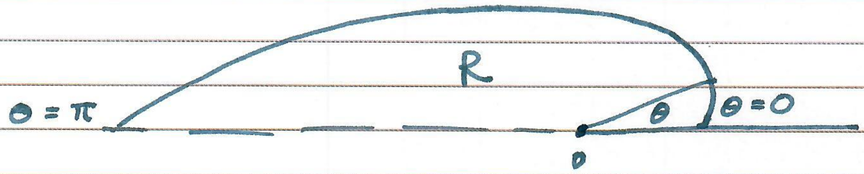
1) (This one ~~may be~~ is easier than it looks)



$$r = 2\sqrt{\sinh\theta + \cosh\theta}$$

Using the standard definitions for sinh and cosh:

$$r = 2\sqrt{\frac{e^{\theta}-e^{-\theta}}{2} + \frac{e^{\theta}+e^{-\theta}}{2}}$$

$$= 2\sqrt{\frac{2e^{\theta}}{2}} = 2\sqrt{e^{\theta}} = 2e^{\frac{\theta}{2}}$$

Area of R is $\displaystyle\int \frac{r^2\,d\theta}{2}$ $\left(2e^{\frac{\theta}{2}}\right)^2$

$$= \int_0^{\pi} \frac{4e^{\theta}\,d\theta}{2}$$

$$= 2\int_0^{\pi} e^{\theta}\,d\theta = 2\left[e^{\theta}\right]_0^{\pi}$$

$$= 2\left(e^{\pi}-1\right).$$

$= 2e^{\pi} - 2$ is the required form $pe^{q} - r$

with $p = 2$ $q = \pi$ $r = 2$

2) a) Maclaurin series for e^x is

$$e^x = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \dots$$

(not sure if we're meant to <u>derive</u> this...)

b) So $e^{(e^x - 1)} = e^{\left(x + \frac{x^2}{2} + \frac{x^3}{6} + \dots\right)}$

$$= (e^x)\left(e^{\frac{x^2}{2}}\right)\left(e^{\frac{x^3}{6}}\right)\dots$$

$$= \left(1 + x + \frac{x^2}{2} + \frac{x^3}{6}\right)\left(1 + \frac{x^2}{2} + \dots\right)\left(1 + \frac{x^3}{6}\right)$$

$$\approx \left(1 + x + \frac{x^2}{2} + \frac{x^3}{6}\right)\left(1 + \frac{x^2}{2}\right)\left(1 + \frac{x^3}{6}\right)$$

$$= \left(1 + x + \frac{x^2}{2} + \frac{x^3}{6}\right)\left(1 + \frac{x^2}{2} + \frac{x^3}{6} + \dots\right)$$

$$= 1 + x + \frac{x^2}{2} + \frac{x^3}{6}$$

$$+ \frac{x^2}{2} + \frac{x^3}{2} + \text{———}$$

$$+ \frac{x^3}{6} + \text{———}$$

$$= 1 + x + x^2 + \frac{5}{6}x^3 + \dots\dots$$

✓

3) $M = \begin{pmatrix} -2 & 5 \\ 6 & k \end{pmatrix}$

$M^2 = \begin{pmatrix} -2 & 5 \\ 6 & k \end{pmatrix}\begin{pmatrix} -2 & 5 \\ 6 & k \end{pmatrix} = \begin{pmatrix} 34 & -10+5k \\ -12+6k & 30+k^2 \end{pmatrix}$

So $M^2 + 11M \qquad = \qquad aI$

$= \begin{pmatrix} 34 & -10+5k \\ -12+6k & 30+k^2 \end{pmatrix} + \begin{pmatrix} -22 & 55 \\ 66 & 11k \end{pmatrix}$

$= \begin{pmatrix} 12 & 45+5k \\ \cancel{54} \; 54+6k & 30+11k+k^2 \end{pmatrix} = a\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

From which we conclude:

$12 = a.1 \qquad \underline{\underline{a = 12}}$

$45+5k = 0 \qquad \underline{\underline{k = -9}}$

(also $54+6k = 0 \qquad k = -9$ ✓)

b) we know $M^2 + 11M - 12I = 0$

So char. eq^n is $\lambda^2 + 11\lambda - 12 = 0$

$(\lambda+12)(\lambda-1) = 0$

$\lambda = -12$ or 1.

