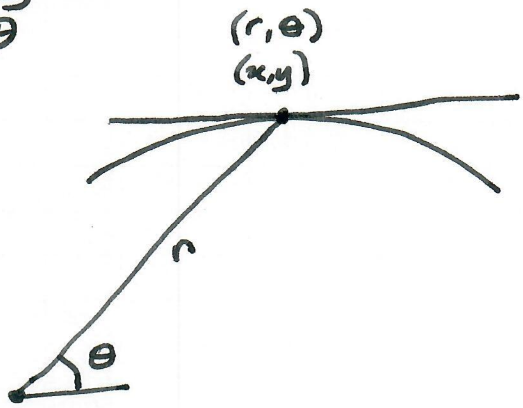
1) First thought is that we need to find where $\frac{dy}{dx} = 0$, but in fact it's easier (and just as valid) to look for $\frac{dy}{d\theta} = 0$.



Given $r = 1 + 2\cos\theta$,

we know

$y = r\sin\theta$

$= (1 + 2\cos\theta)\sin\theta$

$= \sin\theta + 2\cos\theta\sin\theta$

$$\frac{dy}{d\theta} = \cos\theta + 2(\cos^2\theta - \sin^2\theta)$$

$$= \cos\theta + 2(\cos^2\theta - 1 + \cos^2\theta)$$

$$= \cos\theta + 4\cos^2\theta - 2 = 0 \text{ at point P.}$$

So $4\cos^2\theta + \cos\theta - 2 = 0$

$$\cos\theta = \frac{-1 \pm \sqrt{1+32}}{2\cdot 4} = \frac{1}{8}(-1 \pm \sqrt{33})$$

($+\sqrt{33}$ gives the only viable solution)

So at P:

$$r = 1 + 2\cos\theta = 1 + \frac{1}{4}(-1 + \sqrt{33})$$

$$= \frac{1}{4}(3 + \sqrt{33})$$

2) $f(x) = \sinh x \cos x + \sin x \cosh x$ —— ①

$$\frac{df}{dx} = -\sinh x \sin x + \sin x \sinh x + \cosh x \cos x + \cos x \cosh x$$

$$= 2 \cos x \cosh x$$

From ①:

$$\frac{f(x)}{2\cos x \cosh x} = \frac{1}{2}(\tanh x + \tan x)$$

$$\text{So } \frac{f(x)}{f'(x)} = \text{———}$$

$$\frac{1}{f}\cdot\frac{df}{dx} = \frac{f'(x)}{f(x)} = \frac{2}{\tanh x + \tan x}$$

$$\text{So } \int \frac{1}{f}\, df = \int \frac{2}{\tanh x + \tan x}\, dx$$

$$= \ln|f(x)| + C$$

$$= \ln|\sinh x \cos x + \sin x \cosh x| + C$$

3)
$$I = \int_1^4 \frac{3}{(x+9)\sqrt{x}}\,dx$$

Let $u = \sqrt{x}$ then $u^2 = x$

$$2du = \frac{dx}{\sqrt{x}}$$

$$\int_1^4 \frac{3}{(x+9)\sqrt{x}}\,dx$$

$$= 6\int \frac{1}{u^2+9}\,du \quad \text{(ignoring limits)}$$

$$= 6 \times \frac{1}{3}\arctan\left(\frac{u}{3}\right)$$

$$= 2\arctan\left(\frac{\sqrt{x}}{3}\right)\Big|_1^4 \quad \text{(recalling limits)}$$

$$= \underline{\underline{2\left(\arctan\left(\frac{2}{3}\right) - \arctan\left(\frac{1}{3}\right)\right)}}$$

Use the identity $\tan(A-B) = \dfrac{\tan A - \tan B}{1 + \tan A \tan B}$:

$$2\left(\arctan\left(\frac{2}{3}\right) - \arctan\left(\frac{1}{3}\right)\right)$$

$$= 2\,\frac{\tan\left(\arctan\left(\frac{2}{3}\right)\right) - \tan\left(\arctan\left(\frac{1}{3}\right)\right)}{1 + \tan\left(\arctan\left(\frac{2}{3}\right)\right)\tan\left(\arctan\left(\frac{1}{3}\right)\right)}$$

$$= 2\,\frac{\left(\frac{2}{3} - \frac{1}{3}\right)}{1 + \frac{2}{3} \times \frac{1}{3}} = 2 \times \frac{3}{11}$$

This shows $\arctan\frac{2}{3} - \arctan\frac{1}{3} = \arctan\frac{3}{11}$

So $I = \underline{\underline{2\arctan\frac{3}{11}}}$ as required.

