

1)a) $\text{LHS} = \cosh(A-B) = \dfrac{e^{A-B} + e^{-A+B}}{2}$

$\text{RHS} = \cosh A \cosh B - \sinh A \sinh B$

$= \dfrac{(e^{A}+e^{-A})(e^{B}+e^{-B}) - (e^{A}-e^{-A})(e^{B}-e^{-B})}{4}$

$= \dfrac{\cancel{e^{A+B}} + e^{-A+B} + e^{A-B} + \cancel{e^{-A-B}} - \cancel{e^{A+B}} + e^{-A+B} + e^{A-B} - \cancel{e^{-A-B}}}{4}$

$= \dfrac{2e^{A-B} + 2e^{-A+B}}{4}$ = LHS as required.

b) $\cosh(x - \ln 3) = \sinh x$

$\text{LHS} = \cosh(x - \ln 3)$

$= \cosh(x)\cosh(\ln 3) - \sinh(x)\sinh(\ln 3)$

$= \cosh(x)\left(\dfrac{e^{\ln 3} + e^{-\ln 3}}{2}\right) - \sinh(x)\left(\dfrac{e^{\ln 3} - e^{-\ln 3}}{2}\right)$

$= \cosh(x)\left(\dfrac{3+\frac{1}{3}}{2}\right) - \sinh(x)\left(\dfrac{3-\frac{1}{3}}{2}\right)$

$= \dfrac{5}{3}\cosh(x) - \dfrac{4}{3}\sinh(x)$

$= \sinh(x)$ (given)

So $\dfrac{5}{3}\cosh(x) = \dfrac{7}{3}\sinh(x)$

$\dfrac{5}{3}\dfrac{(e^{x}+e^{-x})}{2} = \dfrac{7}{3}\dfrac{(e^{x}-e^{-x})}{2}$

$\dfrac{5}{3}(e^{2x}) + \dfrac{5}{3} = \dfrac{7}{3}(e^{2x}) - \dfrac{7}{3}$

$$\frac{12}{3} = \frac{2}{3}e^{2x}$$

$$e^{2x} = 6$$

$$x = \frac{1}{2}\ln 6 = \ln\sqrt{6}$$

2) $f(x) = \ln(1+\cos 2x)$ $\quad 0 \leqslant x < \frac{\pi}{2}$

$$f'(x) = \frac{1}{(1+\cos 2x)}(-2\sin 2x)$$

$$= \frac{-2\sin 2x}{1+\cos 2x}$$

so in the required expression:

$$\text{RHS} = -2-\frac{1}{2}(f'(x))^2 = -2-\frac{1}{2}\left(\frac{-2\sin 2x}{1+\cos 2x}\right)^2$$

$$= \frac{-2(1+\cos 2x)^2 - 2\sin^2 2x}{(1+\cos 2x)^2}$$

$$= \frac{-2-4\cos 2x \underbrace{-\cancel{2\cos^2 2x}-\cancel{2\sin^2 2x}}_{=-2}}{(1+\cos 2x)^2}$$

$$= \frac{-4-4\cos 2x}{(1+\cos 2x)^2} = \frac{-4}{1+\cos 2x}$$

And the LHS is:

$$f''(x) = \frac{-4(1+\cos 2x)\cos 2x - 2\times 2\sin 2x \sin 2x}{(1+\cos 2x)^2}$$

$$= \frac{-4\cos 2x - \underbrace{\cancel{4\cos^2 2x - 4\sin^2 2x}}_{=-4}}{(1+\cos 2x)^2}$$

$$= \frac{-4(\cos 2x + 1)}{(1+\cos 2x)^2}$$

$$= \frac{-4}{(1+\cos 2x)}$$

So LHS = RHS and the expression is proved:

$$\underline{\underline{f''(x) = -2 - \tfrac{1}{2}(f'(x))^2}}$$

c) This calls for the Taylor expansion, though since it wants it around $x=0$ (it doesn't say this, though) it's actually the Maclaurin expansion.

We know the approximation is

$$f(x) = f(0) + xf'(0) + \frac{x^2}{2}f''(0) + \frac{x^3}{6}f'''(0) + \frac{x^4}{24}f''''(0) + o(5)$$

and the answer given suggests we'll need to go up to the x^4 term.

