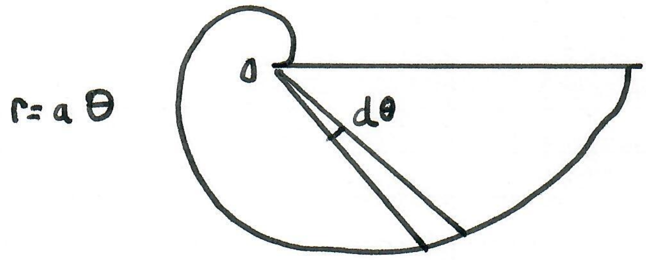
MADAS FP2 Paper J

1)



For calculating areas with polar coords we use $\frac{r^2}{2} d\theta$, so the area here is:

$$\int_0^{2\pi} \frac{(a\theta)^2}{2} \, d\theta$$

$$= \frac{a^2}{2} \left[\frac{\theta^3}{3} \right]_0^{2\pi}$$

$$= \frac{8 a^2 \pi^3}{2 \times 3} = \underline{\underline{\frac{4}{3} a^2 \pi^3}}$$

2) $f(x) = \frac{4x}{1-x^4}$

Factorise $1-x^4$ and break into fractions:

$$\frac{4x}{1-x^4} = \frac{A}{1+x} + \frac{B}{1-x} + \frac{Cx+D}{1+x^2}$$

So $A(1-x)(1+x^2) + B(1+x)(1+x^2) + (Cx+D)(1-x^2)$

$= 4x$

Take coeffs of x^3: $-A+B-C=0$

x^2: $A+B+D=0$

x: $-A+B+C=4$

1: $A+B+D=0$

This boils down to $A=-1,\ B=1,\ C=2,\ D=0.$

so $$\frac{4x}{1-x^4} = \frac{2x}{1+x^2} - \frac{1}{x-1} - \frac{1}{x+1}$$

b) So $\int_0^{\frac{1}{2}} f(x)\,dx =$

$$\int_0^{\frac{1}{2}} \frac{2x}{1+x^2}\,dx - \int_0^{\frac{1}{2}} \frac{1}{x-1}\,dx - \int_0^{\frac{1}{2}} \frac{1}{x+1}\,dx$$

for the first term we note

$$\frac{d}{dx}\left(\ln(1+x^2)\right) = \frac{1}{1+x^2}\cdot 2x,$$

and the other terms are also 'ln' terms,

so the integral is

$$\Big[\ln(1+x^2) - \ln(x-1) - \ln(x+1)\Big]_0^{\frac{1}{2}}$$

(actually all | |)

$$= \ln\left(\tfrac{5}{4}\right) - \ln\left(\tfrac{1}{2}\right) - \ln\left(\tfrac{3}{2}\right) - \cancel{\ln(1)} - \cancel{\ln(1)} - \cancel{\ln(1)}$$

$$= \ln\left(\tfrac{5}{4}\times 2\times\tfrac{2}{3}\right) = \underline{\underline{\ln\left(\tfrac{5}{3}\right)}}$$

3) a) $f(r) = \dfrac{1}{\sqrt{r+2}+\sqrt{r}} = \dfrac{(\sqrt{r+2}-\sqrt{r})}{(\sqrt{r+2})^2-(\sqrt{r^2})}$

$= \dfrac{\sqrt{r+2}-\sqrt{r}}{r+2-r} = \dfrac{\sqrt{r+2}-\sqrt{r}}{2}$

b) To find $\sum_{r=1}^{n} f(r)$ it looks like we should consider cases $n=$ odd and $n=$ even - but in fact it's simpler.