4) This seems to me to be university-level rather than A-level, but here goes...

$$\frac{d^2y}{dx^2} - 3\frac{dy}{dx} + 2y = 20\sin 2x. \qquad \text{——— } ①$$

LHS is a classic 'quadratic' form where the 'auxiliary equation' is:

$$\lambda^2 - 3\lambda + 2 = 0$$

$$\lambda = 1 \text{ or } 2$$

This leads to a Complementary Function

$$y = Ae^{x} + Be^{2x}.$$

(This satisfies $\frac{d^2y}{dx^2} - 3\frac{dy}{dx} + 2y = 0$ for any A and B: it allows many solutions, though it doesn't yet solve the equation with the $20\sin 2x$ term.)

Look for a particular solution of the form

$$y = P\cos 2x + Q\sin 2x$$

then

$$y' = -2P\sin 2x + 2Q\cos 2x$$

$$y'' = -4P\cos 2x - 4Q\sin 2x$$

In ①:

$$-4P\cos 2x - 4Q\sin 2x + 6P\sin 2x - 6Q\cos 2x + 2P\cos 2x + 2Q\sin 2x = 20\sin 2x$$

Equating coeffs of $\sin 2x$ then $\cos 2x$ gives

$$-2P - 6Q = 0 \Rightarrow P = -3Q$$

$$-2Q + 6P = 20 \Rightarrow 10Q = -10$$

$$\Rightarrow Q = -1, \quad P = 3.$$

So the general solution is

$$y = Ae^{x} + Be^{2x} + 3\cos 2x - \sin 2x.$$

At $x = 0$ $y = 1$:

$$1 = A + B + 3 \longrightarrow ②$$

$$y' = Ae^{x} + 2Be^{2x} - 6\sin x - 2\cos x$$

At $x = 0$ $y' = -5$:

$$-5 = A + 2B - 2 \longrightarrow ③$$

② and ③ give
$$\left.\begin{aligned} A + B &= -2 \\ A + 2B &= -3 \end{aligned}\right\} \quad \begin{aligned} B &= -1 \\ A &= -1 \end{aligned}$$

So the solution required is

$$\underline{\underline{y = -e^{x} - e^{2x} + 3\cos 2x - \sin 2x}}$$

5)
$$f(r) = \frac{2}{r(r+1)(r+2)}$$

$$= \frac{A}{r} + \frac{B}{r+1} + \frac{C}{r+2}$$

$$= \frac{A(r+1)(r+2) + Br(r+2) + Cr(r+1)}{\sim}$$

$$= \frac{A(r^2+3r+2) + B(r^2+2r) + C(r^2+r)}{\sim}$$

Equate coeffs:
$$r^2: \; A + B + C = 0$$
$$r: \; 3A + 2B + C = 0$$
$$1: \; 2A = 2$$

So $A = 1$, $B = -2$, $C = 1$: $f(r) = \frac{1}{r} - \frac{2}{r+1} + \frac{1}{r-2}$