So $f'''(x) = 2\left(-\frac{1}{2} f'(x) f''(x)\right)$

$= -f'(x) f''(x)$

$f''''(x) = -f'(x) f'''(x) - \left(f''(x)\right)^2$

Then:

$f(0) = \ln(1+1) = \ln 2$

$f'(0) = 0$

$f''(0) = -2$

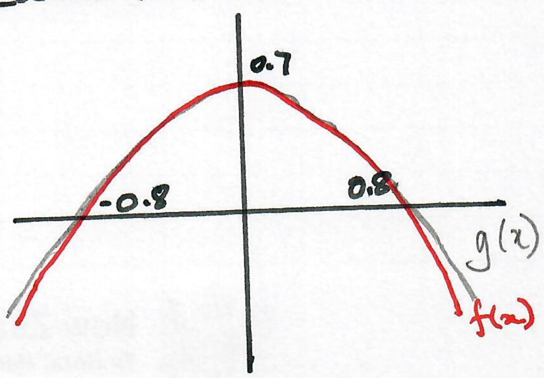
$f'''(0) = 0$

$f''''(0) = -(-2)^2 = -4$

So $f(x) = \ln 2 + 0 - x^2 + 0 - \frac{x^4}{6}$

$= \ln 2 - x^2 - \frac{x^4}{6}$ as required.

* Sometimes it's worth looking at a picture... it's easy to trot out a Taylor expansion but it's well worth using Geogebra to plot both the expansion and the original function. Doing this here gives a beautiful demonstration of how well they match, over a range at least $-0.8 \leqslant x \leqslant 0.8$.



3) a) $u_r = \frac{r}{6}(r+1)(4r+11)$ (1)

$= \frac{r}{6}(4r^2 + 15r + 11)$

$u_{r-1} = (r-1)\frac{r}{6}(4(r-1)+11)$

$= \frac{r}{6}(r-1)(4r+7)$

$= \frac{r}{6}(4r^2 + 3r - 7)$

So $u_r - u_{r-1} = \frac{r}{6}\begin{pmatrix} 4r^2 + 15r + 11 \\ -4r^2 - 3r + 7 \end{pmatrix}$

$= \frac{r}{6}(12r + 18)$

$= r(2r+3)$ —— (2)

b) we note the terms given match the term in (2):

(and $u_0 = 0$ from (1))

$\cancel{u_1} - \cancel{u_0} = 1 \times 5$

$\cancel{u_2} - \cancel{u_1} = 2 \times 7$

$\cancel{u_3} - \cancel{u_2} = 3 \times 9$

$\vdots$

$u_{100} - \cancel{u_{99}} = 100\,(203)$

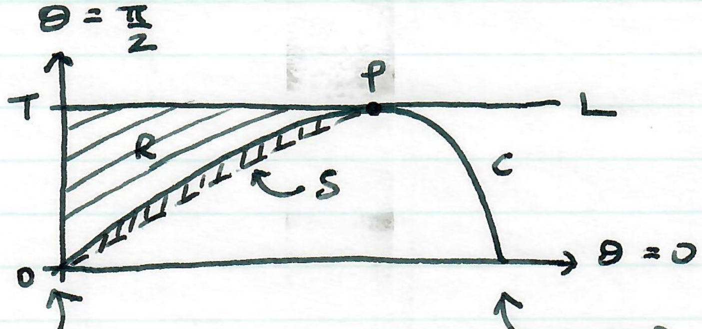
So adding up and cancelling.

$u_{100} - u_0 = 100 \times 203 + 99 \times 201 + \cdots$
$+ 2 \times 7 + 1 \times 5$

$u_{100} - 0 =$

$\frac{1}{6} \cdot 100 \cdot 101 \cdot (400 + 11) = \underline{\underline{691850}}$ and this is the sum required.

