MADAS FP2 Paper M.

1) $\frac{d^2y}{dx^2} + 5\frac{dy}{dx} + 6y = 12(x+e^x)$.

AE: $\lambda^2 + 5\lambda + 6 = 0$

$(\lambda+2)(\lambda+3) = 0 \qquad \lambda = -2 \text{ or } -3$

So the CF is $y = A\cancel{x}e^{-2x} + Be^{-3x}$.

For a PI, consider $y = Px + Qe^x + R$

then $\dot{y} = P + Qe^x$

$\ddot{y} = Qe^x$

So $\ddot{y} + 5\dot{y} + 6y = Qe^x + 5Qe^x + 5P + 6Qe^x + 6R + 6Px$

$= 12Qe^x + (5P+6R) + 6Px$

$= 12x + 12e^x$ (given)

Comparing coeffs:

$e^x: \; 12Q = 12 \Rightarrow Q = 1$

$x: \; 6P = 12 \Rightarrow P = 2$

$1: \; 5P + 6R = 0 \Rightarrow 6R = -10$

$R = -\frac{5}{3}$

So the general solution is

$$\underline{\underline{y = Ae^{-2x} + Be^{-3x} + e^x + 2x - \frac{5}{3}}}$$

2) $$I = \int_e^{\infty} \frac{1-\ln x}{x^2}\,dx$$

'Usually' these problems are best done by 'integration by parts'. This uses the 'product formula' as :

$$\frac{d}{dx}(uv) = u\frac{dv}{dx} + v\frac{du}{dx}$$

So rearranging:

$$\int u\frac{dv}{dx}\,dx = uv - \int v\frac{du}{dx}\,dx.$$

And it's useful to draw a little grid:

u	du
dv	v

In this case: let $u = 1-\ln x$: $\frac{du}{dx} = \frac{-1}{x}$

$v = \frac{1}{x^2}$: $\frac{dv}{dx} = \frac{-1}{x}$

$1-\ln x$	$-\frac{1}{x}$
$-\frac{1}{x}$	$\frac{1}{x^2}$

So $$I = \left[-\frac{(1-\ln x)}{x} - \int \frac{1}{x^2}\,dx \right]_e^{\infty}$$

$$= \lim_{K\to\infty} \left[\cancel{-\frac{1}{x}} + \frac{\ln x}{x} + \cancel{\frac{1}{x}} \right]_e^K$$

$$= \lim_{K\to\infty} \left[\frac{\ln K}{K} - \frac{\ln e}{e} \right]$$

$\ln K$ tends to ∞ slower than K, so $\lim_{K\to\infty} \frac{\ln K}{K} = 0$

So this equals $0 - \frac{\ln e}{e} = \underline{\underline{\frac{-1}{e}}}$

3) $$f(r) = r^2(r+1)^2 - (r-1)^2 r^2$$

$$= r^2\left[(r+1)^2 - (r-1)^2\right]$$

$$= r^2\left[\cancel{r^2} + 2r + \cancel{1} - \cancel{r^2} + 2r - \cancel{1}\right]$$

$$= 4r^3$$

So $4r^3 = r^2(r+1)^2 - (r-1)^2 r^2$

from $r=1$ to 20:

r	$4r^3$		
1	$1^2.2^2$	$-$	$0^2 1^2$
2	$2^2.3^2$	$-$	$1^2 2^2$
3	$3^2.4^2$	$-$	$2^2.3^2$
.			
.			
.			
.			
19	$19^2.20^2$	$-$	$18^2 19^2$
20	$20^2.21^2$	$-$	$19^2.20^2$

$$\Sigma 4r^3 = 20^2.21^2 - 0^2.1^2$$

$$= 400.441 = 176400$$

So dividing by 4: $\sum_1^{20} r^3 = 44100$ as required.

