MADAS FP2 Paper 0.

1) First, look for the 'auxiliary equation' and the 'complementary function':

$$\lambda^2 + 6\lambda + 13 = 0 \quad : \quad \lambda = \frac{-6 \pm \sqrt{36-52}}{2}$$

Unusually, this doesn't factorise, and it even has complex roots:

$$\lambda = -3 \pm 2i$$

Normally this would lead to a general solution $Ae^{(-3+2i)x} + Be^{(-3-2i)x}$

- but because there are imaginary terms here we can re-express it as

$$y = e^{-3x}(A\cos 2x + B\sin 2x)$$

(different A and B).

On the RHS, the polynomial $13x^2 - x + 22$ suggests we consider a 'particular solution'

$$y = Px^2 + Qx + R$$

so $\frac{dy}{dx} = 2Px + Q$

$\frac{d^2y}{dx^2} = 2P$

So $2P + 6(2Px + Q) + 13(Px^2 + Qx + R) = 13x^2 - x + 22$

$13P = 13 : \underline{P = 1}$

$12P + 13Q = -1 : 13Q = -13 : \underline{Q = -1}$

$2P + 6Q + 13R = 22 : 2 - 6 + 13R = 22 : 13R = 26$

$\underline{R = 2}$

So the general solution is $\underline{y = e^{-3x}(A\cos 2x + B\sin 2x) + x^2 - x + 2}$

2) $f(r) = \dfrac{1}{r(r+2)}$

a) Let $f(r) = \dfrac{A}{r} + \dfrac{B}{r+2}$

$$= \frac{A(r+2)+Br}{r(r+2)} = \frac{1}{r(r+2)}$$

Equating coeffs:

$r: \quad A + B = 0$

$1: \quad 2A = 1$

so $A = \frac{1}{2}, \; B = -\frac{1}{2}$

and $f(r) = \dfrac{1}{2r} - \dfrac{1}{2(r+2)}$ or $\frac{1}{2}\left(\frac{1}{r} - \frac{1}{r+2}\right)$

b) Using this expansion,

$$\begin{aligned}\sum_{r=1}^{n} f(r) = f(1) &= \frac{1}{2} - \frac{1}{6}\\ + f(2) &= + \frac{1}{4} - \frac{1}{8}\\ + f(3) &= + \frac{1}{6} - \frac{1}{10}\\ &\;\vdots\\ + f(n-2) &= + \frac{1}{2(n-2)} - \frac{1}{2n}\\ + f(n-1) &= + \frac{1}{2(n-1)} - \frac{1}{2(n+1)}\\ + f(n) &= + \frac{1}{2n} - \frac{1}{2(n+2)}\end{aligned}$$

$$= \frac{1}{2} + \frac{1}{4} - \frac{1}{2(n+1)} - \frac{1}{2(n+2)}$$

$$= \frac{3(n+1)(n+2) - 2(n+2) - 2(n+1)}{4(n+1)(n+2)}$$

$$= \frac{3n^2 + 9n + \cancel{6} - 2n - \cancel{4} - 2n - \cancel{2}}{4(n+1)(n+2)} = \underline{\underline{\frac{n(3n+5)}{4(n+1)(n+2)}}}$$ as required with $\underline{\underline{A=3, B=5}}$

3) Again I think MADAS is a bit remiss in speaking of a 'Maclaurin expansion' when he really wants a 'product of two expansions' - but here goes. We'll do it both ways.

