1) $(x-\alpha)(x-\beta)(x-\gamma) = x^3 - 7x^2 - 12x + 6$

$= (x^2 - (\alpha+\beta)x + \alpha\beta)(x-\gamma)$

$$= \begin{pmatrix} x^3 - (\alpha+\beta)x^2 + \alpha\beta x \\ \quad - \gamma x^2 + (\alpha+\beta)\gamma x - \alpha\beta\gamma \end{pmatrix}$$

$= x^3 - (\alpha+\beta+\gamma)x^2 + (\alpha\beta+\beta\gamma+\gamma\alpha)x - \alpha\beta\gamma$

Equating coeffs:

x^2: $-(\alpha+\beta+\gamma) = -7$ $\qquad \alpha+\beta+\gamma = 7$

x: $\alpha\beta+\beta\gamma+\gamma\alpha = -12$ $\qquad \alpha\beta+\beta\gamma+\gamma\alpha = -12.$

1: $-\alpha\beta\gamma = 6$ $\qquad \alpha\beta\gamma = -6$

We want to reverse this process, to get

$(w-(\alpha+2))(w-(\beta+2))(w-(\gamma+2)) = w^3 + pw^2 + qw + r$
$(x = w)$

(though surely we could just set $w = x-2$, $x = w+2$, and multiply it out...)

So $r = -[(\alpha+2)(\beta+2)(\gamma+2)] = -[\alpha\beta\gamma + 2(\beta\gamma+\gamma\alpha+\alpha\beta) + 4(\alpha+\beta+\gamma) + 8]$

$= -[-6 + 2(-12) + 4(7) + 8]$

$= -[-6 - 24 + 28 + 8] = -[-30 + 36] = \underline{-6}$

$$q = +(\alpha+2)(\beta+2) + (\beta+2)(\gamma+2) + (\gamma+2)(\alpha+2)$$

$$= +(\alpha\beta+\beta\gamma+\gamma\alpha) + (2\alpha+2\beta+2\beta+2\gamma+2\gamma+2\alpha) + 3\times 4$$

$$= -12 + 4(\alpha+\beta+\gamma) + 12$$

$$= +4\times 7 \quad = +\underline{28}.$$

$$p = -[(\alpha+2)+(\beta+2)+(\gamma+2)] = -[\alpha+\beta+\gamma+6]$$

$$= -(7+6) = -\underline{13}.$$

So the required equation is:

$$\underline{\underline{w^3 - 13w^2 + 28w - 6 = 0}}$$

Alternatively just multiply it out:

$$(w-2)^3 - 7(w-2)^2 - 12(w-2) + 6 = 0$$

$$w^3 - 6w^2 + 12w - 8 - 7w^2 + 28w - 28 - 12w + 24 + 6 = 0$$

$$\underline{\underline{w^3 - 13w^2 + 28w - 6 = 0}}$$

If giving us w is a 'hint', then it seems a bit cruel to let us go down the first path.

2) a) 'Completing the Square':

$$x^2 + 4x - 5 = (x+2)^2 - 9$$

(Check: $x^2 + 4x + 4 - 9$ ✓)

b) We know: $\displaystyle\int \frac{1}{\sqrt{x^2-a^2}}\,dx = \ln\left(x + \sqrt{x^2-a^2}\right) + c$

Let $u = x+2$: then $du = dx$

So using the result above,

$$\int \frac{1}{\sqrt{x^2+4x-5}}\,dx = \int \frac{1}{\sqrt{(x+2)^2-9}}\,dx$$

$$= \int \frac{1}{\sqrt{u^2-9}}\,du$$

$$= \ln\left(u + \sqrt{u^2-9}\right) + c$$

$$= \ln\left(x+2 + \sqrt{(x+2)^2-9}\right) + c.$$

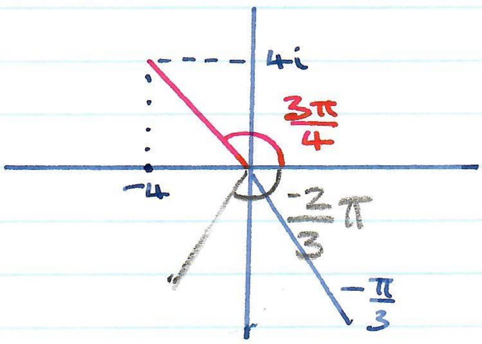
c) The mean value between the limits given is:

$$\frac{1}{13-3}\left[\ln\left(x+2+\sqrt{(x+2)^2-9}\right)\right]_3^{13}$$

$$= \frac{1}{10}\left[\ln\frac{\left(13+2+\sqrt{15^2-9}\right)}{\left(3+2+\sqrt{5^2-9}\right)}\right] = \frac{1}{10}\ln\frac{\left(15+\sqrt{216}\right)}{\left(9\right)}$$

$$= \frac{1}{10}\ln\left(\frac{5+2\sqrt{6}}{3}\right)$$

3)



a) $|z_1| = \sqrt{4^2+4^2}$

$= 4\sqrt{2}$

$\arg z_1 = \frac{3\pi}{4}$

$z_1 = 4\sqrt{2} \text{ cis} \left(\frac{3\pi}{4}\right)$

b) i) $z_2 = 3 \text{ cis } \frac{17\pi}{12}$

So $\frac{z_1}{z_2} = \frac{4\sqrt{2}}{3} \text{ cis} \left(\frac{3\pi}{4} - \frac{17\pi}{12}\right)$

$= \frac{4\sqrt{2}}{3} \text{ cis} \left(\frac{9\pi - 17\pi}{12}\right)$

$= \frac{4\sqrt{2}}{3} \text{ cis} \left(-\frac{2}{3}\pi\right)$

$= \frac{4\sqrt{2}}{3} \left(-\frac{1}{2} - \frac{\sqrt{3}}{2}i\right)$

$= -\frac{2\sqrt{2}}{3} - \frac{2\sqrt{6}}{3}i$

ii) $(z_2)^4 = 3^4 \text{ cis} \left(4 \times \frac{17\pi}{12}\right)$

$= 81 \text{ cis} \left(\frac{17}{3}\pi\right)$

$= 81 \text{ cis} \left(\frac{5}{3}\pi\right)$

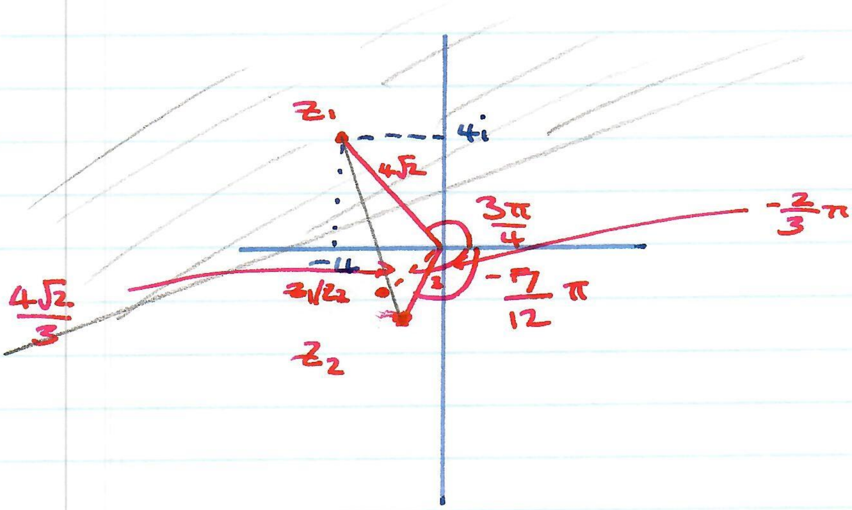
$$= 81 \operatorname{cis}\left(-\frac{\pi}{3}\right)$$

$$= 81\left(\cos\frac{\pi}{3} - i\sin\frac{\pi}{3}\right)$$

$$= \frac{81}{2} - \frac{81\sqrt{3}}{2}i$$

$$= \frac{81}{2} + i\left(-\frac{81\sqrt{3}}{2}\right)$$

c)



z_2 is $3\operatorname{cis}\left(-\frac{7}{12}\pi\right)$, which we hadn't noted yet.

