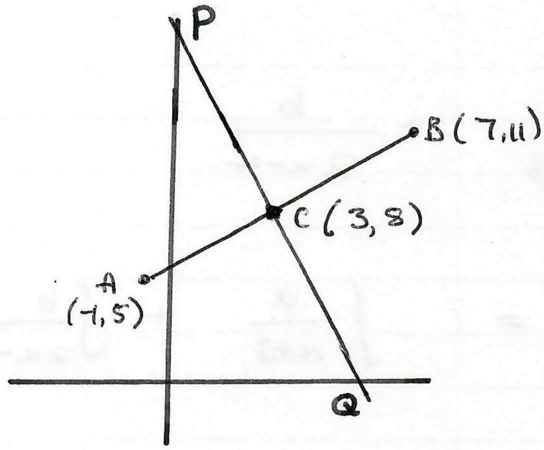
MADAS SYN Paper A

1) a)



Midpoint of AB is $\left(\frac{-1+7}{2}, \frac{5+11}{2}\right)$

$= \left(\frac{6}{2}, \frac{16}{2}\right) = (3,8)$. (Point C).

Gradient of AB $= \frac{11-5}{7-(-1)} = \frac{6}{8} = \frac{3}{4}$.

So the perp. bisector has gradient $-\frac{4}{3}$ and passes through C (3, 8).

General form of this line is $y = mx + c$ where $m = -\frac{4}{3}$, so $y = -\frac{4}{3}x + c$. (c = a constant)

Subs. for the point C:

$8 = -\frac{4}{3}.3 + c = -4 + c$

So $c = 12$

the perp. bisector is $y = -\frac{4}{3}x + 12$

i.e. $3y = -4x + 36$

or $4x + 3y = 36$ as required.

b) Rewrite the eq^n as

$$\frac{4}{36}x + \frac{3}{36}y = 1$$

or $$\frac{x}{9} + \frac{y}{12} = 1$$

Then when $x = 0,\ y = 12$ (point P)

$y = 0,\ x = 9$ (point Q).

So $|OP| = 12,\ |OQ| = 9$ and $OP \perp OQ$,

So $\triangle OPQ = \frac{12 \times 9}{2} = \underline{\underline{54}}$

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2) $f(x) = x^3 + x^2 - x + k \qquad x \in \mathbb{R}$

If $(x-k)$ is a factor then $f(k) = 0$, so:

$$k^3 + k^2 - k + k = 0$$

$$k^2(k+1) = 0$$

So $\underline{k = 0 \text{ or } k = -1.}$

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3) a) i) $\sqrt{98} + \sqrt{2} = \sqrt{2}\sqrt{49} + \sqrt{2} = 7\sqrt{2} + \sqrt{2} = \underline{\underline{8\sqrt{2}}}$

ii) $(\sqrt{2}+3)(2-3\sqrt{2}) = 2\sqrt{2} + 6 - 3\sqrt{2}\sqrt{2} - 9\sqrt{2}$

$= 2\sqrt{2} + 6 - 6 - 9\sqrt{2} = \underline{\underline{-7\sqrt{2}}}$

b) $\frac{27^t}{3^{t-1}} = \frac{3^{3t}}{3^{t-1}} = \frac{3^{2t+1}}{3^0} = 3^{2t+1} = 3\sqrt{3}$

So $3^{2t} = \sqrt{3} = 3^{\frac{1}{2}}$: $\quad 2t = \frac{1}{2}$ so $\underline{\underline{t = \frac{1}{4}}}$

4) a) This one's been made to look more difficult that it really is...

b) ...but that said, modulus and inequality problems are always a bit tricky.