So invariant lines are given by

$M\underline{x} = \lambda\underline{x}$ for $\lambda = -12, 1$.

So firstly, $\lambda = -12$:

$$\begin{pmatrix} -2 & 5 \\ 6 & -9 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = -12\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -12x \\ -12y \end{pmatrix}$$

Equating top line: $-2x + 5y = -12x$

$$5y = -10x$$

$$y = -2x$$

So $\begin{pmatrix} 1 \\ -2 \end{pmatrix}$ is an eigenvector. ✓

Then, $\lambda = 1$:

$$\begin{pmatrix} -2 & 5 \\ 6 & -9 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix}$$

Equating bottom line (why not):

$$6x - 9y = y$$

$$6x = 10y$$

$$y = \frac{6}{10}x = \frac{3}{5}x$$

So $\begin{pmatrix} 5 \\ 3 \end{pmatrix}$ is also an eigenvector. ✓

c) Since one eigenvalue is 1, this leaves all points on the line where they were. To show this:

$$\begin{pmatrix} -2 & 5 \\ 6 & -9 \end{pmatrix}\begin{pmatrix} 5 \\ 3 \end{pmatrix} = \begin{pmatrix} -10+15 \\ 30-27 \end{pmatrix} = \begin{pmatrix} 5 \\ 3 \end{pmatrix} ✓$$

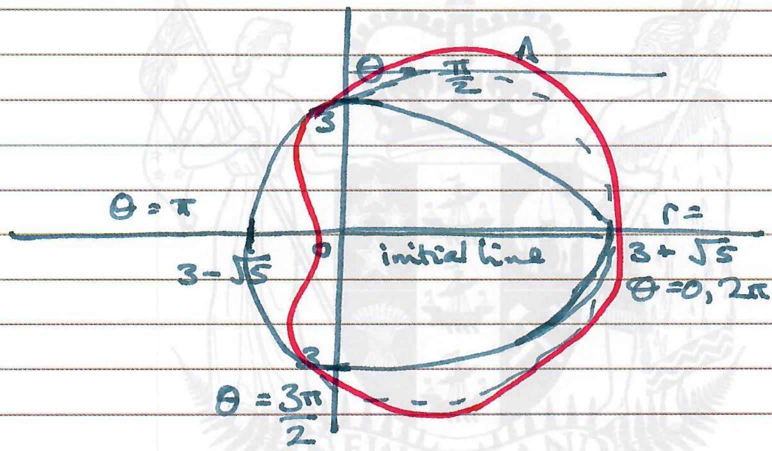
4) a) when $\theta = 0$: $r = 3 + \sqrt{5}$

$\frac{\pi}{2}$: $r = 3$

π: $r = 3 - \sqrt{5}$

$\frac{3\pi}{2}$: $r = 3$

2π: $r = 3 + \sqrt{5}$



Is it a circle? Or just a lump? With pointy bits?

It's actually a softened cardioid, with a 'dimple'. Apparently students are expected to 'know'/'recognise'/'recall' this.

b) It would seem A is on the dotted line.

This means at A $\frac{dy}{dx} = 0$

but also $\frac{dy}{d\theta} = 0$, which might be useful.

$r = 3 + \sqrt{5}\cos\theta$

$y = (3 + \sqrt{5}\cos\theta)\sin\theta = 3\sin\theta + \sqrt{5}\cos\theta\sin\theta$

$\left(= 3\sin\theta + \frac{\sqrt{5}}{2}\sin 2\theta\right)$

So $\frac{dy}{d\theta} = 3\cos\theta + \sqrt{5}(\cos\theta\cos\theta - \sin\theta\sin\theta)$

$= 3\cos\theta + \sqrt{5}(2\cos^2\theta - 1)$

$= 3\cos\theta + 2\sqrt{5}\cos^2\theta - \sqrt{5}$

$= 0$

$2\sqrt{5}c^2 + 3c - \sqrt{5} = 0$

(This looks horrible)