4) a)



curve ends here

$(\theta = \frac{\pi}{4})$

curve starts here

$(\theta = 0)$

$$C: r^2 = 2\cos 2\theta \qquad 0 \leq \theta \leq \frac{\pi}{4} \quad \text{——} \ ①$$

Looking for the point P the obvious thing to say is $\frac{dy}{dx} = 0$. But running this into ① is horrible. Instead we note $\frac{dy}{d\theta}$ is also 0, so we can say

$$y = r\sin\theta \qquad \text{(known)}$$

$$y^2 = r^2\sin^2\theta$$

$$= 2\cos 2\theta \times \sin^2\theta \qquad \text{from ①.}$$

Differentiating:

$$2y\frac{dy}{d\theta} = 2\left(2\cos 2\theta \sin\theta\cos\theta - 2\sin 2\theta \sin^2\theta\right)$$

Since at P $\frac{dy}{d\theta} = 0$, this gives:

$$0 = \cos 2\theta \cos\theta - \sin 2\theta \sin\theta$$

$$= (\cos^2\theta - \sin^2\theta)\cancel{\cos}\theta - 2\sin\theta\cancel{\cos}\theta\sin\theta$$

$$= \cos^2\theta - 3\sin^2\theta$$

so $\tan\theta = \left(\frac{1}{\sqrt{3}}\right)$: $\underline{\underline{\theta = \frac{\pi}{6}}}$

(or using trig identities:

$$\cos(2\theta + \theta) = 0$$

So $3\theta = \frac{\pi}{2}$

$$\underline{\underline{\theta = \frac{\pi}{6}}}$$

)

b) Area S under the curve

$$= \int_{\frac{\pi}{6}}^{\frac{\pi}{4}} r^2 \, d\theta$$

$$= \int_{\frac{\pi}{6}}^{\frac{\pi}{4}} \cos 2\theta \, d\theta = \frac{1}{2}\sin 2\theta \Big|_{\frac{\pi}{6}}^{\frac{\pi}{4}}$$

$$= \frac{1}{2}\left(\sin\frac{\pi}{2} - \sin\frac{\pi}{3}\right) = \frac{1}{2}\left(1 - \frac{\sqrt{3}}{2}\right)$$

Area of $\triangle$ TOP is $1 \times \overset{\sin}{\left(\frac{\pi}{6}\right)} \times \frac{1}{2} = =$

$$\frac{xy}{2} = \frac{1}{2}\cos\left(\frac{\pi}{6}\right)\sin\left(\frac{\pi}{6}\right) = \frac{1}{2}\cdot\frac{\sqrt{3}}{2}\cdot\frac{1}{2} = \frac{\sqrt{3}}{8}$$

So area $R = \frac{\sqrt{3}}{8} - \left(\frac{1}{2} - \frac{\sqrt{3}}{4}\right) = \frac{1}{8}(3\sqrt{3} - 4)$ as required.

5) $(2x - 4y^2)\dfrac{dy}{dx} + y = 0$

Not sure what 'reversing x and y' means,

but if we cast it in terms of $\dfrac{dx}{dy}$ ...

$$(2x - 4y^2)\frac{dy}{dx} = -y$$

$$\frac{dx}{dy} = \frac{2x-4y^2}{y} = -2\frac{x}{y} + 4y$$

So $\dfrac{dx}{dy} + 2\dfrac{x}{y} = 4y$

Use an IF $e^{\int \frac{2}{y}dy} = e^{2\ln y} = y^2$

Apply the IF:

$$y^2\frac{dx}{dy} + 2xy = 4y^3$$

$$\int \frac{d}{dy}(xy^2)\,dy = \frac{4}{4}y^4 + c$$

$$xy^2 = y^4 + c$$

$$x = y^2 - \frac{c}{y^2}$$

or $y^4 - xy^2 + c = 0$ so

$$y^2 = \frac{x \pm \sqrt{x^2 - 4c}}{2}.$$

6) a) $f(x) = (2x^2-1)\arcsin x + x\sqrt{1-x^2}$

$-1 \leqslant x \leqslant 1$

$$f'(x) = \frac{2(2x^2-1)}{2\sqrt{1-x^2}} + 4x\arcsin x$$

$$+ \frac{x(-2x)}{2\sqrt{1-x^2}} + \frac{2(1-x^2)}{2\sqrt{1-x^2}}$$

$$= \frac{2(2x^2-1) - 2x^2 + 2(1-x^2)}{2\sqrt{1-x^2}} + 4x\arcsin x$$

$$= \frac{\cancel{4x^2} \cancel{-2} \cancel{-2x^2} \cancel{+2} \cancel{-2x^2}}{2\sqrt{1-x^2}} + 4x\arcsin x$$