A. As a proper Maclaurin expansion.

$$f(x) = (1-x)^2 \ln(1-x) \qquad \text{So } f(0) = 1 \times \ln(1) = \underline{0}$$

Note here that $\dfrac{d}{dx} \ln(1-x) = \dfrac{1}{1-x} \times -1$, i.e $-\dfrac{1}{1-x}$

$$f'(x) = -\frac{(1-x)^2}{1-x} + 2(-1)(1-x)\ln(1-x)$$

$$= -(1-x) - 2(1-x)\ln(1-x)$$

$$\text{So } f'(0) = -1 - 2 \times 1 \times \ln(1) = \underline{-1}$$

$$f''(x) = +1 - 2\left[\frac{(1-x)}{(1-x)} \times (-1) - 1 \ln(1-x)\right]$$

$$= 1 + 2 + 2\ln(1-x) \qquad \text{So } f''(0) = \underline{3}$$

$$= 3 + 2\ln(1-x)$$

$$f'''(x) = \frac{-2}{1-x} \qquad \text{So } f'''(0) = \underline{-2}$$

In the Maclaurin expansion:

$$f(x) = 0 - 1x + \frac{3}{2}x^2 - \frac{2}{6}x^3 + o(x^4)$$

$$\underline{\underline{\approx -x + \frac{3}{2}x^2 - \frac{1}{3}x^3 + o(x^4).}}$$

B. Done with standard expansions

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + o(x^5)$$ (it's very slow to converge)

So

$$\ln(1-x) = -x - \frac{x^2}{2} - \frac{x^3}{3} - \frac{x^4}{4} - \dots$$ (even slower)

So $(1-x)^2 \ln(1-x) = (1-2x+x^2)\ln(1-x)$

$$= -x - \frac{x^2}{2} - \frac{x^3}{3} + o(x^4)$$

$$+2x^2 + x^3 + o(x^4)$$

$$-x^3 + o(x^4)$$

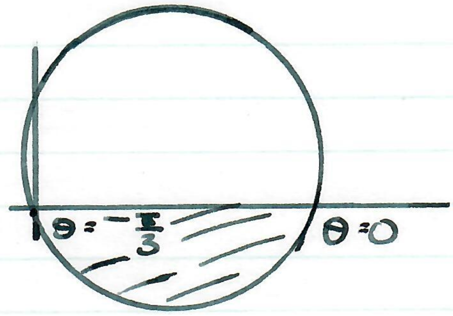
$$= -x + \frac{3}{2}x^2 - \frac{x^3}{3} + o(x^4)$$ as required.

$$= 4\sinh^{-1}(2)$$

which is the required form ... with $a=4$
$b=2$.

5) Required area is

$$\int_{-\frac{\pi}{3}}^{0} \frac{r^2}{2}\, d\theta$$



$$r = \sqrt{3}\cos\theta + \sin\theta$$

$$r^2 = 3\cos^2\theta + 2\sqrt{3}\cos\theta\sin\theta + \sin^2\theta$$

$$= 2\cos^2\theta + 2\sqrt{3}\cos\theta\sin\theta + 1$$

And $\cos 2\theta = \cos^2\theta - \sin^2\theta = \cos^2\theta - (1-\cos^2\theta)$

$$= 2\cos^2\theta - 1$$

So $2\cos^2\theta = \cos 2\theta + 1$

And $\sin 2\theta = 2\cos\theta\sin\theta$.

So the integral is $\frac{1}{2}\int_{-\frac{\pi}{3}}^{0} \cos 2\theta + 1 + \sqrt{3}\sin 2\theta + 1 \, d\theta$

$$= \frac{1}{2}\int_{-\frac{\pi}{3}}^{0} \cos 2\theta + \sqrt{3}\sin 2\theta + 2 \, d\theta$$

$$= \frac{1}{2}\left[\frac{1}{2}\sin 2\theta - \frac{\sqrt{3}}{2}\cos 2\theta + 2\theta\right]_{-\frac{\pi}{3}}^{0}$$

$$= \frac{1}{2}\left(0 - \frac{\sqrt{3}}{2} + 0 \quad - \frac{1}{2}\sin\left(\frac{-2\pi}{3}\right) + \frac{\sqrt{3}}{2}\cos\left(-\frac{2\pi}{3}\right) + \frac{2\pi}{3}\right)$$

$$= \frac{1}{2}\left(- \frac{\sqrt{3}}{2} + \cancel{\frac{\sqrt{3}}{4}} - \cancel{\frac{\sqrt{3}}{4}} + \frac{2\pi}{3}\right)$$