$c = \frac{-3 \pm \sqrt{9 + 4.2\sqrt{5}.\sqrt{5}}}{2\sqrt{5}}$

$= \frac{-3 \pm \sqrt{9 + 40}}{2\sqrt{5}}$

$= \frac{-3 \pm 7}{2.2\sqrt{5}}$ = (discard -7)

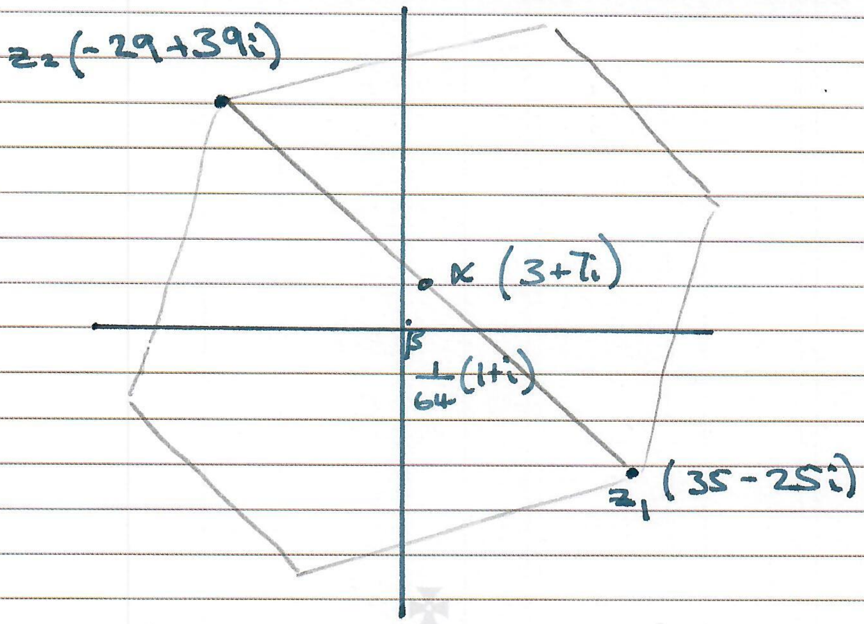
$\frac{-3+7}{2.2\sqrt{5}} = \frac{4}{2.2\sqrt{5}} = \frac{2}{2\sqrt{5}}$.

$\cos\theta = \frac{1}{\sqrt{5}}$ ✓ as required.

c) So at A: $r = 3 + \sqrt{5}\cos\theta = 3 + \frac{\sqrt{5}}{\sqrt{5}}$

$= \underline{\underline{4}}$. ✓

5)



Because z_1 and z_2 are opposite vertices of the hexagon, the centre must lie on the midpoint of z_1z_2:

i.e. $\alpha = \frac{1}{2}(z_1+z_2) = \frac{1}{2}(35-25i + -29+39i)$

$= \frac{1}{2}(6+14i)$

$= \underline{\underline{3+7i}}$ as required.

b) $\beta = \dfrac{1+i}{64}$

$\beta(z_1-\alpha) = \dfrac{1+i}{64}(35-25i-3-7i)$

$= \dfrac{1+i}{64}(32-32i)$

$= \dfrac{32(1+i)(1-i)}{64} = \frac{1}{2}(1+i)(1-i)$

$= \frac{1}{2}(1+1) = \frac{2}{2} = \underline{\underline{1}}$

as required.

c) i) If $w^6 = 1$ then $w = 1e^{\frac{2\pi}{6}i \cdot n}$ for $n = 0 \ldots 5$

$$= e^{\frac{n}{3}\pi i}$$

ii) So $\beta(z-\alpha) = e^{\frac{n}{3}\pi i}$ ⊛

and $\beta = \sqrt{2}\,\dfrac{e^{i\frac{\pi}{4}}}{64}$ (missed the 64!) (and the $\sqrt{2}$ is there)

