MADAS FP2 Paper N

1) a) $$\frac{1}{r(r+1)} - \frac{1}{(r+1)(r+2)}$$

$$= \frac{(r+2) - r}{r(r+1)(r+2)} = \frac{2}{r(r+1)(r+2)}$$ (note factor 2)

Reverting to the original form:

$$\sum_{r=1}^{20} \frac{1}{r(r+1)} - \frac{1}{(r+1)(r+2)} =$$

$$\frac{1}{1\times2} - \cancel{\frac{1}{2\times3}}$$

$$+ \cancel{\frac{1}{2\times3}} - \cancel{\frac{1}{3\times4}}$$

$$\vdots \qquad \vdots$$

$$+ \cancel{\frac{1}{19\times20}} - \cancel{\frac{1}{20\times21}}$$

$$+ \cancel{\frac{1}{20\times21}} - \frac{1}{21\times22}$$

$$= \frac{1}{2} - \frac{1}{21\times22} = \frac{231-1}{462} = \frac{230}{462}$$

So dividing by the factor 2,

$$\sum_{r=1}^{20} \left(\frac{1}{r(r+1)(r+2)} \right) = \frac{1}{2} \times \frac{230}{462}$$

$$= \frac{115}{462}$$ as required.

2) $f(x) = e^{-2x}\cos 4x$

$$f'(x) = -4e^{-2x}\sin 4x - 2e^{-2x}\cos 4x$$

$$= -2e^{-2x}(2\sin 4x + \cos 4x)$$

$$= -4e^{-2x}\sin 4x - 2f(x), \text{ if that's useful.}$$

$$f''(x) = -4\times 4e^{-2x}\cos 4x + 8e^{-2x}\sin 4x - 2f'(x)$$

$$= -16f(x) - 2f'(x) + 8e^{-2x}\sin 4x$$

$$f'''(x) = -16f'(x) - 2f''(x) + 32e^{-2x}\cos 4x - 16e^{-2x}\sin 4x$$

$$= 32f(x) - 16f'(x) - 2f''(x) - 16e^{-2x}\sin 4x$$

$$f''''(x) = 32f'(x) - 16f''(x) - 2f'''(x) - 64e^{-2x}\cos 4x + 32e^{-2x}\sin 4x$$

$$= -64f(x) + 32f'(x) - 16f''(x) - 2f'''(x) + 32e^{-2x}\sin 4x$$

So $f(0) = e^0\cos 0 = 1\times 1 = 1$

$f'(0) = \ldots\sin 0 - 2f(0) = -2$

$f''(0) = -16\times 1 - 2\times(-2) + \ldots\sin 0 = -12$

$f'''(0) = 32\times 1 - 16\times(-2) + 2\times 12 - \ldots\sin 0$
$= 32 + 32 + 24 = 88$

$f''''(0) = -64\times 1 + 32\times(-2) - 16\times 12 - 2\times 88 + \ldots\sin 0$
$= -64 - 64 + 192 - 176 = 112$

So the Maclaurin expansion is:

$$f(x) = f(0) + x f'(0) + \frac{x^2}{2} f''(0) + \frac{x^3}{6} f'''(0) + \frac{x^4}{24} f''''(0) + \cdots$$

$$= 1 + (-2)x + (-12)\frac{x^2}{2} + \frac{88}{6}x^3 + \frac{112}{24}x^4 + \cdots$$

$$= 1 - 2x - 6x^2 + \frac{44}{3}x^3 + \frac{14}{3}x^4 + \cdots$$

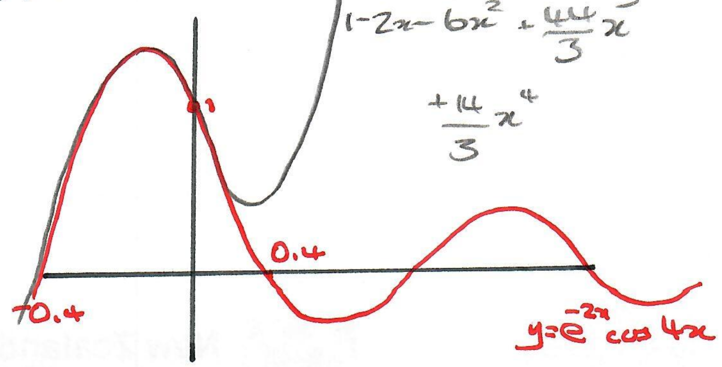
It turns out in fact that MADAS didn't want a Maclaurin expansion but simply the multiple of the two standard expressions