5b) $$f(r) = \frac{1}{r} - \frac{2}{r+1} + \frac{1}{r+2}$$

$$\sum_{r=1}^{n} f(r) = \frac{1}{1} - \frac{2}{2} + \cancel{\frac{1}{3}}$$

$$+ \frac{1}{2} - \cancel{\frac{2}{3}} + \cancel{\frac{1}{4}}$$

$$+ \cancel{\frac{1}{3}} - \cancel{\frac{2}{4}} + \cancel{\frac{1}{5}}$$

$$+ \cancel{\frac{1}{4}} - \cancel{\frac{2}{5}} + \cancel{\frac{1}{6}}$$

$$- \; - \; - \; - \; -$$

$$+ \cancel{\frac{1}{n-2}} - \cancel{\frac{2}{n-1}} + \cancel{\frac{1}{n}}$$

$$+ \cancel{\frac{1}{n-1}} - \cancel{\frac{2}{n}} + \frac{1}{n+1}$$

$$+ \cancel{\frac{1}{n}} - \frac{2}{n+1} + \frac{1}{n+2}$$

$$= \frac{1}{1} - \frac{2}{2} + \frac{1}{2} + \frac{1}{n+1} - \frac{2}{n+1} + \frac{1}{n+2}$$

$$= \frac{1}{2} - \frac{1}{n+1} + \frac{1}{n+2}$$

$$= \frac{1}{2} - \frac{n+2-n-1}{(n+1)(n+2)}$$

$$= \frac{1}{2} - \frac{1}{(n+1)(n+2)}$$ as required

5c) As $r \to \infty$, $\frac{1}{(n+1)(n+2)} \to 0$

So $\sum_{r=1}^{\infty} f(r) = \frac{1}{2} - 0 = \frac{1}{2}$.

Also we know

$$\sum_{r=1}^{4} f(r) = \frac{1}{2} - \frac{1}{5 \times 6} = \frac{1}{2} - \frac{1}{30}$$

So the infinite sum given, which is $\frac{1}{2}\sum_{r=4}^{\infty} f(r)$,

is $\frac{1}{2}\Big(\underset{(\sum_1^\infty)}{\frac{1}{2}} - \underset{(\sum_1^4)}{\left(\frac{1}{2} - \frac{1}{30}\right)}\Big) = \frac{1}{2} \cdot \frac{1}{30} = \underline{\underline{\frac{1}{60}}}$

6) $y = 2x \arcsin(2x) + \sqrt{1-4x^2}$ —— ①

$$y' = 2x \frac{1}{\sqrt{1-4x^2}} \times 2 + 2\arcsin(2x) + \frac{1}{2} \cdot \frac{1}{\sqrt{1-4x^2}} \cdot (-8x)$$

$$= \frac{4x}{\sqrt{1-4x^2}} - \frac{4x}{\sqrt{1-4x^2}} + 2\arcsin 2x$$

$$= 2\arcsin 2x. \quad \text{—— ②}$$

Using a standard identity,

$$y'' = 2\frac{1}{\sqrt{1-4x^2}} \cdot 2 = \frac{4}{\sqrt{1-4x^2}}$$

So $\sqrt{1-4x^2} = \frac{4}{y''}$ —— ③

Substitute 2 and 3 into 1:

$$y = xy' + \frac{4}{y''}$$

$$(y - xy')y'' = 4$$

Differentiate again:

$$(\cancel{y'} - xy'' - \cancel{y'})y'' + (y - xy')y''' = 0$$

Rearrange:

$$\underline{\underline{(y - xy')y''' = x(y'')^2}} \quad \text{as required.}$$

7) $I = \int_0^\infty \frac{2}{1+2x} - \frac{x}{1+x^2}\,dx$

Both terms here have the form $\frac{dx}{x}$, which suggests the result will have ln terms - but if you consider the terms separately you end up with "ln ∞" which isn't good.

But if you just smash into it...

What we want is

$$\lim_{k\to\infty}\left(\int_0^k \frac{2}{1+2x} - \frac{x}{1+x^2}\,dx\right)$$

$$= \lim_{k\to\infty}\left(\ln(1+2x) - \frac{1}{2}\ln(1+x^2)\right)_0^k$$

$$= \lim_{k\to\infty}\frac{1}{2}\left[\ln(1+2x)^2 - \ln(1+x^2)\right]_0^k$$

$$= \lim_{k\to\infty} \frac{1}{2}\left[\ln \frac{(1+2x)^2}{1+x^2}\right]_0^k$$

$$= \lim_{k\to\infty} \frac{1}{2}\left[\ln \frac{(1+4k+4k^2)}{1+k^2} - \cancel{\ln\left(\frac{1}{1}\right)}\right]$$