4) De Moivre's theorem gives $\text{cis}\,4\theta = (\text{cis}\,\theta)^4$

so (letting $c = \cos\theta$, $s = \sin\theta$):

$$\cos 4\theta + i\sin\theta = c^4 + 4ic^3s - 6c^2s^2 - 4ics^3 + s^4$$

Taking real and imaginary parts:

$$\cos 4\theta = c^4 - 6c^2s^2 + s^4$$

$$\sin 4\theta = 4c^3s - 4cs^3$$

so $$\tan 4\theta = \frac{\sin 4\theta}{\cos 4\theta}$$

$$= \frac{4c^3s - 4cs^3}{c^4 - 6c^2s^2 + s^4}$$

Divide through by c^4:

$$= \frac{4\frac{s}{c} - 4\frac{s^3}{c^3}}{1 - 6\frac{s^2}{c^2} + \frac{s^4}{c^4}}$$

$$= \frac{4\tan\theta - 4\tan^3\theta}{1 - 6\tan^2\theta + \tan^4\theta}$$ as required.

5) a) 'standard results' we take to mean standard expansions for e^x and $\sin x$, without needing to develop Maclaurin or Taylor results from scratch.

So $y = e^{2x} \sin 3x$

$$= \left(1 + 2x + \frac{(2x)^2}{2} + \frac{(2x)^3}{6} + \frac{(2x)^4}{24} + o(x^5)\right)$$

$$\times \left(3x - \frac{(3x)^3}{6} + o(x^5)\right)$$

Multiplying out:

$$= 3x + 6x^2 + \frac{3 \times 4x^3}{2} + \frac{3 \times 8x^4}{6} + o(x^5)$$

$$- \frac{27x^3}{6} - \frac{27 \times 2x^4}{6} + o(x^5)$$

$$= 3x + 6x^2 + \left(6 - \frac{9}{2}\right)x^3 + (4 - 9)x^4 + o(x^5)$$

$$= \underline{\underline{3x + 6x^2 + \frac{3}{2}x^3 - 5x^4 + o(x5)}}$$

b) Using the approximation:

$$\int_0^{0.1} e^{2x} \sin 3x \, dx \approx \int_0^{0.1} 3x + 6x^2 + \frac{3}{2}x^3 - 5x^4 dx$$

$$= \left[\frac{3}{2}x^2 + \frac{6}{3}x^3 + \frac{3}{8}x^4 - \frac{5}{5}x^5\right]_0^{0.1}$$

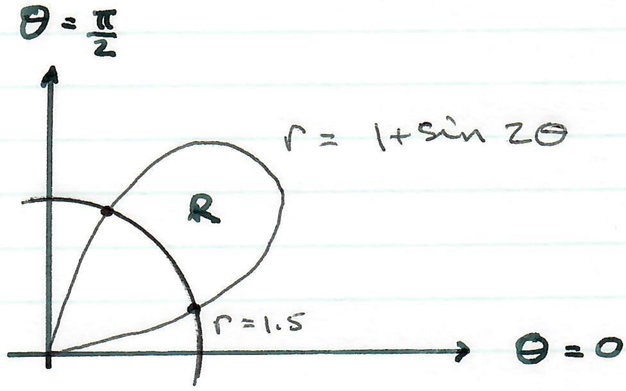
$$= 0.01 \times 1.5 + 0.001 \times 2 + 0.0001 \times 0.375 - 0.00001$$

$$= \begin{array}{r} 0.015 \\ + \ 0.002 \\ + \ 0.0000375 \\ \hline 0.0170375 \\ - \ 0.00001 \end{array}$$

$= \cancel{0.00}$

$= \underline{\underline{0.170275}}$

6)



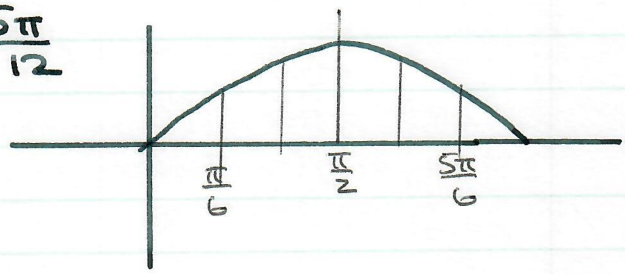
a) Points of intersection are given by:

$$r = 1.5 = 1 + \sin 2\theta$$

$$\sin 2\theta = 0.5$$

$$2\theta = \frac{\pi}{6} \text{ or } \frac{5\pi}{6}$$

So $\quad \theta = \frac{\pi}{12}$ or $\frac{5\pi}{12}$



So $\quad R = \frac{1}{2}\int_{\frac{\pi}{12}}^{\frac{5\pi}{12}} (1 + \sin 2\theta)^2 \, d\theta - \frac{1.5^2}{2}\left(\frac{\pi}{3}\right)$

$$= (1 + \sin 2\theta)^2 = 1 + 2\sin 2\theta + \sin^2 2\theta$$

And $\quad \cos 4\theta = \cos^2 2\theta - \sin^2 2\theta$

so $\quad \sin^2 2\theta = \cos^2 2\theta - \cos 4\theta$

$$= 1 - \sin^2 2\theta - \cos 4\theta$$

so $\quad 2\sin^2 2\theta = 1 - \cos 4\theta$

$$\sin^2 2\theta = \frac{1}{2}(1-\cos 4\theta)$$

$$\text{So } R = \frac{1}{2}\int_{\frac{\pi}{12}}^{\frac{5\pi}{12}} 1 + 2\sin 2\theta + \frac{1}{2} - \frac{1}{2}\cos 4\theta \, d\theta - \frac{3}{8}\pi$$

$$= \frac{1}{2}\int_{\frac{\pi}{12}}^{\frac{5\pi}{12}} \frac{3}{2} + 2\sin 2\theta - \frac{1}{2}\cos 4\theta \, d\theta - \frac{3}{8}\pi$$

$$= \frac{1}{2}\left[\frac{3}{2}x - \frac{2}{2}\cos 2\theta - \frac{1}{8}\sin 4\theta\right]_{\frac{\pi}{12}}^{\frac{5\pi}{12}} - \frac{3\pi}{8}$$

$$= \frac{1}{2}\left[\frac{15\pi}{24} - \cos\frac{5\pi}{6} - \frac{1}{8}\sin\frac{5\pi}{3}\right.$$

$$\left. - \frac{3\pi}{24} + \cos\frac{\pi}{6} + \frac{1}{8}\sin\frac{\pi}{3}\right] - \frac{3\pi}{8}$$

$$= \frac{1}{2}\left[\frac{\pi}{2} + \frac{\cancel{2}\sqrt{3}}{\cancel{2}} + \frac{\cancel{2}\sqrt{3}}{8\cdot\cancel{2}}\right] - \frac{3\pi}{8}$$

$$= -\frac{\pi}{8} + \frac{9}{16}\sqrt{3}$$

$$= \underline{\underline{\frac{1}{16}(9\sqrt{3} - 2\pi)}} \quad \text{as required.}$$

17) $5\cosh x + 3\sinh x = 12$ —— ①

[We want to use the identity

$$\cosh(x+\alpha) = \cosh x \cosh\alpha + \sinh x \sinh\alpha.$$

This isn't given in the formula book but it's easily proved by saying

$$RHS = \frac{(e^{x}+e^{-x})(e^{\alpha}+e^{-\alpha})}{4} + \frac{(e^{x}-e^{-x})(e^{\alpha}-e^{-\alpha})}{4}$$

$$= \frac{e^{x+\alpha} + \cancel{e^{-x+\alpha}} + \cancel{e^{x-\alpha}} + e^{-x-\alpha} + e^{x+\alpha} - \cancel{e^{-x+\alpha}} - \cancel{e^{x-\alpha}} + e^{-x-\alpha}}{4}$$

$$= \frac{2e^{x+\alpha} + 2e^{-x-\alpha}}{4} = \cosh(x+\alpha) = LHS.]$$

That done, we can now recast ① as:

$$R\cosh x \cosh\alpha + R\sinh x \sinh\alpha = 12R$$

(we'll probably end up with a different R)