$$e^{-2x} = 1 - 2x + \frac{(-2x)^2}{2} + \frac{(-2x)^3}{6} + \frac{(-2x)^4}{24} + \cdots$$

and $$\cos 4x = 1 - \frac{(4x)^2}{2} + \frac{(4x)^4}{24} + \cdots$$

This does indeed give the same result - but since the question explicitly mentions Maclaurin I think MADAS is misleading us.

As usual, plotting it on Geogebra gives a beautiful match.



3) $f(y) = \dfrac{4y}{y^4-1} = \dfrac{4y}{(y^2-1)(y^2+1)} = \dfrac{4y}{(y-1)(y+1)(y^2+1)}$

a) Let $f(y) = \dfrac{A}{(y-1)} + \dfrac{B}{(y+1)} + \dfrac{Cy+D}{(y^2+1)}$

$$= \frac{A(y+1)(y^2+1) + B(y-1)(y^2+1) + (Cy+D)(y-1)(y+1)}{(\quad)(\quad)(\quad)}$$

$$= \frac{A(y^3+y^2+y+1) + B(y^3-y^2+y-1) + C(y^3-y) + D(y^2-1)}{(\quad)(\quad)(\quad)}$$

Equating coefficients:

y^3: $A+B+C = 0$

y^2: $A-B+D = 0$

y: $A+B-C = 4$

1: $A-B-D = 0$

$2(A+B) = 4$

$2(A-B) = 0$

$A+B = 2$

$A-B = 0$

$2A = 2$: $A = 1$, $B = 1$, $C = -2$, $D = 0$

So $f(y) = \dfrac{1}{y-1} + \dfrac{1}{y+1} - \dfrac{2y}{y^2+1}$

2 points to note:

a) $\dfrac{1}{y-1}$ and $\dfrac{1}{y+1}$ will both integrate as ln:

$\dfrac{2y}{y^2+1}$ will too, since $\dfrac{d}{dy}(y^2+1) = 2y$.

b) We need to do a $\lim\limits_{k\to\infty}$ process; this doesn't work for each term in isolation, so keep them together.

b) $\displaystyle\int_2^{\infty} f(y)\,dy = \int_2^{\infty} \frac{1}{y-1} + \frac{1}{y+1} - \frac{2y}{y^2+1}\,dy$

$$= \lim_{k\to\infty} \int_2^{k} \frac{1}{y-1} + \frac{1}{y+1} - \frac{2y}{y^2+1}\,dy$$

$$= \lim_{k\to\infty} \Big[\ln(y-1) + \ln(y+1) - \ln(y^2+1) \Big]_2^k$$

$$= \lim_{k\to\infty} \left[\ln\frac{(y^2-1)}{(y^2+1)} \right]_2^k$$

$$= \lim_{k\to\infty} \left[\ln\frac{(k^2-1)}{(k^2+1)} - \ln\left(\frac{3}{5}\right) \right]$$

$$= \ln(1) - \ln\left(\frac{3}{5}\right)$$

$$= \underline{\underline{\ln\left(\frac{5}{3}\right)}}$$

c) In similar form,

$$\int_2^4 f(y)\,dy = \left[\ln\frac{(y^2-1)}{(y^2+1)} \right]_2^4$$

$$= \ln\left(\frac{15}{17}\right) - \ln\left(\frac{3}{5}\right) = \ln\left(\frac{25}{17}\right)$$

So the mean value over $[2,4]$ is

$$\underline{\underline{\frac{1}{2}\ln\left(\frac{25}{17}\right)}} = \underline{\underline{\ln\left(\frac{5}{\sqrt{17}}\right)}}$$

4) $z = e^{i\theta}$

a) It's not exactly clear what we're allowed to assume, but...