$$= \underline{\underline{\frac{1}{12}\left(4\pi - 3\sqrt{3}\right)}}$$

6) $$I = \int_0^{\ln 2} \frac{e^x}{\cosh x}\, dx$$

$$= \int_0^{\ln 2} \frac{2e^x}{e^x + e^{-x}}\, dx$$

Let $t = e^x$:

$$= \int_0^{\ln 2} \frac{2t}{t + t^{-1}}\, dx$$

$dt = e^x dx = t dx$

$$I = \int \frac{2}{t + t^{-1}}\, dt$$

$$= \int \frac{2t}{t^2 + 1}\, dt$$

Let $u = t^2 + 1$: $\quad du = 2t\,dt$

$$I = \int \frac{du}{u} = \ln|u| \Big|_?^?$$

$$= \ln(t^2+1)\Big|_{?}^{?}$$

$$= \ln(e^{2x}+1)\Big|_{0}^{\ln 2}$$

$$= \ln(e^{2\ln 2}+1) - \ln(1+1)$$

$$= \ln(4+1) - \ln(2)$$

$$= \underline{\ln\left(\frac{5}{2}\right)}$$

7) De Moivre's Theorem states $\text{cis}\,5\theta = (\text{cis}\,\theta)^5$

Expanding:

$$(\cos 5\theta + i\sin 5\theta) = (\cos\theta + i\sin\theta)^5$$

$$= \cos^5\theta + 5i\cos^4\theta\sin\theta + 10i^2\cos^3\theta\sin^2\theta$$
$$+ 10i^3\cos^2\theta\sin^3\theta + 5i^4\cos\theta\sin^4\theta + i^5\sin^5\theta$$

Taking the imaginary component:

$$\sin 5\theta = 5\cos^4\theta\sin\theta - 10\cos^2\theta\sin^3\theta + \sin^5\theta$$

$$= 5(1-\sin^2\theta)^2\sin\theta - 10(1-\sin^2\theta)\sin^3\theta + \sin^5\theta$$

$$= 5(1-2\sin^2\theta+\sin^4\theta)\sin\theta - 10\sin^3\theta + 10\sin^5\theta + \sin^5\theta$$

$$= 5\sin\theta - 10\sin^3\theta + 5\sin^5\theta - 10\sin^3\theta + 11\sin^5\theta$$

So finally:

$$\sin 5\theta = 16\sin^5\theta - 20\sin^3\theta + 5\sin\theta$$

b) if $\sin 5\theta = 0$ then $\theta = 0$ or $\pm\frac{\pi}{5}$ or $\pm\frac{2\pi}{5}$

so in $16x^4 - 20x^2 + 5 = 0$ ——— ①

the $x=0$ root has been discarded, and we can see $x = \sin\frac{\pi}{5}$, $\sin\frac{2\pi}{5}$,

$\sin -\frac{\pi}{5}$, $\sin -\frac{2\pi}{5}$.

① is a quadratic in x^2, so

$$x^2 = \frac{20 \pm \sqrt{20^2 - 4.16.5}}{32}$$

$$= \frac{20 \pm \sqrt{400-320}}{32} = \frac{20 \pm \sqrt{80}}{32}$$

$$= \frac{5 \pm \sqrt{5}}{8}$$

The '+' form is infeasible if this is to equal $\sin^2\left(\frac{\pi}{5}\right)$, so we can say

$$\underbrace{\sin^2\left(\frac{\pi}{5}\right)}_{=36^\circ} = \frac{5-\sqrt{5}}{8} \qquad \left(\text{also } -\frac{\pi}{5}\right)$$

on the other hand $\underbrace{\sin^2\left(\frac{2\pi}{5}\right)}_{=72^\circ} = \frac{5+\sqrt{5}}{8}$. $\left(\text{also } -\frac{2\pi}{5}\right)$