$f(1) = \frac{1}{2}(\sqrt{3} - \sqrt{1})$

$f(2) = \frac{1}{2}(\sqrt{4} - \sqrt{2})$

$f(3) = \frac{1}{2}(\sqrt{5} - \sqrt{3})$

$\vdots$

$f(n-2) = \frac{1}{2}(\sqrt{n} - \sqrt{n-2})$

$f(n-1) = \frac{1}{2}(\sqrt{n+1} - \sqrt{n-1})$

$f(n) = \frac{1}{2}(\sqrt{n+2} - \sqrt{n})$

So $\sum_{r=1}^{n} f(r) = \frac{1}{2}\left(\sqrt{n+2}+\sqrt{n+1} - \sqrt{2} - 1\right)$

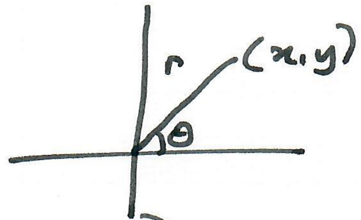
c) So $\sum_{r=1}^{48} f(r) = \dfrac{1}{2}\left(\sqrt{50}+\sqrt{49}-\sqrt{2}-1\right)$

$= \frac{1}{2}(5\sqrt{2}+7-\sqrt{2}-1)$

$= \underline{3+2\sqrt{2}}$ as required.

4) Treat this not as a generic trigonometric identity but as polar coordinates (which it is) and use

$$x = r\cos\theta$$
$$y = r\sin\theta$$



Then $r^2 = 2(r\cos\theta - r\sin\theta)$

$= 2(x-y)$

And $r^2 = x^2+y^2$, so

$$x^2+y^2 = 2(x-y)$$

or $x^2+y^2-2x+2y=0$

This is the usual equation for a ⊙ centre $(1,-1)$, radius $\sqrt{1^2+1^2+0} = \sqrt{2}$

and can be expressed

$$\underline{\underline{(x-1)^2+(y+1)^2 = 2}}$$

5) Maclaurin expansion is the special case of the Taylor expansion around $f(a)$, where 0. We use

$$f(x) = f(0) + x f'(0) + \frac{x^2}{2!}f''(0) + \frac{x^3}{3!}f'''(0) \ ---$$

So with $f(x) = \ln(2-e^x)$

$$f'(x) = \frac{-e^x}{2-e^x} = \frac{e^x}{e^x-2}$$

$$f''(x) = \frac{(e^x-2)e^x - e^x.e^x}{(e^x-2)^2}$$

$$= \frac{-2e^x}{(e^x-2)^2}$$

$$f'''(x) = \frac{-2(e^x-2)^{\cancel{2}}e^x + 2e^x.2(e^x-2)e^x}{(e^x-2)^{\cancel{4}3}}$$

$$= \frac{-2e^{2x} + 4e^x + 4e^{2x}}{(e^x-2)^3}$$

$$= \frac{2e^{2x} + 4e^x}{(e^x-2)^3}$$

When $x=0$: $f(0) = \ln 1 = 0$

$$f'(0) = \frac{-1}{2-1} = -1$$

$$f''(0) = \frac{-2.1}{(1-2)^2} = -2$$

$$f'''(0) = \frac{2.1+4.1}{(-1)^3} = -6$$

So the Maclaurin expansion gives

$$\ln(2-e^x) = 0 - 1.x - \frac{2}{2!}x^2 - \frac{6}{3!}x^3 + O(x^4)$$

$$= -x - x^2 - x^3 + O(x^4)$$

6) a) The identity to be proved here is often 'given' as a 'known'... so it can be hard to stay clear on what has to be developed in a 'proof.' Here goes...

$$y = \operatorname{arctanh} x$$

so $$x = \tanh y = \frac{\sinh y}{\cosh y} = \frac{e^{y}-e^{-y}}{e^{y}+e^{y}}$$

(by definition)

Rearranging,

$$x = \frac{e^{2y}-1}{e^{2y}+1}$$

$$x + xe^{2y} = e^{2y} - 1$$

$$x+1 = e^{2y}(1-x)$$

$$e^{2y} = \frac{1+x}{1-x}$$

So $$2y = \ln\left(\frac{1+x}{1-x}\right)$$

$$y = \frac{1}{2}\ln\left(\frac{1+x}{1-x}\right)$$ as required.