$|z - z_1| < |z - z_2|$ in the region above the perp. bisector of the line between z_1 and z_2 - shown shaded.

4) Proof by induction.

Assume $\begin{pmatrix} 1 & -2 \\ 0 & 1 \end{pmatrix}^n = \begin{pmatrix} 1 & -2n \\ 0 & 1 \end{pmatrix}$

Then $\begin{pmatrix} 1 & -2 \\ 0 & 1 \end{pmatrix}^{n+1} = \begin{pmatrix} 1 & -2 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & -2 \\ 0 & 1 \end{pmatrix}^n$

$$= \begin{pmatrix} 1 & -2 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & -2n \\ 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1\times 1 - 2\times 0 & 1\times -2n - 2\times 1 \\ 1\times 0 + 0\times 1 & 0\times -2n + 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & -2n-2 \\ 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & -2(n+1) \\ 0 & 1 \end{pmatrix} \text{ as required.}$$

Also, the result is true for $n=1$:

$$\begin{pmatrix} 1 & -2 \\ 0 & 1 \end{pmatrix}^1 = \begin{pmatrix} 1 & -2\times 1 \\ 0 & 1 \end{pmatrix}$$

So by induction the result holds $\forall n$.

(But apparently we should have done it with k, not n)

5) $l: \frac{x+5}{1} = \frac{y+4}{-3} = \frac{z-3}{5}$

This is a line through $(-5,-4,3)$ with a gradient vector $(1,-3,5)$, so we can express it parametrically as

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -5 \\ -4 \\ 3 \end{pmatrix} + t\begin{pmatrix} 1 \\ -3 \\ 5 \end{pmatrix} = \begin{pmatrix} -5+t \\ -4-3t \\ 3+5t \end{pmatrix}$$

Substitute this into the eq^n for Π_1:

$$2x + 3y - 2z = 6$$

So $2(-5+t) + 3(-4-3t) - 2(3+5t) = 6$

$$= -10 + 2t - 12 - 9t - 6 - 10t = 6$$

$$-17t = 28 + 6 = 34$$

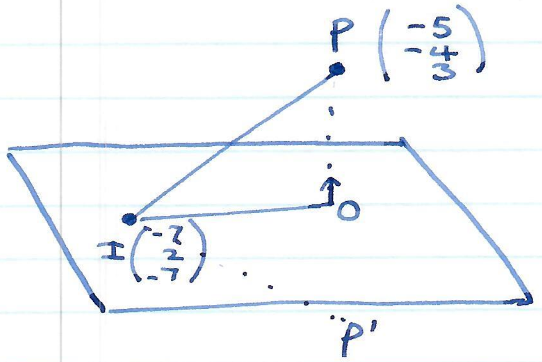
$$t = -2.$$

So the point of intersection is $\begin{pmatrix} -5 \\ -4 \\ 3 \end{pmatrix} - 2\begin{pmatrix} 1 \\ -3 \\ 5 \end{pmatrix}$

$$= \begin{pmatrix} -5-2 \\ -4+6 \\ 3-10 \end{pmatrix} = \underline{\underline{\begin{pmatrix} -7 \\ 2 \\ -7 \end{pmatrix}}}$$

b) (This stuff is hard)

Start with the point we know:



We know the normal to the plane is $\begin{pmatrix} 2 \\ 3 \\ -2 \end{pmatrix}$

So the point O must be $P + \lambda \underline{n}$

ie $O = \begin{pmatrix} -5 \\ -4 \\ 3 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ 3 \\ -2 \end{pmatrix}$ for some λ.

Subs. this into the eq^n for Π_1:

$$2(-5+2\lambda) + 3(-4+3\lambda) - 2(3-2\lambda) = 6$$

$$-10 + 4\lambda - 12 + 9\lambda - 6 + 4\lambda = 6$$

$$-28 + 17\lambda = 6$$

$$17\lambda = 34 \qquad \underline{\lambda = 2.}$$

Since at P, $\lambda = 0$, we know that at its reflection P', $\lambda = 2(2) = 4$.