3 ways to solve it:

$$12 - 2|2x-3| \geq 7$$

Do what you can with normal equations, being careful:

$$-2|2x-3| \geq -5$$

$$2|2x-3| \leq 5$$

$$|2x-3| \leq \frac{5}{2}$$

A Consider 2 cases where the 'modulused' expression is +ve or −ve:

+ve: $0 \leq 2x-3 \leq \frac{5}{2}$

$$3 \leq 2x \leq \frac{11}{2}$$

$$\frac{3}{2} \leq x \leq \frac{11}{4}$$

−ve: $0 \leq 3-2x \leq \frac{5}{2}$

$$-3 \leq -2x \leq -\frac{1}{2}$$

$$\frac{1}{2} \leq 2x \leq 3$$

$$\frac{1}{4} \leq x \leq \frac{3}{2}$$

Both conditions work, so we conclude solutions are given by $\frac{1}{4} \leq x \leq \frac{11}{4}$

(B) An alternative is to square the modulus term:

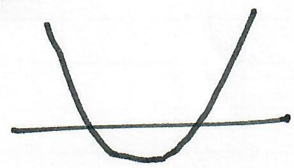
$$|2x-3| \leqslant \frac{5}{2}$$

$$(2x-3)^2 \leqslant \frac{25}{4}$$

$$4x^2-12x+9 \leqslant \frac{25}{4}$$

$$16x^2-48x+36 \leqslant 25$$

$$16x^2-48x+11 \leqslant 0$$ — A quadratic, with graph:



This factorises!

$$(4x-11)(4x-1)=0$$

So $x = \frac{1}{4}$ or $\frac{11}{4}$

(C) And with separation into squares, we get

$$(4x-6)(4x-6)-36+11=0$$

$$(4x-6)^2-25=0$$

$$(4x-6)^2-5^2=0$$

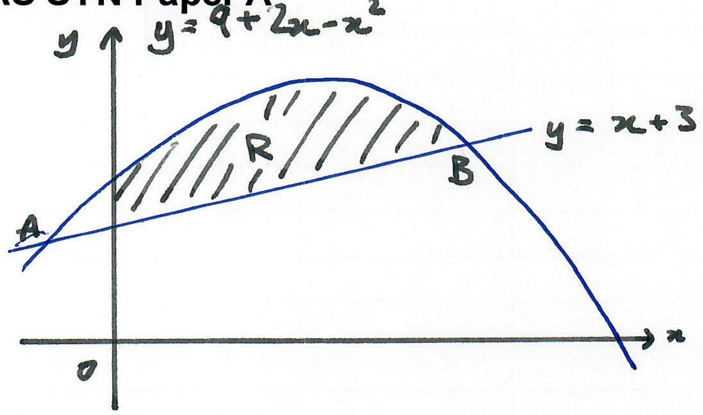
This shows the curve is centred on

$x = \frac{3}{2}$ $(4x-6=0)$ and its 'span'

between roots is $\pm\frac{5}{4}$ —

hence $\frac{1}{4}$ to $\frac{11}{4}$.

5)



Start by finding the point(s) of intersection (A and) B. At these:

$$9 + 2x - x^2 = x + 3$$

$$0 = x^2 - x - 6$$

$$\text{So } (x-3)(x+2) = 0$$

$$x = -2 \quad (A)$$

$$x = 3 \quad (B).$$

R is given by $\displaystyle\int_0^3 (9 + 2x - x^2) - (x+3)\,dx$

$$= \int_0^3 -x^2 + x + 6 \, dx$$

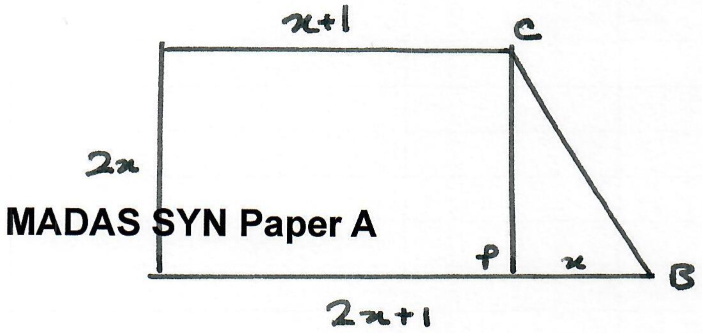
$$= \left[-\frac{x^3}{3} + \frac{x^2}{2} + 6x \right]_0^3$$

$$= -\frac{27}{3} + \frac{9}{2} + 18 - 0$$

$$= -9 + 4.5 + 18$$

$$= 13.5 \text{ as required.}$$

6)



$$\text{Area} = 2x\left(\frac{(x+1)+(2x+1)}{2}\right) = x(3x+2) = 3x^2+2x = 16$$

So $3x^2+2x-16 = 0$

$(3x+8)(x-2) = 0$

So $x = 2$.