b) $$x = \tanh\left(\ln\sqrt{6x}\right)$$

So $\text{artanh}\, x = \ln\sqrt{6x}$

using the equation derived in (a):

$$\frac{1}{2}\ln\left(\frac{1+x}{1-x}\right) = \ln\sqrt{6x}$$

Removing ln and $\sqrt{\ }$:

$$\frac{1+x}{1-x} = 6x$$

$$1 + x = 6x - 6x^2$$

$$6x^2 - 5x + 1 = 0$$

$$(2x-1)(3x-1) = 0$$

$$x = \frac{1}{3} \text{ or } \frac{1}{2}$$

7) $$x^2 \frac{dy}{dx} + xy(x+3) = 1$$

Express this in 'standard form':

$$\frac{dy}{dx} + \frac{(x+3)}{x}y = \frac{1}{x^2}$$

and consider an 'integrating factor' $e^{\int \frac{x+3}{x} dx}$.

$$\int \frac{x+3}{x}\,dx = \int 1+\frac{3}{x}\,dx$$

$$= x+\ln x^3$$

So the integrating factor is $e^{x+\ln x^3}$

$$= x^3e^x.$$

So we rewrite the equation as

$$x^3e^x\frac{dy}{dx} + x^3e^x\frac{(x+3)}{x}y = \frac{d}{dx}\left(x^3e^xy\right)$$

$$= xe^x.$$

Integrating both sides, this means

$$x^3e^xy = \int xe^x\,dx. \quad \text{———} \; ①$$

To calculate $\int xe^x\,dx$, consider integrating by parts: $\frac{d}{dx}(xe^x) = xe^x + e^x$

So $xe^x = \int xe^x\,dx + e^x + A$

So $\int xe^x\,dx = xe^x - e^x + A$

So in ①: $x^3e^xy = xe^x - e^x + A$

$$\underline{\underline{y = \frac{1}{x^2} - \frac{1}{x^3} + \frac{Ae^{-x}}{x^3}}}$$

Now substitute for point (1,1):

$$1 = 1 - 1 + \frac{A}{1}e^{-1}$$

$$1 = Ae^{-1}$$

$$\underline{A = e}$$

So $$y = \frac{1}{x^2} - \frac{1}{x^3} + \frac{e^{1-x}}{x^3}$$

Substitute for point (2, k):

$$k = \frac{1}{4} - \frac{1}{8} + \frac{e^{1-2}}{8}$$

$$= \frac{1}{8}\left(\frac{1}{e} + 1\right)$$

$$k = \frac{1+e}{8e}$$

8) This one uses various hyperbolic function identities... not all of which are given in the formula book.

$$f(x) = \tanh^2(x)$$

Identity ①: $1 - \tanh^2(x) = \text{sech}^2(x)$

So $f(x) = 1 - \text{sech}^2(x)$

Identity ②: $\frac{d}{dx}(\tanh(x)) = \text{sech}^2(x)$

So $\displaystyle\int_0^{\ln 3} \tanh^2(x)\,dx = \int_0^{\ln 3} 1 - \operatorname{sech}^2(x)\,dx$

$$= \Big[x - \tanh(x)\Big]_0^{\ln 3}$$

Identity 3: $\quad \tanh(x) = \dfrac{e^x - e^{-x}}{e^x + e^{-x}} = \dfrac{e^{2x} - 1}{e^{2x} + 1}$

So the integral is

$$\left[x - \frac{e^{2x} - 1}{e^{2x} + 1}\right]_0^{\ln 3}$$

$$= \ln 3 - \cancel{0} - \frac{e^{2\ln 3} - 1}{e^{2\ln 3} + 1} + \cancel{\frac{e^0 - 1}{e^0 + 1}}$$

$$= \ln 3 - \frac{9-1}{9+1}$$

$$= \underline{\underline{\ln 3 - \frac{4}{5}}}$$

To find the mean over the domain $[0, \ln 3]$ divide by $\ln 3$:

$$\underline{\underline{1 - \frac{4}{5\ln 3}}} \quad \text{or} \quad \underline{\underline{1 - \frac{4}{\ln(243)}}}$$