So $\sqrt{2}\,\dfrac{e^{i\frac{\pi}{4}}}{64}(z-\alpha) = e^{\frac{n}{3}\pi i}$

$$z - \alpha = \frac{64}{\sqrt{2}} e^{\left(\frac{n}{3} - \frac{1}{4}\right)\pi i}$$

$$z = \frac{64}{\sqrt{2}} e^{\left(\frac{n}{3} - \frac{1}{4}\right)\pi i} + (3+7i)$$

$$= e^{-\frac{1}{4}\pi i},\ e^{\frac{\pi}{12}i},\ e^{\frac{5\pi}{12}i},\ e^{\frac{9\pi}{12}i},\ e^{\frac{13\pi}{12}i},$$

$$e^{\frac{17\pi}{12}i}$$

It actually works out that instead of all these $\frac{\pi}{12}$ terms we should work from ⊛ and say

$$z - \alpha = \frac{1}{\beta} e^{\frac{n}{3}\pi i} \qquad n = 0, 1, 2, 3, 4, 5)$$

$$= \frac{1}{\beta} \operatorname{cis}\left(0, \frac{\pi}{3}, \frac{2\pi}{3}, \pi, \frac{4\pi}{3}, \frac{5\pi}{3}\right)$$

$$= \frac{1}{\beta} \begin{cases} 1 + 0i \\ \frac{1}{2} + \frac{\sqrt{3}}{2}i \\ -\frac{1}{2} + \frac{\sqrt{3}}{2}i \\ -1 + 0i \\ -\frac{1}{2} - \frac{\sqrt{3}}{2}i \\ +\frac{1}{2} - \frac{\sqrt{3}}{2}i \end{cases}$$

and $\frac{1}{\beta} = \frac{64}{1+i} = \frac{64(1-i)}{(1+i)(1-i)} = \frac{64}{2}(1-i)$

$$= 32(1-i)$$

So $z - \alpha = 32(1-i) \times 6$ values

$$= 32 \begin{cases} 1 - i \\ \frac{1}{2}(1+\sqrt{3}) + \frac{1}{2}(-1+\sqrt{3})i \\ \frac{1}{2}(-1+\sqrt{3}) + \frac{1}{2}(1+\sqrt{3})i \\ -1 + i \\ \frac{1}{2}(-1-\sqrt{3}) + \frac{1}{2}(1-\sqrt{3})i \\ \frac{1}{2}(1-\sqrt{3}) + \frac{1}{2}(-1-\sqrt{3})i \end{cases}$$

$$= \begin{array}{l} 32 - 32i \\ 16 + 16\sqrt{3} + (-16 + 16\sqrt{3})i \\ -16 + 16\sqrt{3} + (16 + 16\sqrt{3})i \\ -32 + 32i \\ -16 - 16\sqrt{3} + (16 - 16\sqrt{3})i \\ +16 - 16\sqrt{3} + (-16 - 16\sqrt{3})i \end{array}$$

Adding $\alpha = (3+7i)$ onto each of these gives:

$$35 - 25i \qquad (z_1)$$
$$19 + 16\sqrt{3} + (-9 + 16\sqrt{3})i$$
$$13 + 16\sqrt{3} + (23 + 16\sqrt{3})i$$
$$-29 + 39i \qquad (z_2)$$
$$13 - 16\sqrt{3} + (23 - 16\sqrt{3})i$$
$$19 - 16\sqrt{3} + (-9 - 16\sqrt{3})i$$

This gives 6 vertices: z_1 and z_2 as expected, and the other 4 as required.

(A neat and interesting question - though tedious.)

6) This looks horrible but I suspect it isn't - and in particular most things with hyperbolic functions become nicer when turned into e^x terms.

But note for further maths they expect proper use of k (or r) for proofs by induction: not just a fudgy use of n to get to n+1.

So:

$$y = e^{2x} \sinh x$$

We could try using $\frac{d}{dx}(\sinh x) = \cosh x$ etc

... in fact let's do that.

<u>To prove</u> $\quad \frac{d^n y}{dx^n} = e^{2x}\left(\frac{3^n+1}{2}\sinh x + \frac{3^n-1}{2}\cosh x\right)$

Assume the result is true for $n=k$.