So P' is the point $\begin{pmatrix} -5 \\ -4 \\ 3 \end{pmatrix} + 4 \begin{pmatrix} 2 \\ 3 \\ -2 \end{pmatrix} = \begin{pmatrix} -5+8 \\ -4+12 \\ 3-8 \end{pmatrix}$

$$= \begin{pmatrix} 3 \\ 8 \\ -5 \end{pmatrix}$$

Now we calculate $\vec{IP'}$

$$= \begin{pmatrix} 3 \\ 8 \\ -5 \end{pmatrix} - \begin{pmatrix} -7 \\ 2 \\ -7 \end{pmatrix} = \begin{pmatrix} 3+7 \\ 8-2 \\ -5+7 \end{pmatrix} = \begin{pmatrix} 10 \\ 6 \\ 2 \end{pmatrix}$$

And so the line l_2 is given by

$$\underline{r} = \begin{pmatrix} -7 \\ 2 \\ -7 \end{pmatrix} + \mu \begin{pmatrix} 10 \\ 6 \\ 2 \end{pmatrix} \text{ as required.}$$

c) The gradient vector of l_1 is $\begin{pmatrix} 1 \\ -3 \\ 5 \end{pmatrix}$

l_2 is $\begin{pmatrix} 10 \\ 6 \\ 2 \end{pmatrix}$

So the normal to Π_2 is $\begin{pmatrix} 1 \\ -3 \\ 5 \end{pmatrix} \times \begin{pmatrix} 10 \\ 6 \\ 2 \end{pmatrix}$

$$= \begin{pmatrix} -3\times2 - 5\times6 \\ -(1\times2 - 5\times10) \\ 1\times6 + 30 \end{pmatrix} = \begin{pmatrix} -36 \\ 48 \\ 36 \end{pmatrix}$$

Scaling, this can be used as $\underline{\begin{pmatrix} -3 \\ 4 \\ 3 \end{pmatrix}}$

So the equation of Π_2 is

$$-3x + 4y + 3z = c$$

Substitute for the point $\begin{pmatrix} -7 \\ 2 \\ -7 \end{pmatrix}$:

$$21 + 8 - 21 = C \quad \text{so } C = 8$$

And the equation of Π_2 is $\underline{-3x + 4y + 3z = 8}$.

So: Π_2 is $-3x + 4y + 3z = 8$: normal $(-3, 4, 3)$

Π_1 is $2x + 3y - 2z = 6$: normal $(2, 3, -2)$

So l_3 is $\underline{r}$ (point) + λ $(^-3, 4, 3) \times (2, 3 - 2)$

$$= (\quad) + \cdots \begin{pmatrix} -8-9 \\ -6+6 \\ -9-8 \end{pmatrix}$$

$$\begin{pmatrix} -17 \\ 0 \\ -17 \end{pmatrix}$$

Again we can scale: $\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$

For the (point), use $I \begin{pmatrix} -7 \\ 2 \\ -7 \end{pmatrix}$

and again we can scale:

$$l_3 = \begin{pmatrix} -7 \\ 2 \\ -7 \end{pmatrix} - 7 \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \lambda$$

$$= \begin{pmatrix} 0 \\ 2 \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$$

$$= \begin{pmatrix} \lambda \\ 2 \\ \lambda \end{pmatrix}.$$

So the equation of l_3 is $\underline{l} = \begin{pmatrix} 0 \\ 2 \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$.

For Π_3, we know $\underline{r} \begin{pmatrix} 1 \\ 1 \\ a \end{pmatrix} = b$

and we can pick any points on $\underline{l}$:

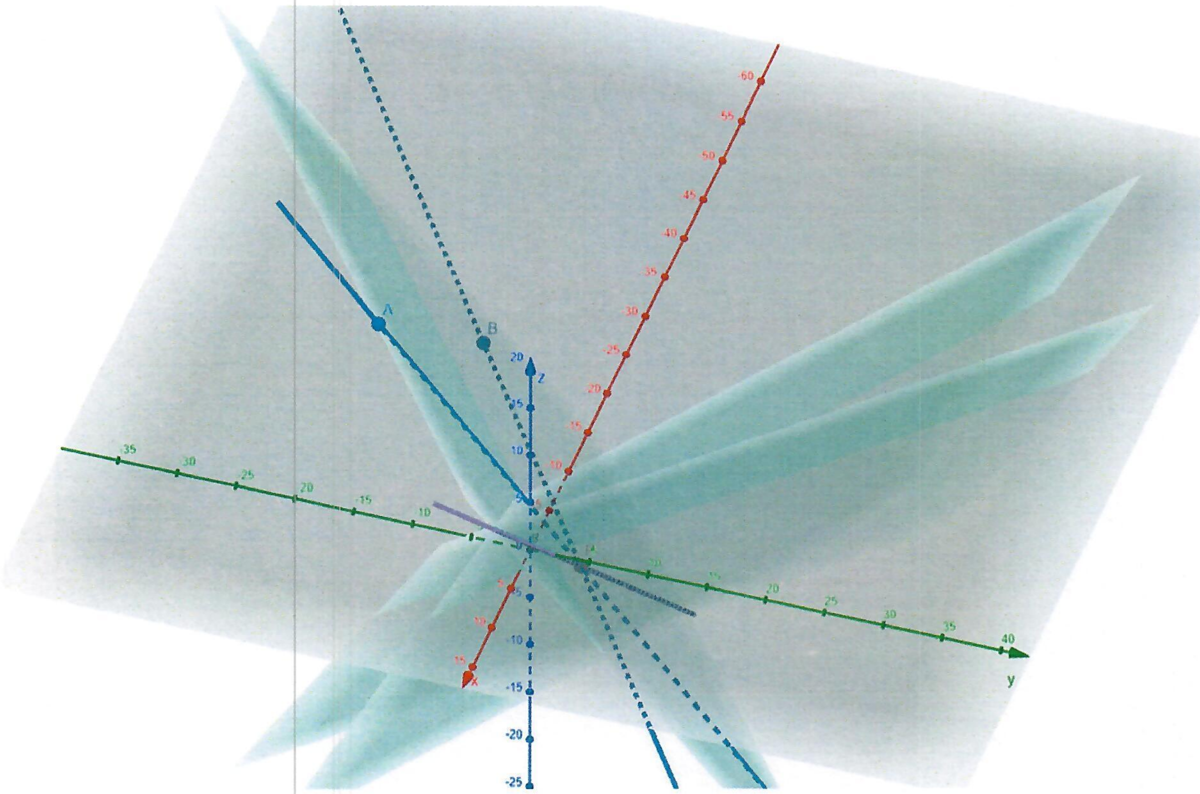
when $\lambda = 2$: $\begin{pmatrix} 2 \\ 2 \\ 2 \end{pmatrix}$ so $(2,2,2)\begin{pmatrix} 1 \\ 1 \\ a \end{pmatrix} = b$