So $\triangle$ CPB has sides 2 and 4 at right angles:

$$CB = \sqrt{2^2+4^2} = \sqrt{20} = \underline{\underline{2\sqrt{5}}}$$

7) $f(x) = \left(1+\frac{1}{8}\right)^{\frac{1}{2}}$

Binomial expansion:

$$\approx 1 + \binom{\frac{1}{2}}{1}\left(\frac{x}{8}\right)^1 + \binom{\frac{1}{2}}{2}\left(\frac{x}{8}\right)^2 + \cdots$$

$$= 1 + \frac{x}{16} - \frac{x^2}{512} + \cdots$$

$$\frac{\frac{1}{2}\times-\frac{1}{2}}{2}\cdot\frac{x^2}{64}$$

Substituting $x = 1$:

$$\sqrt{1+\frac{1}{8}} = \sqrt{\frac{9}{8}} = \frac{3}{2}\sqrt{\frac{1}{2}}$$

$$\approx 1+\frac{1}{16}-\frac{1}{512} = \frac{512+32-1}{512} = \frac{543}{512}$$

So $\sqrt{\frac{1}{2}} \approx \frac{543}{512}\times\frac{2}{3} = \frac{181}{256}$

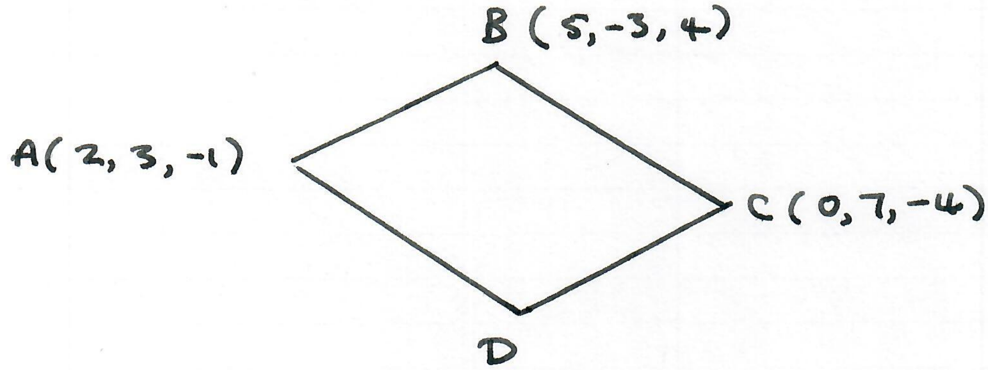
So $\sqrt{2} \approx \frac{256}{181}$

Also, divide by 2:

$\frac{1}{\sqrt{2}} \approx \frac{128}{181}$ so $\sqrt{2} \approx \frac{181}{128}$

(gives 1.41406 as opposed to 1.41421)

8) Drawing this as a picture seems difficult given that it's in 3D – but knowing that ABCD is a $\parallel$gram we can draw a 'schematic' in its plane:



a) In vector terms, we need to add $\vec{BC}$ onto $\vec{OA}$:

$$\vec{BC} = (0, 7, -4) - (5, -3, 4) = (-5, +10, -8)$$

$$\text{So } \vec{OD} = (2, 3, -1) + (-5, +10, -8)$$

$$= (-3, 13, -9)$$

Position vector of D is $-3\underline{i} + 13\underline{j} - 9\underline{k}$.