$$= \underline{\underline{4x\arcsin x.}}$$

... which is convenient:

b) $$\int_0^{\frac{\sqrt{2}}{2}} x\arcsin x \, dx = \left[\frac{f(x)}{4}\right]_0^{\frac{\sqrt{2}}{2}}$$

$$= \frac{1}{4}\left\{ \left(\frac{2\times 2}{4} - 1\right)\arcsin\left(\frac{\sqrt{2}}{2}\right) + \frac{\sqrt{2}}{2}\sqrt{1-\frac{2}{4}} \right.$$

$$\left. - (-1)\arcsin(0) + 0 \right\}$$

$$= \frac{1}{4}\left\{ 0 + \frac{\cancel{\sqrt{2}}}{2\cancel{\sqrt{2}}} + 0 + 0 \right\} = \underline{\underline{\frac{1}{8}}}$$

7) a) $\text{cis } 5\theta = (\text{cis}\theta)^5$

$$= \cos^5\theta + 5i\cos^4\theta\sin\theta - 10\cos^3\theta\sin^2\theta$$
$$-10i\cos^2\theta\sin^3\theta + 5\cos\theta\sin^4\theta$$
$$+ i\sin^5\theta$$

Separating real and imaginary parts:

① $\cos 5\theta = \cos^5\theta - 10\cos^3\theta\sin^2\theta + 5\cos\theta\sin^4\theta$
$= c^5 - 10c^3s^2 + 5cs^4$

② $\sin 5\theta = 5\cos^4\theta\sin\theta - 10\cos^2\theta\sin^3\theta + \sin^5\theta$
$= 5c^4s - 10c^2s^3 + s^5$

② ÷ ① gives:

$$\tan 5\theta = \frac{\sin 5\theta}{\cos 5\theta} = \frac{5c^4s - 10c^2s^3 + s^5}{c^5 - 10c^3s^2 + 5cs^4}$$

Divide through by c^5: $= \dfrac{5t - 10t^3 + t^5}{1 - 10t^2 + 5t^4}$

as required.

b) $t^4 - 10t^2 + 5 = 0$

This is the numerator of $\tan 5\phi$ divided by 0, so we can say

$\tan 5\phi = 0$ when $t(t^4 - 10t^2 + 5) = 0$
i.e. $t^5 - 10t^3 + 5t = 0$

i.e. when $\tan 5\phi = 0$.

Now $\tan 5\phi = 0$ when $5\phi = 0$ or $\pi, 2\pi, \ldots$
Discount $\phi = 0$ (outside the range $0 < \phi < \pi$)
so $\phi = \frac{\pi}{5}, \frac{2\pi}{5}, \frac{3\pi}{5}, \frac{4\pi}{5}$.

and $t = \tan\frac{\pi}{5}, \tan\frac{2\pi}{5}, \tan\frac{3\pi}{5}, \tan\frac{4\pi}{5}$.

c) $t^4 - 10t^2 + 5 = 0$ and we know its roots:

$$\tan\frac{4\pi}{5} = \frac{\sin\frac{4\pi}{5}}{\cos\frac{4\pi}{5}} = \frac{\sin\frac{\pi}{5}}{-\cos\frac{\pi}{5}} = -\tan\frac{\pi}{5}$$

Similarly $\tan\frac{3\pi}{5} = -\tan\frac{2\pi}{5}$

So since the product of the roots of the polynomial is the constant term,

$$(t - t_1)(t - t_2)(t - t_3)(t - t_4) = t^4 - 10t^2 + 5$$

gives $t_1 t_2 t_3 t_4 = 5$

or $\tan\frac{\pi}{5}\tan\frac{2\pi}{5}\tan\frac{3\pi}{5}\tan\frac{4\pi}{5} = 5$

$$\tan\frac{\pi}{5}\tan\frac{2\pi}{5}\left(-\tan\frac{2\pi}{5}\right)\left(-\tan\frac{\pi}{5}\right) = 5$$

$$\left(\tan\frac{\pi}{5}\tan\frac{2\pi}{5}\right)^2 = 5$$

So $\tan\frac{\pi}{5}\tan\frac{2\pi}{5} = \sqrt{5}$

as required.