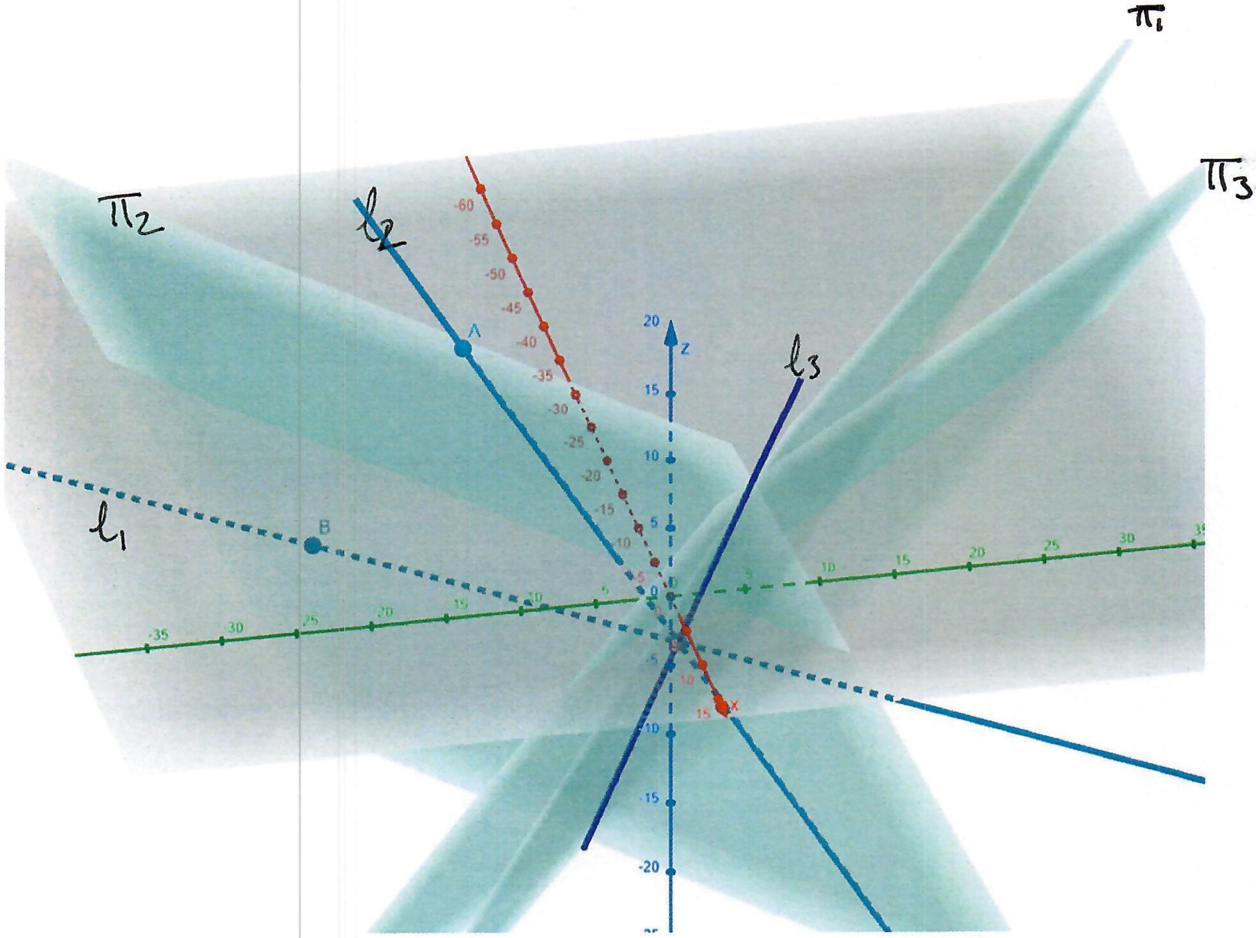
$4 + 2a = b$

$\lambda = 1$: $\begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}$ so $(1,2,1)\begin{pmatrix} 1 \\ 1 \\ a \end{pmatrix} = b$

$3 + a = b$

So $a = -1$ and $b = 2$.


A
B
z
y
x


π_1
π_2
π_3
l_1
l_2
l_3
A
B
z
x

6) a) $V(0) = 10$

$$\frac{dV}{dt} = \left(3 - \frac{4}{1+e^{0.8t}}\right) - kV \quad \text{for some } V.$$

when $t=0$, $\frac{dV}{dt} = -3$

So $-3 = 3 - \frac{4}{1+e^0} - 10k$

$$10k = 3 + 3 - \frac{4}{2} = 6 - 2 = 4$$

So $k = 0.4$ and we have:

$$\frac{dV}{dt} = 3 - \frac{4}{1+e^{0.8t}} - 0.4V \quad \text{as required.}$$

b) From given formulae,

$$\frac{d}{dt}(\arctan t) = \frac{1}{1+x^2}$$

So $$\frac{d}{dt}\left(\arctan e^{0.4t}\right) = \frac{0.4e^{0.4t}}{1+\left(e^{0.4t}\right)^2}$$

c) from (a),

$$\frac{dV}{dt} + 0.4V = 3 - \frac{4}{1+e^{0.8t}}$$

Using an integrating factor $e^{0.4t}$:

$$e^{0.4t}\frac{dV}{dt} + 0.4Ve^{0.4t} = 3e^{0.4t} - \frac{4e^{0.4t}}{1+e^{0.8t}}$$