b) Suspect we'll need much more than just the distance AC, but here goes:

$$|AC| = \sqrt{(0-2)^2 + (7-3)^2 + (-4+1)^2} = \sqrt{4+16+9} = \sqrt{29}$$

$$|AB| = \sqrt{(2-5)^2 + (3+3)^2 + (-1-4)^2} = \sqrt{9+36+25} = \sqrt{70}$$

$$|BC| = \sqrt{(5-0)^2 + (-3-7)^2 + (4+4)^2} = \sqrt{25+100+64} = \sqrt{189}$$

Now we use the cosine rule:

$$\cos ABC = \frac{70 + 189 - 29}{2\sqrt{70}\sqrt{189}} = 0.9998$$

This gives $ABC = 1.11°$, which makes ABCD an extremely thin parallelogram.

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9) $$\frac{x^2+3}{x-1} = Ax + B + \frac{C}{x-1}$$

This is long division of polynomials, though we can also do it by inspection:

$$x^2+3 = x(x-1) + x + 3$$

$$= x(x-1) + (x-1) + 4$$

So $$\frac{x^2+3}{x-1} = \frac{x(x-1)}{x-1} + \frac{x-1}{x-1} + \frac{4}{x-1}$$

$$= x + 1 + \frac{4}{x-1}$$

So $A = 1$, $B = 1$, $C = 4$.

10) $$\frac{\cos 2x}{1+\cos 2x} = 1 - 2\tan x$$

$$\text{LHS} = \frac{\cos^2 x - \sin^2 x}{1 + \cos^2 x - \sin^2 x} = \frac{\cos^2 x - \sin^2 x}{2\cos^2 x}$$

$$= \frac{1}{2} - \frac{1}{2}\tan^2 x = 1 - 2\tan x \quad (\text{RHS})$$

So $$-\frac{1}{2}\tan^2 x = \frac{1}{2} - 2\tan x$$

$$-\tan^2 x = 1 - 4\tan x$$

$$\tan^2 x - 4\tan x + 1 = 0$$

$$\tan x = \frac{4 \pm \sqrt{16-4}}{2} = 2 \pm \sqrt{3}$$

$$= 2 + \sqrt{3} \quad \text{or} \quad 2 - \sqrt{3}.$$

Neither of these gives a neat answer in terms of π,

but apparently $\arctan(2 \pm \sqrt{3}) = \frac{5\pi}{12}$ or $\frac{\pi}{12}$

$$\sin(15^\circ) = \sin(60^\circ) - 45^\circ) = \sin 60 \cos 45 - \cos 60 \sin 45$$

$$= \frac{\sqrt{3}}{2} \cdot \frac{1}{\sqrt{2}} - \frac{1}{2} \cdot \frac{1}{\sqrt{2}} = \frac{1}{2\sqrt{2}}(\sqrt{3}-1)$$

$$\cos(15^\circ) = \cos(60^\circ - 45^\circ) = \cos 60 \cos 45 + \sin 60 \sin 45$$

$$= \frac{1}{2} \cdot \frac{1}{\sqrt{2}} + \frac{\sqrt{3}}{2}\frac{1}{\sqrt{2}} = \frac{1}{2\sqrt{2}}(\sqrt{3}+1)$$

$\tan(15^\circ) = (\sqrt{3}-1)(\sqrt{3}+1) = 2 - \sqrt{3}$ (who knew??)

b) $\int_2^4 \frac{x^2+3}{x-1}\,dx = \int_2^4 x + 1 + \frac{4}{x-1}\,dx$

$= \frac{x^2}{2}\Big|_2^4 + x\Big|_2^4 + \int_2^4 \frac{4}{x-1}\,dx$

Let $x-1 = t$ then $dx = dt$, limits on t are 1 and 3,

and the integral term is $4\int_1^3 \frac{1}{t}\,dt$

$= 4(\ln 3 - \ln 1) = 4\ln 3$ or $\ln(3^4)$

So the overall integral is

$\frac{16}{2} - \frac{4}{2} + 4 - 2 + 4\ln 3$

$= \underline{\underline{8 + 4\ln 3}}$

11) Technically the 'surface' includes both inside and outside - but we'll assume the question means only the outside...