$z^n = e^{in\theta} = \cos(n\theta) + i\sin(n\theta)$ (1)

$\frac{1}{z^n} = z^{-n} = e^{-in\theta} = \cos(-n\theta) + i\sin(-n\theta)$

$= \cos(n\theta) - i\sin(n\theta)$ (2)

(1)+(2): $z^n + \frac{1}{z^n} = \underline{\underline{2\cos n\theta}}$ $(+\, 0 i \sin n\theta)$

as required.

Similarly

(1)-(2): $z^n - \frac{1}{z^n} = (0\cos n\theta +)\ \underline{\underline{2i\sin n\theta}}$

as required.

b) Again it's not quite clear what direction to go in, but here goes:

LHS $= \cos^4\theta \sin^2\theta$

Consider $z = \cos\theta + i\sin\theta$

Then $\frac{1}{2}(z + z^{-1}) = \cos\theta$

$\frac{1}{2}(z - z^{-1}) = i\sin\theta$

So $\cos^4\theta \sin^2\theta$

$= \frac{1}{64}(z + z^{-1})^4 (z - z^{-1})^2 (-i^2)$

$= \frac{1}{64}(z^4 + 4z^3z^{-1} + 6z^2z^{-2} + 4zz^{-3} + z^{-4})$
$(z^2 - 2zz^{-1} + z^{-2})$

$$= \frac{-1}{64}\left(z^4 + 4z^2 + 6 + 4z^{-2} + z^{-4}\right)\left(z^2 - 2 + z^{-2}\right)$$

$$= \frac{-1}{64}\big(z^6 + 4z^4 + 6z^2 + 4 + z^{-2}$$
$$-2z^4 - 8z^2 - 12 - 8z^{-2} - 2z^{-4}$$
$$+ z^2 + 4 + 6z^{-2} + 4z^{-4} + z^{-6}\big)$$

$$= \frac{-1}{64}\big(z^6 + 2z^4 - z^2 - 4$$
$$+ z^{-6} + 2z^{-4} - z^{-2}\big)$$

$$= -\frac{1}{64}\left(\left(z^6 + z^{-6}\right) + 2\left(z^4 + z^{-4}\right) - 1\left(z^2 + z^{-2}\right) - 4\right)$$

Using the result in (a):

$$= -\frac{1}{32}\cos 6\theta - \frac{1}{16}\cos 4\theta + \frac{1}{32}\cos 2\theta + \frac{1}{16}$$

$=$ RHS as required

5) This one can get very complex unless you make some simplifications and use the right identities:

$$f(x) = e^{2x+2}\left(e^{2x} - 4\right)$$

If $x = \ln\left(2\cosh\frac{1}{2}\right)$ then

$$e^{2x} = e^{2\ln\left(2\cosh\frac{1}{2}\right)} = e^{\ln\left(2\cosh\frac{1}{2}\right)^2}$$