Then $\quad \frac{d^k y}{dx^k} = e^{2x}\left(\frac{3^n+1}{2}\sinh x + \frac{3^n-1}{2}\cosh x\right)$

Differentiating using the product rule:
(and using standard hyp. func. derivatives)

$$\frac{d^{k+1}y}{dx^{k+1}} = e^{2x}\left(\frac{3^k+1}{2}\cosh x + \frac{3^k-1}{2}\sinh x\right)$$

$$+ 2e^{2x}\left(\frac{3^k+1}{2}\sinh x + \frac{3^k-1}{2}\cosh x\right)$$

$$= e^{2x}\left\{\left(\frac{3^k+1}{2} + 2\left(\frac{3^k-1}{2}\right)\right)\cosh x + \left(\left(\frac{3^k-1}{2}\right) + 2\left(\frac{3^k+1}{2}\right)\right)\sinh x\right\}$$

$$= e^{2x}\left\{\frac{1}{2}\left(3^k+1+2\times 3^k-2\right)\cosh x + \frac{1}{2}\left(3^k-1+2\times 3^k+2\right)\sinh x\right\}$$

$$= e^{2x}\left\{\frac{3\times 3^k-1}{2}\cosh x + \frac{3\times 3^k+1}{2}\sinh x\right\}$$

$$= e^{2x}\left\{\frac{3^{k+1}+1}{2}\sinh x + \frac{3^{k+1}-1}{2}\cosh x\right\}$$

which is exactly as required:

if the result is true for $k=n$ then it is true for $k=n+1$.

Moreover, when $k=1$,

$$\frac{dy}{dx} = \frac{d}{dx}\left\{e^{2x}\sinh x\right\} = 2e^{2x}\sinh x + e^{2x}\cosh x$$

$$= e^{2x}\left\{2\,\frac{e^x-e^{-x}}{2} + \frac{e^x+e^{-x}}{2}\right\}$$

$$= e^{2x}\left\{\frac{3e^x-e^{-x}}{2}\right\}$$

And $e^{2x}\left\{ \frac{\cancel{3^1+1}^{\,4}}{\cancel{2}^{\,2}} \sinh x + \frac{\cancel{3^1-1}^{\,2}}{\cancel{2}^{\,2}} \cosh x \right\}$

$= 2e^{2x}\sinh x + e^{2x}\cosh x$ which

shows $\frac{dy}{dx}$ = expression in the required form.

So the result is a) true when $k=1$

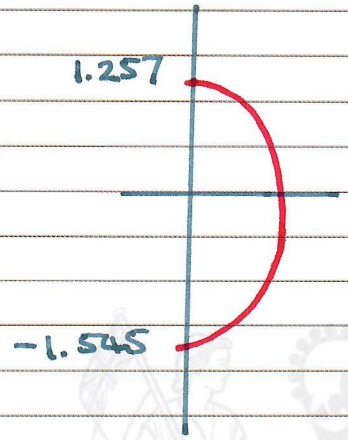
b) true for $k=n+1$ if true when $k=n$.

And so it is proved in general by induction for all $n \in \mathbb{N}$ ✓

important to say this.

7) (Suspect this involves some tedious calculating).

And it's a curious case of integrating by y, not x.



$$16x^2 + 3y^2 - y\cos\left(\frac{5}{2}y\right) = 6 \qquad x \geq 0$$

Applying integration by y, the volume of revolution about the y axis is

$$2\pi \int_{-1.545}^{1.257} \frac{x^2}{2}\, dy$$

. which happily we can do from

$$x^2 = \frac{1}{16}\left(6 - 3y^2 + y\cos\frac{5}{2}y\right).$$

So the integral is $\displaystyle\int_{-1.545}^{1.257} \frac{6}{16} - \frac{3}{16}y^2 + \frac{1}{16}y\cos\frac{5}{2}y\, dy.$

for $\int y \cos ky \, dy$, consider integration by parts:

$$\frac{d}{dy}(uv) = u\frac{dv}{dy} + v\frac{du}{dy}$$

So if $u = y$ $\qquad du = 1$

$v = \sin ky$ $\qquad dv = k \cos ky$

$$\text{So } \frac{d}{dy}\left(y \sin ky\right) = ky \cos ky + \sin ky$$

$$y \sin ky = \int ky \cos ky \, dy - \frac{1}{k}\cos ky$$

$$\frac{2}{5} y \sin \frac{5}{2}y = \int y \cos \frac{5}{2} dy - \frac{4}{25} \cos \frac{5}{2}y$$