Capacity = $\pi r^2 h = 1500$ So $h = \dfrac{1500}{\pi r^2}$

a) Area = area of base $= \pi r^2$

+ area of sides $= (2\pi r)h$

$= 2\pi r \cdot \dfrac{1500}{\pi r^2}$

$= \dfrac{3000}{r}$

So Area $= \underline{\underline{\pi r^2 + \dfrac{3000}{r}}}$

b) Stationary value: $\dfrac{dA}{dr} = 0$

$\dfrac{dA}{dr} = 2\pi r + -\left(\dfrac{3000}{r^2}\right) = 0$

So $2\pi r = \dfrac{3000}{r^2}$

$r^3 = \dfrac{1500}{\pi}$

$r = \sqrt[3]{\dfrac{1500}{\pi}} =$ 7.82 cm for this Turning Point to be a minimum, $\frac{d^2A}{dr^2} > 0$

$$\frac{d^2A}{dr^2} = 2\pi - 2\frac{(-3000)}{r^3}$$

$$= 2\pi + \frac{6000}{r^3}$$

This is +ve even without having to calculate it,

so this r gives the minimum value of A.

d) For this value of r (7.82)

$$A = \pi r^2 + \frac{3000}{r} = \underline{\underline{576 \text{ cm}^3}}$$

12) $f: x \to x^2 - 2x - 3 \qquad 0 \leq x \leq 5$

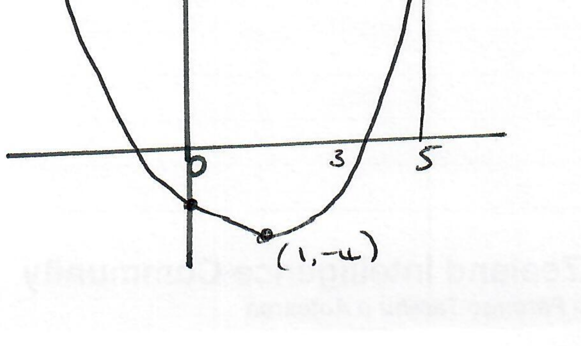
$g: x \to ax^2 + 2$

Curve-plotting:

$x^2 - 2x - 3 = (x-3)(x+1)$ so $f(x) = 0$ at $x = -1$, $x = 3$.

also $(x^2 - 2x - 3) = (x-1)^2 - 1 - 3 = (x-1)^2 - 2^2$

So the curve is a parabola:



The function is defined over domain $[0,5]$,

so the range is $[-4, 12]$

b) $f(1) = 1-2-3 = -4$

so $gf(1) = g(-4) = a(-4)^2+2$

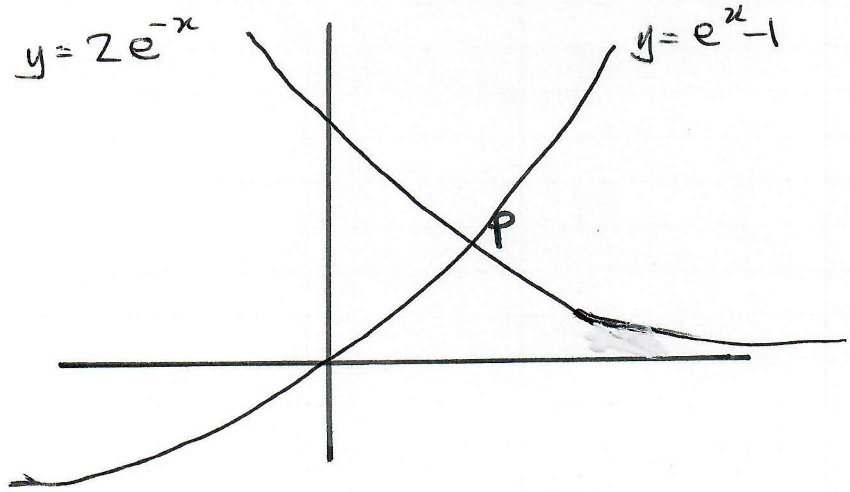
$= 16a+2$

$= 6$ (given)