So $$\frac{d}{dt}\left(e^{0.4t}V\right) = 3e^{0.4t} - \frac{4}{0.4}\left(\frac{0.4e^{0.4t}}{1+e^{0.8t}}\right)$$

Integrating,

$$e^{0.4t}V = \frac{3}{0.4}e^{0.4t} - 10\int \frac{0.4e^{0.4t}}{1+e^{0.8t}}\,dt$$

And from (b),

$$e^{0.4t}V = 7.5e^{0.4t} - 10\arctan(e^{0.4t}) + c$$

Using the $t=0$ condition:

$$10 = 7.5 - 10\arctan(1) + c$$

$$10 = 7.5 - 10\left(\frac{\pi}{4}\right) + c$$

$$2.5 + 2.5\pi = c$$

$$c = 2.5(1+\pi)$$

So $$V = 7.5 - \frac{10\arctan(e^{0.4t})}{e^{0.4t}} + \frac{2.5(1+\pi)}{e^{0.4t}}$$

(= f(t) as specified)

d) for $t=10$ the model gives $V = 7.2984$ l, which is some way from the 8l observed: the model seems to be over-estimating the flow out (or under-- flow in.)

7) a) LHS $= \sum_{r=1}^{2n} (-1)^r f(r)$

$= \underbrace{(-1 f(1) + 1 f(2))} + \underbrace{(-1 . f(3) + 1 f(4))} + \cdots$

$+ \underbrace{(-1 f(2n-1) + 1 f(2n))}$

RHS $= \sum_{r=1}^{n} f(2r) - f(2r-1)$

$= \underbrace{(f(2) - f(1))} + \underbrace{(f(4) - f(3)} + \cdots$

$+ \underbrace{(f(2n) - f(2n-1))}$

The terms can clearly be grouped in pairs which are equal:

$-f(1) + f(2) = f(2) - f(1)$

$-f(2n-1) + f(2n) = f(2n) - f(2n-1)$

etc.

So the two formulation are equivalent and equality is demonstrated.

b) $\displaystyle\sum_{r=1}^{2n} r\left((-1)^r + 2r\right)^2$

We could use the form in (a) straightaway, but it's better to multiply this out first.

$$= \sum_{r=1}^{2n} \left[r(-1)^{2r} + 4r^2(-1)^r + 4r^3 \right]$$

$$= \sum_{r=1}^{2n} r + 4\sum_{r=1}^{2n} (-1)^r r^2 + 4\sum_{r=1}^{2n} r^3$$

$$= \frac{2n(2n+1)}{2}^{*}$$

$$+ 4\,\frac{(2n+1)^2(2n)^2}{4}^{*}$$

(* from standard formulas)

using (a)

$$= 4\sum_{r=1}^{n} (2r)^2 - (2r-1)^2$$

$$= 4\sum_{r=1}^{n} \cancel{4r^2} - \cancel{4r^2} + 4r - 1$$

$$= 4\sum_{r=1}^{n}(4r-1)$$

$$= 16\sum_{r=1}^{n} r - 4\sum_{r=1}^{n} \quad = \frac{16n(n+1)}{2}^{*} - 4n$$

Putting this all together,

Required expression =

$$\frac{2n(n+1)}{2} + \frac{16n(n+1)}{2} - 4n + (2n+1)^2(2n)^2$$

$$= n(2n+1) + 8n(n+1) - 4n + 4n^2(4n^2+4n+1)$$

$$= 2n^2 + n + 8n^2 + 8n - 4n + 16n^4 + 16n^3 + 4n^2$$

$$= 16n^4 + 16n^3 + 14n^2 + 5n$$

$$= n(16n^3 + 16n^2 + 14n + 5)$$

$$= \underline{\underline{n(2n+1)(8n^2+4n+5)}} \text{ by factorisation.}$$

8) a) The categories may not accurately reflect the stages of development of individuals.