9) a) De Moivre's theorem says

$$\text{cis}(n\theta) = \text{cis}^n(\theta)$$

So in this case

$$\cos 4\theta + i\sin 4\theta$$

$$= \cos^4\theta + 4i\sin\theta\cos^3\theta - 6\sin^2\theta\cos^2\theta$$

$$- 4i\sin^3\theta\cos\theta + \sin^4\theta$$

Taking real and imaginary parts,

$$\cos 4\theta = \cos^4\theta - 6\sin^2\theta\cos^2\theta + \sin^4\theta$$

$$\sin 4\theta = 4\sin\theta\cos^3\theta - 4\sin^3\theta\cos\theta$$

So $\cot 4\theta = \dfrac{\cos 4\theta}{\sin 4\theta}$

$$= \frac{\cos^4\theta - 6\sin^2\theta\cos^2\theta + \sin^4\theta}{4\sin\theta\cos^3\theta - 4\sin^3\theta\cos\theta}$$

Dividing throughout by $\sin^4\theta$ this gives:

$$\frac{\cot^4\theta - 6\cot^2\theta + 1}{4\cot^3\theta - 4\cot\theta}$$ as required.

b) Consider $\theta = \frac{\pi}{8}$.

Then $4\theta = \frac{\pi}{2}$ and $\cot(4\theta) = \cot\left(\frac{\pi}{2}\right) = 0$.

from the identity proved: if $\cot(4\theta) = 0$

then $\cot^4\theta - 6\cot^2\theta + 1$ (numerator)

also $= 0$.

So in this case:

$$0 = \cot\left(\frac{\pi}{2}\right) = \cot^4\left(\frac{\pi}{8}\right) - 6\cot^2\left(\frac{\pi}{8}\right) + 1$$

$$= x^2 - 6x + 1$$

$$= 0.$$

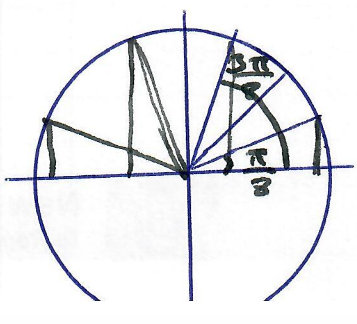
So $x = \cot^2\left(\frac{\pi}{8}\right)$ is a root of $x^2 - 6x + 1$.

We also note that

$\cot\left(\frac{\pi}{2}\right) = \cot\left(\frac{3\pi}{2}\right) = \cot\left(\frac{5\pi}{2}\right) = \dots = 0$

So $x = \cot^2\left(\frac{\pi}{8}\right), \cot^2\left(\frac{3\pi}{8}\right), \cot^2\left(\frac{5\pi}{8}\right), \dots$ are also roots.

We also note that many of these are the same: the only different values are essentially $\cot^2\left(\frac{\pi}{8}\right)$ and $\cot^2\left(\frac{3\pi}{8}\right)$.



This means we can use the standard sum of roots of a quadratic $ax^2+bx+c=0$ as $\frac{-b}{a}$, and get:

$$\cot^2\left(\frac{\pi}{8}\right)+\cot^2\left(\frac{3\pi}{8}\right) = -\frac{6}{1}$$

And since $\cot^2 A + 1 = \operatorname{cosec}^2 A$,

$$\operatorname{cosec}^2\left(\frac{\pi}{8}\right)-1+\operatorname{cosec}^2\left(\frac{3\pi}{8}\right)-1 = 6$$

or $$\underline{\underline{\operatorname{cosec}^2\left(\frac{\pi}{8}\right)+\operatorname{cosec}^2\left(\frac{3\pi}{8}\right) = 8}}$$

as required.