So $16a+2=6$

$16a = 4$

$a = \frac{4}{16} = \frac{1}{4}$

13) 

At P, $2e^{-x} = e^{x}-1$

So multiplying by e^{x}:

$2 = e^{2x} - e^{x}$

$e^{2x} - e^{x} - 2 = 0$

This is a quadratic in e^{x}:

$(e^{x}-2)(e^{x}+1) = 0.$

e^x can never be negative, so

$$e^x = 2$$

$$x = \ln 2.$$

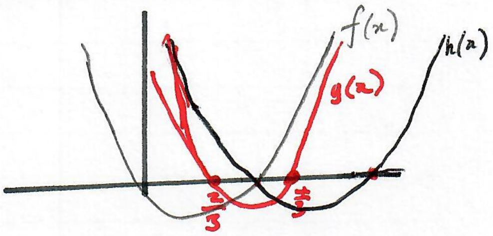
when $x = \ln 2$, $y = e^{\ln 2} - 1 = 2 - 1 = 1.$

So P is $\underline{\underline{(\ln 2, 1)}}$

14) $f(x) = x(x-1)$

a) Transformations involve changing x (or y) by the 'reverse' of the transformation. So

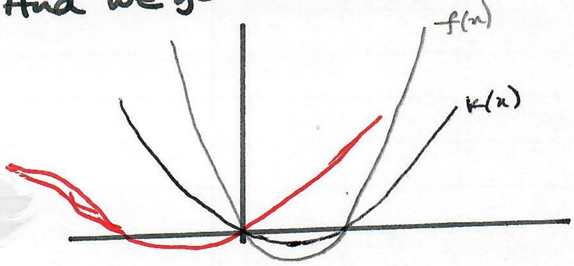
$n(x) = (x-1)(x-2)$ (not the h in part b)



$$g(x) = \left(\frac{3x}{2} - 1\right)\left(\frac{3x}{2} - 2\right)$$

$$= \frac{1}{4}(3x-2)(3x-4)$$

b) But this is the opposite. And we go backwards.



$f(x) = x(x-1)$

$k(x) = \frac{1}{2}x(x-1)$

$h(x) = \frac{1}{2}(x+1)x = \frac{1}{2}x(x+1)$

(It helps to draw pictures)

15) $x^2+y^2-12x-2y+33=0$.

a) 'weknow' for a circle $x^2+y^2+2px+2qy+r=0$,

centre is $(-p, -q)$

radius is $\sqrt{p^2+q^2-r}$

But to show this, recast the equation:

$$(x-6)^2-36+(y-1)^2-1+33=0$$

$$(x-6)^2+(y-1)^2=4$$

This is the equation of a circle centre $(6,1)$, radius 2.

b) When $y=x-3$,

$$(x-6)^2+(x-4)^2=4$$

$$x^2-12x+36+x^2-8x+16=4$$

$$2x^2-20x+52=4$$

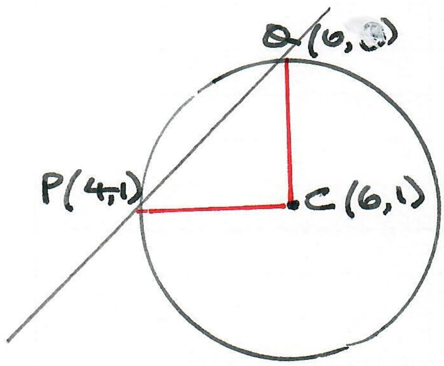
$$2x^2-20x+48=0$$

$$x^2-10x+24=0$$

$$(x-6)(x-4)=0 \qquad x=4 \quad y=1$$

or $\quad x=6 \quad y=3$.

c) Now draw this as:



Evidently PQC is a right-angled $\triangle$

So Sector $PCQ = \frac{\pi}{2}\cdot 2 = \pi$

$\triangle PQC = \frac{2\times 2}{2} = 2$

So the minor segment is $\underline{\underline{\pi-2}}$

as required.