So the integral required is:

$$\frac{\pi}{16}\left[6y - \quad y^3 + \frac{2}{5} y \sin \frac{5}{2}y + \frac{4}{25} \cos \frac{5}{2}y \right]_{-1.545}^{1.257}$$

= (using a spreadsheet):

$$\frac{\pi}{16}\left(5.40 + 6.11\right) = \underline{\underline{2.26 \text{ cm}^3}}$$

b) $100 \times 2.26 \text{ cm}^3 = 226 \text{ cm}^3$ which is $\frac{1}{8}$ over the 200cm^3 required – but (a) not all berries will be this large (none more will) and (b) parts of the berries (pips, skin etc) won't make juice. So it seems risky to think there'll be enough.

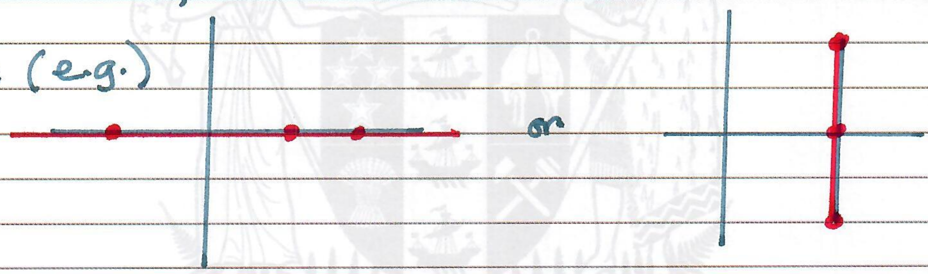
8) We assume the question intends all the equation's coefficients are real. Then...

a) If all 3 roots are real they lie on the line $y=0$ (x-axis)

Otherwise 1 root will be real and the others will form a conjugate pair - which means those two lie on a 'vertical' line $x=k$ for some k.

And since we know all 3 roots lie on the same line, we know all 3 lie on $x=k$.

So: (e.g.)



b) $f(z) = 8z^3 + bz^2 + cz + d$

one root is $\frac{3}{2} + \frac{3}{2}i$

Using (a), the other two roots are

$$\frac{3}{2} - \frac{3}{2}i$$

and $\frac{3}{2}$

c) $g(z) = z^3 + Pz^2 + Qz + 12$

has 3 distinct real roots – i.e. on the real axis.

i) if $f(z)$ and $g(z)$ have roots on distinct lines, but have one root in common, that root must be $\underline{\underline{\frac{3}{2}}}$

ii) Knowing that $\frac{3}{2}$ and -4 are both roots,

and using the fact -12 is the product of the roots, we know

$$\frac{3}{2} \times -4 \times \alpha = -12 \qquad (\alpha = \text{third root})$$

$$-3 \times 2\alpha = -12$$

$$\alpha = 2$$

d) $f(z) = g(z)$ I wonder if there's some quick fix here, but apparently we just have to do it:

$$f(z) = (2z-3)(2z-(3+3i))(2z-(3-3i))$$

$$= (2z-3)(4z^2 - 2z(3+3i+3-3i) + (3+3i)(3-3i))$$

$$= (2z-3)(4z^2 - 12z + 9+9)$$

$$= (2z-3)(4z^2 - 12z + 18)$$

$$= 2(z-\tfrac{3}{2})(4z^2 - 12z + 18)$$

and $g(z) = \left(z - \frac{3}{2}\right)(z+4)(z-2)$

So if $f(z) = g(z)$:

$$2(4z^2 - 12z + 18) = (z+4)(z-2)$$

$$8z^2 - 24z + 36 = z^2 + 2z - 8$$

$$7z^2 - 26z + 44 = 0.$$

$$z = \frac{26 \pm \sqrt{26^2 - 4.44.7}}{14}$$

$$= \frac{26 \pm \sqrt{-556}}{14}$$

$$= \frac{13 \pm i\sqrt{139}}{7}$$

So the solutions to $f(z) = g(z)$ are:

$$z = \frac{3}{2}, \quad \frac{13 - i\sqrt{139}}{7}, \quad \frac{13 + i\sqrt{139}}{7}$$

(Rather a nasty question)