx} = 2$$

$$\frac{dy}{dx} = \frac{2}{\cos y}$$

$$= \frac{2}{\sqrt{1-\sin^2 y}}$$

$$= \frac{2}{\sqrt{1-4x^2}}$$ as required.

24) $e^2 - e^{3x} - 1 = \left(\dfrac{e^{x+1}}{e^{-2x}}\right)^2$

$$= \frac{e^{2x+2}}{e^{-4x}} = e^{2+6x}$$

Let $e^{3x} = p$ then

$$e^2 - p - 1 = e^2 p^2$$

$$e^2p^2 + p - e^2 + 1 = 0$$

$$e^2p^2 + p + (1-e^2) = 0$$

$$p = \frac{-1 \pm \sqrt{1^2 - 4e^2(1-e^2)}}{2e^2}$$

$$= \frac{-1 \pm \sqrt{1 - 4e^2 + 4e^4}}{2e^2}$$

$$= \frac{-1 \pm (1-2e^2)}{2e^2}$$

$$= \frac{-2e^2}{2e^2} \quad \text{or} \quad \frac{-2+2e^2}{2e^2}$$

$$= 1 \quad \text{or} \quad \frac{e^2-1}{e^2} = 1 - \frac{1}{e^2}$$

$e^{3x} = 1 : x = 0$

$$e^{3x} = 1 - \frac{1}{e^2}$$

$$3x = \ln(1-e^{-2})$$

$$x = \frac{1}{3}\ln\left(1 - \frac{1}{e^2}\right)$$