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16) $C: x^2 - \dfrac{y^2}{2} = 1$

$L: y = x + c$

Substitute L into C:

$$x^2 - \frac{(x+c)^2}{2} = 1$$

$$2x^2 - x^2 - 2xc - c^2 = 1$$

$$x^2 - 2xc - (1+c^2) = 0$$

$$x = \frac{2c \pm \sqrt{4c^2 + 4(1+c^2)}}{2}$$

The discriminant $= 4 + 8c^2$ is always positive, (assume $c \in \mathbb{R}$) so there is always a solution – in fact 2. So the curve and straight line intersect for all c.

17) 'General solution' and an equation in y are reminiscent of tough university-level problems, but this one turns out rather easier.

$$5\frac{dy}{dx} = 2y^2 - 7y + 3 = (2y-1)(y-3)$$

So $$\frac{5}{(2y-1)(y-3)}\,dy = dx$$

Separate the LHS:

$$\frac{5}{(2y-1)(y-3)} = \frac{A}{2y-1} + \frac{B}{y-3} = \frac{A(y-3)+B(2y-1)}{(2y-1)(y-3)}$$

$$5 = Ay + 2By - 3A - B$$

So $A + 2B = 0$ $\quad A = -2B.$

$3A + B = -5$

$-6B + B = -5$

$-5B = -5$

$B = 1.$

$A = -2.$

$$\text{LHS} = \frac{-2}{2y-1} + \frac{1}{y-3}$$

Let $2y-1 = s$ $\quad 2dy = ds$ $\quad dy = \frac{ds}{2}$

$y-3 = t$ $\quad dy = dt.$

So we have: $\frac{-\cancel{2}}{s}\frac{ds}{\cancel{2}} + \frac{1}{t}dt = dx$

Integrate throughout: $-\ln s + \ln t = x + C$

$$\ln\frac{t}{s} = x + C$$

$$\frac{t}{s} = e^{x+C}$$

$$\frac{y-3}{2y-1} = e^{x+C} = Ae^{x} \text{ where } A = e^{c}.$$

$y - 3 = Ae^{x}(2y-1)$

$y(1 - 2Ae^{x}) = 3 - Ae^{x}$ so $y = \dfrac{3 - Ae^{x}}{1 - 2Ae^{x}}$

as required.

MADAS SYN Paper A

18) $\theta, \frac{\pi}{4}, \phi$ are in arithmetic progression.

Let the difference be δ: then

$$\phi - \theta = 2\delta$$

$$\phi + \theta = \frac{\pi}{2}$$

Using standard identities:

$$\sin\theta - \sin\phi = 2\cos\frac{\theta+\phi}{2}\sin\frac{\theta-\phi}{2}$$

$$= 2\cos\frac{\pi}{4}\sin(-\delta)$$

$$= -\sqrt{2}\sin\delta$$

$$\cos\theta + \cos\phi = 2\cos\frac{\pi}{4}\cos\delta$$

$$= \sqrt{2}\cos\delta$$

So $(\sin\theta - \sin\phi)^2 + (\cos\theta + \cos\phi)^2$

$$= 2\sin^2\delta + 2\cos^2\delta = 2$$

So $\underline{\underline{k = 2}}$

19) $x = a + \tan t$

$y = b + \cot^2 t$ $\qquad 0 < t < \frac{\pi}{2}$

a) i) $\frac{dx}{dt} = \sec^2 t$

$\frac{dy}{dt} = -2\operatorname{cosec}^2 t \cot t$

So $\frac{dy}{dx} = \frac{-2\operatorname{cosec}^2 t \cot t}{\sec^2 t} = \frac{-2\cos^3 t}{\sin^3 t}$

$= \underline{\underline{-2\cot^3 t}}$

a) ii) Show this simply by substitution:

$(y-b)(x-a)^2 = (\cot^2 t)(\tan t)^2$

$= \cot^2 t \tan^2 t$

$= 1$ as required

(that was easy!)

b) on the line $y = 6x + 2$

when $y = 2$ $\quad x = 0$: point $(0, 2)$

when $y = 5$ $\quad x = \frac{1}{2}$: point $(\frac{1}{2}, 5)$

substitute into $(y-b)(x-a)^2 = 1$:

$(5-b)(\frac{1}{2}-a)^2 = 1$ ——— ①

$(2-b)(-a)^2 = 1$ ——— ②

$$2-b = \frac{1}{a^2}$$

$$5-b = \frac{1}{a^2}+3$$

$$\left(\frac{1}{a^2}+3\right)\left(\frac{1}{2}-a\right)^2 = 1$$

$$(1+3a^2)\left(\frac{1}{2}-a\right)^2 = a^2$$

$$(1+3a^2)(1-2a)^2 = 4a^2$$

$$(1+3a^2)(1-4a+4a^2) = 4a^2$$

$$1+3a^2-4a-12a^3+\cancel{4a^2}+12a^4 = \cancel{4a^2}$$

$$12a^4-12a^3+3a^2-4a+1 = 0$$

$a=1$ makes this 0, so factorising...

$$(a-1)(12a^3+3a-1) = 0 \text{ as required.}$$

so a possible value for a is 1, giving

$$2-b = \frac{1}{1} \qquad b=1$$

(there's at least 1 other possible value, but that would mean finding 1/2/3 roots of the cubic)

20) $y = x\sqrt{\ln x}$

$4y = bx - a$

at $x = a$: $\quad a\sqrt{\ln a} = \frac{1}{4}(ba - a)$

$= \frac{1}{4}a(b-1)$

gradient of tangent $= \frac{b}{4}$.

and this $= \frac{dy}{dx}$ on the curve:

$x \cdot \frac{1}{2}\frac{1}{\sqrt{\ln x}} \cdot \frac{1}{x} + \sqrt{\ln x}$

$= \sqrt{\ln x} + \frac{1}{2\sqrt{\ln x}}$

$= \sqrt{\ln a} + \frac{1}{2\sqrt{\ln a}}$ —— 1

$= \frac{b}{4}$

Also $\quad 4a\sqrt{\ln a} = ba - a$

$4\sqrt{\ln a} = b - 1$

So $\quad \sqrt{\ln a} = \frac{b-1}{4}$

$\sqrt{\ln a} + \frac{1}{4} = \frac{b}{4}$ —— 2

So $\sqrt{\cancel{\ln a}} + \dfrac{1}{2\sqrt{\ln a}} = \sqrt{\cancel{\ln a}} + \dfrac{1}{4}$

$$2 = \sqrt{\ln a}$$

$$\ln a = 2^2 = 4$$

$$\underline{\underline{a = e^4}}$$

21) $\displaystyle\int \frac{1-x}{\sqrt{x}(x+1)^2}\,dx$

Let $\sqrt{x} = \tan\theta \qquad x = \tan^2\theta$

$$\frac{dx}{d\theta} = 2\tan\theta\sec^2\theta$$

So the integral is

$$\int \frac{(1-\tan^2\theta)}{\cancel{\tan\theta}\,(\tan^2\theta+1)^2} \cdot 2\,\cancel{\tan\theta}\,\sec^2\theta\,d\theta$$

Now $\cos^2 + \sin^2 = 1$

So $1 + \tan^2 = \sec^2$

The integral is:

$$\int \frac{(1-\tan^2\theta)}{(\sec^2\theta)^2} \cdot 2\sec^2\theta\,d\theta$$

$$= \int \frac{1-\tan^2\theta}{\sec^2\theta} \cdot 2\,d\theta = 2\int \cos^2\theta - \sin^2\theta\,d\theta$$

$$= 2\int \cos 2\theta\,d\theta$$

$$= \frac{2}{2} \sin 2\theta + C$$

$$= \underline{\underline{\sin 2\theta + C}}$$