8) $y = \cos^{-1} x$

a) $x = \cos y$

$dx = -\sin y \, dy$

$$\frac{dy}{dx} = \frac{-1}{\sin y} = -\frac{1}{\sqrt{1-\cos^2 y}}$$

$$= -\frac{1}{\sqrt{1-x^2}}$$

as required.

b) $y = \cos^{-1} x - \frac{1}{2} \ln (1-x^2)$ ——— ①

$$\frac{dy}{dx} = \frac{1}{\sqrt{1-x^2}} - \frac{1}{2} \frac{1}{(1-x^2)} (-2x)$$

$$= \frac{1}{\sqrt{1-x^2}} + \frac{x}{1-x^2}$$

For a stationary point: (ignore $\sqrt{1-x^2}$ denominator)

$$0 = 1 + \frac{x}{\sqrt{1-x^2}}$$

$$\sqrt{1-x^2} = -x$$

$$1-x^2 = x^2$$

$$x = \frac{1}{\sqrt{2}}$$

So in ①, $y = \cos^{-1}\left(\frac{1}{\sqrt{2}}\right) - \frac{1}{2} \ln\left(\frac{1}{2}\right) = \frac{\pi}{4} + \frac{1}{2} \ln 2$

$$= \underline{\frac{1}{4}(\pi + \ln 4)}$$ as required.

16

a) $$(1-x^2)\frac{dy}{dx} + y = (1-x^2)(1-x)^{\frac{1}{2}}$$

$$\frac{dy}{dx} + \frac{1}{1-x^2}y = (1-x)^{\frac{1}{2}} \qquad \text{———} \quad (1)$$

Use an IF: $$e^{\int \frac{1}{1-x^2}dx} = e^{\frac{1}{2}\ln\left|\frac{1+x}{1-x}\right|}$$

(standard formula)

$$= \sqrt{\frac{1+x}{1-x}}$$

(always >0 in these

Assuming there are no divisions by 0, this makes (1):

$$(1+x)^{\frac{1}{2}}(1-x)^{-\frac{1}{2}}\frac{dy}{dx} + \frac{(1+x)^{\frac{1}{2}}(1-x)^{-\frac{1}{2}}}{(1-x)\cdot(1+x)}y = (1+x)^{\frac{1}{2}}$$

$$\frac{d}{dx}\left(\sqrt{\frac{(1+x)}{(1-x)}}\,y\right) = (1+x)^{\frac{1}{2}}$$

So $$\sqrt{\frac{1+x}{1-x}}\,y = \int (x+1)^{\frac{1}{2}}dx$$

$$= \frac{2}{3}(x+1)^{\frac{3}{2}} + C$$

So $$y = \frac{2}{3}(x+1)(1-x)^{\frac{1}{2}} + C\frac{(1-x)^{\frac{1}{2}}}{(1+x)^{\frac{1}{2}}}$$

Initial condition $\left(\frac{1}{2}, \frac{\sqrt{2}}{2}\right)$:

$$\frac{3\sqrt{2}}{3\cdot 2} = \frac{\cancel{2}}{3}\frac{1}{\cancel{2}}\times\frac{1\sqrt{2}}{\sqrt{2}\sqrt{2}} + C\frac{1}{\cancel{\sqrt{2}}\sqrt{3}}\frac{\cancel{\sqrt{2}}\sqrt{2}\sqrt{2}}{\sqrt{3}\cdot 2}$$

$$3\sqrt{2} = \sqrt{2} + C\sqrt{6} \qquad C = 0$$

So $y = \frac{2}{3}(x+1)(1-x)^{\frac{1}{2}}$

$= \frac{2}{3}(1+x)^{\frac{1}{2}}\ (1+x)^{\frac{1}{2}}(1-x)^{\frac{1}{2}}$

$= \frac{2}{3}(1+x)^{\frac{1}{2}}(1-x^2)^{\frac{1}{2}}$

$= \frac{2}{3}\sqrt{(1-x^2)(1+x)}$ as required.