So we can say:

$$R\cosh\alpha = 5 = \frac{R}{2}(e^{\alpha}+e^{-\alpha}) \quad ②$$

$$R\sinh\alpha = 3 = \frac{R}{2}(e^{\alpha}-e^{-\alpha}) \quad ③$$

② + ③ : $Re^{\alpha} = 8$

② − ③ : $Re^{-\alpha} = 2$

$$e^{2\alpha} = \frac{8}{2} = 4$$

$$e^{\alpha} = 2$$

$$\underline{\alpha = \ln 2}$$

$$R.2 = Re^{\ln 2} = 8 \Rightarrow R = 4.$$

So $4\cosh x \cosh(\ln 2) + 4\sinh x \sinh(\ln 2)$

$$= 4\cosh(x+\ln 2) = 12$$

$$\Rightarrow \cosh(x+\ln 2) = 3$$

$$\Rightarrow \frac{e^{(x+\ln 2)} + e^{(x-\ln 2)}}{2} = 3$$

$$= \frac{2e^{x} + \frac{1}{2}e^{-x}}{2} = 3$$

$$2e^{x} + \frac{1}{2}e^{-x} = 6$$

$$e^{2x} - 3e^{x} + \frac{1}{4} = 0$$

$$e^{x} = \frac{3 \pm \sqrt{9 - \frac{4.1}{4}}}{2} = \frac{3 \pm \sqrt{8}}{2}$$

So $x = \ln\left(\frac{3}{2} \pm \sqrt{2}\right)$ as required.

$$A = \frac{3}{2}$$

$$B = \sqrt{2}.$$

8) a) $y = \arctan(x)$

(Formula book gives $\frac{dy}{dx} = \frac{1}{1+x^2}$, but here we'll do a proof)

$x = \tan y$

As a standard result, (or by using the 'quotient rule'):

$$\frac{dx}{dy} = \sec^2 y = 1 + \tan^2 y$$
$$= 1 + x^2$$

So $\frac{dy}{dx} = \frac{1}{1+x^2}$ as required.

b) Here $f(x) = \arctan \sqrt{x}$

So using the chain rule:

$$f'(x) = \frac{df}{dx} = \frac{1}{1+x} \cdot \frac{d(\sqrt{x})}{dx} = \frac{1}{1+x} \; \frac{1}{2\sqrt{x}}$$
$$= \frac{1}{2}(1+x)^{-1}(x)^{-\frac{1}{2}}$$

$$f''(x) = \frac{d^2f}{dx^2} = \frac{1}{2}\left((1+x)^{-1}\left(-\frac{1}{2}\right)\left(x^{-\frac{3}{2}}\right) + (x)^{-\frac{1}{2}}\left((-1)(1+x)^{-2}\right) \right)$$
$$= \frac{1}{4}\left(-\frac{(1+x)}{(1+x)^2 \, 2\sqrt{x^3}} \right) - \frac{1}{2}\left(\frac{x}{(1+x)^2 \, 2\sqrt{x^3}} \right)$$
$$= \frac{\frac{1}{4}(-1-x-2x)}{(1+x)^2(\sqrt{x})^3}$$
$$= \underline{\underline{-\frac{1}{4}(3x+1)(x+1)^{-2}\, x^{-\frac{3}{2}}}} \text{ as required}$$

So RHS $= \dfrac{-x}{x^2+2} + \dfrac{4x}{4x^2+3}$

$$= -\frac{1}{2}\frac{2x}{x^2+2} + \frac{1}{2}\frac{8x}{4x^2+3}$$

Integrating ... each of these has the form

$\dfrac{\frac{du}{dx}}{u}$ so the integral is:

$$\frac{1}{2}\left(\ln(4x^2+3) - \ln(x^2+2)\right) + C$$

$$= \frac{1}{2}\ln\left(\frac{4x^2+3}{2x^2+4}\right) + C$$

So combining with the LHS:

$$y = \frac{1}{2x}\ln\left(\frac{4x^2+3}{2x^2+4}\right) + \frac{C}{x}$$

Using the initial condition:

$$\frac{1}{2}\ln\frac{7}{6} = \frac{1}{2}\ln\left(\frac{4+3}{2+4}\right) + C = \frac{1}{2}\ln\frac{7}{6} + C$$

So $C = 0$, and the answer required

is: $\underline{\underline{y = \dfrac{1}{2x}\ln\left(\dfrac{4x^2+3}{2x^2+4}\right)}}$