$$= (\text{as } k\to\infty) \quad \frac{1}{2}\ln 4$$

$$= \underline{\underline{\ln 2}} \quad \text{as required.}$$

8) $f(z) = z^6 + 8z^3 + 64$

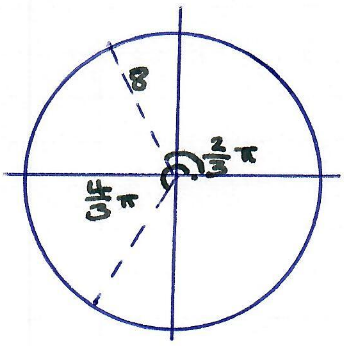
a) Let $w = z^3$: $\quad w^2 + 8w + 64 = 0$

$$w = \frac{-8 \pm \sqrt{8^2 - 4\times 64}}{2}$$

$$= \underline{\underline{-4 \pm 4\sqrt{3}\,i}} \quad \text{as required.}$$

In polar terms these are

$8\,\text{cis}\left(\frac{2\pi}{3}\right)$ and $8\,\text{cis}\left(\frac{4\pi}{3}\right)$



ie $8e^{i\frac{2\pi}{3}}$ and $8e^{i\frac{4\pi}{3}}$

and also (adding $2n\pi$) $\quad 8e^{i\frac{8}{3}\pi}$, $8e^{i\frac{14}{3}\pi}$

$8e^{i\frac{10\pi}{3}}$, $8e^{i\frac{16}{3}\pi}$

So if $z^3 = -4 + 4\sqrt{3}\,i = 8e^{i\frac{2\pi}{3}}$ etc,

$$z = \left(8e^{i\frac{2\pi}{3}}\right)^{\frac{1}{3}} \text{etc} = 2e^{i\frac{2\pi}{9}},\ 2e^{i\frac{8\pi}{9}},\ 2e^{i\frac{14\pi}{9}}$$

or $\underline{2e^{i\frac{2}{9}\pi},\ 2e^{i\frac{8}{9}\pi},\ 2^{-i\frac{4}{9}\pi}}$

expressing the arguments within $[-\pi, \pi]$.

Similarly if $z^3 = -4-4\sqrt{3}\,i$,

$$z = \underline{2e^{-i\frac{2}{9}\pi},\ 2e^{-i\frac{8}{9}\pi},\ 2^{i\frac{4}{9}\pi}}$$

c) i) If we express the polynomial as
$(z-\alpha)(z-\beta)\ \cdots\ (z-\xi)$ (6 factors: 6 roots)

we know that

$$-\frac{\text{coeff of } z^5}{\text{coeff of } z^6} = \text{sum of the 6 roots.}$$

So in this case, sum of the 6 roots $= 0$.

ii) Writing this out,

$$2e^{i\frac{2}{9}\pi} + 2e^{-i\frac{2}{9}\pi} + 2e^{i\frac{4}{9}\pi} + 2e^{-i\frac{4}{9}\pi} + 2e^{i\frac{8}{9}\pi} + 2e^{-i\frac{8}{9}\pi} = 0$$

Extracting the real part:

$$\cos\frac{2\pi}{9} + \cos\frac{2\pi}{9} + \cos\frac{4\pi}{9} + \cos\frac{4\pi}{9} + \cos\frac{8\pi}{9} + \cos\frac{8\pi}{9} = 0$$

$$\cos\frac{2\pi}{9} + \cos\frac{4\pi}{9} + \cos\frac{8\pi}{9} = 0$$

and $\cos\frac{6\pi}{9} = \cos\frac{2}{3}\pi = -\frac{1}{2}$

so adding: $\underline{\cos\frac{2\pi}{9} + \cos\frac{4\pi}{9} + \cos\frac{6\pi}{9} + \cos\frac{8\pi}{9} = -\frac{1}{2}}$