$$= e^{\ln\left(4\cosh^2\frac{1}{2}\right)}$$

$$= \underline{4\cosh^2\frac{1}{2}}$$

So $f(x) = e^2 . 4\cosh^2(\tfrac{1}{2})\left(4\cosh^2(\tfrac{1}{2}) - 4\right)$

$= 16e^2 \cosh^2\tfrac{1}{2}\left(\cosh^2\tfrac{1}{2} - 1\right)$

$(\cosh^2 - \sinh^2 \equiv 1)$

$= 16e^2 \cosh^2\tfrac{1}{2}\left(\sinh^2\tfrac{1}{2}\right)$

$(\sinh 2A \equiv 2\cosh A \sinh A)$

$= 4e^2 \sinh^2 1$

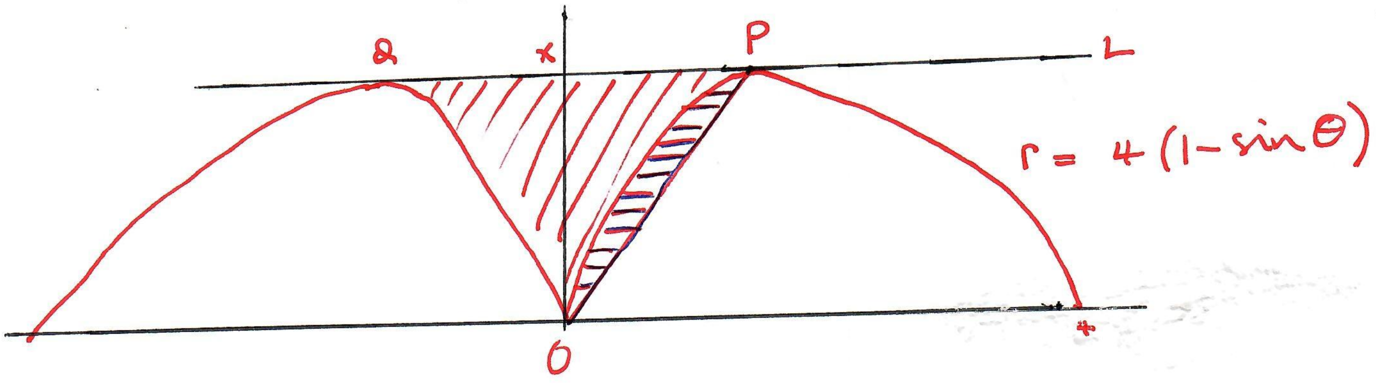
$= 4e^2 \left(\frac{e - e^{-1}}{2}\right)^2$

$= e^2(e^2 - 2 + e^{-2})$

$= e^4 - 2e^2 + 1$

$= \underline{\underline{(e^2-1)^2}}$ as required.

b) This question strikes me as truly 'weird' though perhaps mainly because the diagram is so very 'not drawn to scale'.



The condition on P can mean $\frac{dy}{dx} = 0$ or $\frac{dy}{d\theta} = 0$.

Usually it's best to go with $\frac{dy}{d\theta}$:

$$y = r\sin\theta = 4(\sin\theta - \sin^2\theta)$$

$$\frac{dy}{d\theta} = 4\cos\theta - 8\sin\theta\cos\theta$$
$$= 4\cos\theta(1 - 2\sin\theta) = 0$$

so $2\sin\theta = 1$

$$\sin\theta = \frac{1}{2}$$

$$\theta = \frac{\pi}{6}$$

$$r = 4\left(1 - \sin\frac{\pi}{6}\right) = 2$$

So P is $\left(2, \frac{\pi}{6}\right)$

and Q is $\left(2, \frac{5\pi}{6}\right)$ (so even my diagram is slightly wrong)

We'll calculate the area <u>inside</u> the curve from P to O: i.e. the  shape.

This is $\displaystyle\int_{\frac{\pi}{6}}^{\frac{\pi}{2}} \frac{r^2}{2}\,d\theta = \int_{\frac{\pi}{6}}^{\frac{\pi}{2}} 8(1 - \sin^2\theta)\,d\theta$

$$= \int_{\frac{\pi}{6}}^{\frac{\pi}{2}} 8 - 16\sin\theta + 8\sin^2\theta\,d\theta$$

And $\cos 2\theta = \cos^2\theta - \sin^2\theta = 1 - 2\sin^2\theta$

so $8\sin^2\theta = 4 - 4\cos 2\theta$

So the integral is

$$\int_{\frac{\pi}{6}}^{\frac{\pi}{2}} 12 - 16\sin\theta - 4\cos 2\theta \, d\theta$$

$$= \Big[12x + 16\cos\theta - 2\sin 2\theta\Big]_{\frac{\pi}{6}}^{\frac{\pi}{2}}$$

$$= 12\left(\frac{\pi}{2} - \frac{\pi}{6}\right) + 16\left(0 - \frac{\sqrt{3}}{2}\right) - 2\left(0 - \frac{\sqrt{3}}{2}\right)$$

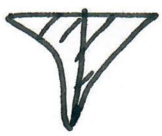
$$= 4\pi - \frac{14\sqrt{3}}{2} = \underline{4\pi - 7\sqrt{3}}$$

To find $\triangle OXP$:

At P, $x = 2\cos\frac{\pi}{6} = \frac{2\sqrt{3}}{2} = \sqrt{3}$

$y = 2\sin\frac{\pi}{6} = \frac{2 \cdot 1}{2} = 1$

So $\triangle OXP = \frac{\sqrt{3}}{2}$

So the required  area is

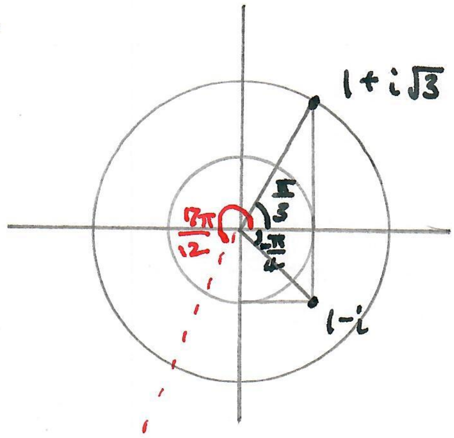
$$2 \times \left(\frac{\sqrt{3}}{2} - \left(4\pi - 7\sqrt{3}\right)\right)$$

$$= 2\left(\frac{\sqrt{3}}{2} + \frac{14\sqrt{3}}{2} - 4\pi\right)$$

$$= \underline{\underline{15\sqrt{3} - 8\pi}}$$ as required.

7) $z^3 = (1+i\sqrt{3})^8 (1-i)^5$

write these in polar coords:



$z^3 = \left(2e^{i\frac{\pi}{3}}\right)^8 \left(\sqrt{2}e^{-i\frac{\pi}{4}}\right)^5$

$= 256 \times 4\sqrt{2}\, e^{i\left(\frac{8\pi}{3} - \frac{5\pi}{4}\right)}$

$= 1024\sqrt{2}\, e^{i\frac{17\pi}{12}}$ and also $e^{i\left(\frac{17\pi}{12}+2n\pi\right)}$, particularly

$e^{i\frac{41\pi}{12}}$ and $e^{i\frac{65}{12}\pi}$

Taking $\sqrt[3]{}$s:

So $z = \underline{\underline{8\sqrt{2}\, e^{i\frac{17}{36}\pi}}}$

or $8\sqrt{2}\, e^{i\frac{41}{36}\pi} = \underline{\underline{8\sqrt{2}\, e^{-i\frac{31}{36}\pi}}}$

or $8\sqrt{2}\, e^{i\frac{65}{36}\pi} = \underline{\underline{8\sqrt{2}\, e^{-i\frac{7}{36}\pi}}}$

(In the form required, $k = 8$)

8) 'Coupled' ODES are an interesting idea... essentially we substitute one into the other and then back again - but we also differentiate one to get a 'third' reference point.

$$\frac{dx}{dt} = x + \frac{2}{3}y \quad \text{———} \quad ①$$

$$\frac{dy}{dt} = 3y - \frac{3}{2}y \quad \text{———} \quad ②$$

From ①:

$$\frac{d^2x}{dt^2} = \frac{dx}{dt} + \frac{2}{3}\frac{dy}{dt}$$

From ②:

$$= \frac{dx}{dt} + \frac{2}{3}\left(3y - \frac{3}{2}x\right)$$

$$= \frac{dx}{dt} + 3\left(\frac{dx}{dt} - x\right) - x$$

$$= 4\frac{dx}{dt} - 4x.$$

So $\frac{d^2x}{dt^2} - 4\frac{dx}{dt} + 4x = 0$

This has auxiliary equation

$$\lambda^2 - 4\lambda + 4 = 0$$

$$(\lambda - 2)^2 = 0$$

which points to a general solution

$$x = Ae^{2t} + Bte^{2t} = e^{2t}(A + Bt) \quad \text{———} \quad ③$$

So $\frac{dx}{dt} = 2Ae^{2t} + Be^{2t} + 2Bte^{2t}$

$= e^{2t}(2A + B + 2Bt)$

and we know from ① this equals $x + \frac{2}{3}y$

$= e^{2t}(A + Bt) + \frac{2}{3}y$

So $\frac{2}{3}y = e^{2t}(2A + B + 2Bt - A - Bt)$

$= e^{2t}(A + Bt + B)$ ——— ④

By now we know enough to use the initial condition $t = 0,\ x = 1,\ y = 3$:

From ③: $1 = 1.A$ $\qquad A = 1$

from ④: $3 \times \frac{2}{3} = 1(A + B)$ $\qquad B = 1$

So the solution is

$$\underline{\underline{x = (1+t)e^{2t}}}$$

$\frac{2}{3}y = (2+t)e^{2t}$ $\qquad \underline{\underline{y = \left(3 + \frac{3t}{2}\right)e^{2t}}}$