$$\begin{pmatrix} N_{n+1} \\ J_{n+1} \\ B_{n+1} \end{pmatrix} = \begin{pmatrix} 0 & 0 & 2 \\ a & b & 0 \\ 0 & 0.48 & 0.96 \end{pmatrix}\begin{pmatrix} N_n \\ J_n \\ B_n \end{pmatrix}$$

When $n=0$: $N_0 = J_0 = 0$ $B_n = B_0$

when $n=2$: $N_2 = 48$ $J_2 = 40$

$$\begin{pmatrix} 0 & 0 & 2 \\ a & b & 0 \\ 0 & 0.48 & 0.96 \end{pmatrix}\begin{pmatrix} 0 & 0 & 2 \\ a & b & 0 \\ 0 & 0.48 & 0.96 \end{pmatrix}\begin{pmatrix} 0 \\ 0 \\ B_0 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 2\times 0.48 & 2\times 0.96 \\ ab & b^2 & 2a \\ 0.48a & \begin{matrix} 0.48b + \\ 0.48\times 0.96 \end{matrix} & 0.96^2 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 0.96 & 1.92 \\ ab & b^2 & 2a \\ 0.48a & \begin{matrix} 0.48b \\ + 0.4608 \end{matrix} & 0.9216 \end{pmatrix}\begin{pmatrix} 0 \\ 0 \\ B_0 \end{pmatrix} = \begin{pmatrix} 48 \\ 40 \\ B_2 \end{pmatrix}$$

So $1.92B_0 = 48$: $B_0 = 25$

$2aB_0 = 40$: $a = 0.8$

$0.9216B_0 = B_2$: $B_2 = 23.04$ (?)

i) So there are $0+0+25 = \underline{\underline{25}}$ mammals at the start.

ii) We have shown $\underline{\underline{a = 0.8}}$.

c) $\det\begin{pmatrix} 0 & 0 & 2 \\ 0.8 & b & 0 \\ 0 & 0.48 & 0.96 \end{pmatrix} = 0(\quad) - 0(\quad) + 2(0.8\times 0.48 - 0\times b)$

$= 0.768$

$\operatorname{adj}\begin{pmatrix} 0 & 0 & 2 \\ 0.8 & b & 0 \\ 0 & 0.48 & 0.96 \end{pmatrix} = \begin{pmatrix} 0.96b & 0.96 & -2b \\ -0.768 & 0 & 1.6 \\ 0.384 & 0 & 0 \end{pmatrix}$

So the inverse required is

$$\frac{1}{0.768}\begin{pmatrix} 0.96b & 0.96 & -2b \\ -0.768 & 0 & 1.6 \\ 0.384 & 0 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 1.25b & 1.25 & -2.6042b \\ -1 & 0 & 2.0833 \\ 0.5 & 0 & 0 \end{pmatrix}$$

d) Suppose the figures of 596, 464, 437 relate to month N+1.

Then $N_N + J_N + B_N = 1015$

and $\begin{pmatrix} N_N \\ J_N \\ B_N \end{pmatrix} = \begin{pmatrix} 1.25b & 1.25 & -2.6042b \\ -1 & 0 & 2.0833 \\ 0.5 & 0 & 0 \end{pmatrix}\begin{pmatrix} 596 \\ 464 \\ 437 \end{pmatrix}$

So $B_N = 0.5\times 596 = 298$

$J_N = -596 + 2.0833\times 437 = 314.40$

So $N_N = 1015 - 298 - 314.40 = 402.6$

Also $N_N = (1.25\times 596)b + 1.25\times 464 - (437\times 2.6042)b$

$= (745 - 1138.04)b + 580$

So $402.6 = 580 - 393.04b$

$-177.4 = -393.04b$

$b = \underline{\underline{0.4514.}}$

e) Propose a 4×4 matrix and variable set,

Total = 2

with $\begin{pmatrix} N_m \\ N_f \\ J \\ B \end{pmatrix}$ and the matrix $\begin{pmatrix} 0 & 0 & 0 & 0.84 \\ 0 & 0 & 0 & 1.16 \\ ? & ? & 0.45 & 0 \\ 0 & 0 & 0.48 & 0.96 \end{pmatrix} \begin{pmatrix} N_m \\ N_f \\ J \\ B \end{pmatrix}$