MEI Differential Equations 2017

1) $\frac{d^2x}{dt^2} + 2\frac{dx}{dt} - 3x = 6\cos t$

(i) AE: $\lambda^2 + 2\lambda - 3 = 0$

$(\lambda+3)(\lambda-1) = 0 \qquad \lambda = -3 \text{ or } 1$

CF: $x = Ae^{-3t} + Be^{t}$

PI: Try $x = P\cos t + Q\sin t$

$\frac{dx}{dt} = -P\sin t + Q\cos t$

$\frac{d^2x}{dt^2} = -P\cos t - Q\sin t$

So

$$\begin{array}{rcll} \frac{d^2x}{dt^2} & = & -P\cos t & -Q\sin t \\ +2\frac{dx}{dt} & = & 2Q\cos t & -2P\sin t \\ -3x & = & -3P\cos t & -3Q\sin t \\ \hline & & -4P+2Q & -2P-4Q \\ & & =6 & =0 \end{array}$$

So $\left.\begin{array}{l} Q - 2P = 3 \\ P + 2Q = 0 \end{array}\right\} \quad Q = 0.6 \quad P = -1.2$

So the General Solution is:

$$\underline{\underline{x = Ae^{-3t} + Be^{t} - 1.2\cos t + 0.6\sin t.}}$$

(ii) • ~~Point (0,0) ⇒~~ $\cancel{0 = A + B - 1.2}$ $\qquad A + B = 1.2$

• 'x bounded' ⇒ Be^t is bounded, so $B = 0$

• The question seems misleading: apparently although we're told the particle is at rest when $t=0$, we're not told x is 0. Apparently we're to understand $x = \underline{\underline{x_0}}$, but $\frac{dx}{dt} = 0$.

It's hard to believe most students weren't caught out.

Anyway: $\frac{dx}{dt} = -3Ae^{-3t} + Be^{t} + 1.2\sin t + 0.6\cos t$

So $0 = -3A + 0.6 \quad : \quad A = \frac{0.6}{3} = 0.2.$

So the particular solution is:

$$\underline{\underline{x = 0.2e^{-3t} - 1.2\cos t + 0.6\sin t}}$$

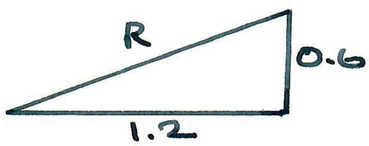
(iii) For large t the e^{-3t} term tends to 0 and

$$x \approx -1.2\cos t + 0.6\sin t.$$

This can be expressed as

$$R\cos(t+\alpha)$$

for some α, and where R is the hypotenuse of the right triangle



So $R = \sqrt{0.6^2 + 1.2^2} = \sqrt{0.36 + 1.44} = \sqrt{1.80}$

$$= \underline{0.6\sqrt{5}} \quad (\approx 1.342)$$

R is then the amplitude of the oscillation for large t.

(iv) Now $\frac{d^3x}{dt^3} + 2\frac{d^2x}{dt^2} - 3\frac{dx}{dt} = 0$

(and note the particle is <u>now</u> at the <u>origin</u> when $t = 0$, though with non-negative $\frac{dx}{dt}$ and $\frac{d^2x}{dt^2}$.)

The Auxiliary Function is similar to before but with an extra λ factor:

$$\lambda^3 + 2\lambda^2 - 3\lambda = 0$$

$$\lambda(\lambda+3)(\lambda-1) = 0 \qquad \lambda = -3, 1 \text{ or } 0$$

CF: $Re^{-3t} + Se^{t} + T$ where T = a constant because the Te^{0x} term $= T$.

There's no Particular Integral here because the RHS is 0 (the equation is homogeneous) ... though presumably we could integrate the $\frac{d^2x}{dt^2} + \cdots$ form and equate it to a constant.

At $t=0$:

So $x = Re^{-3t} + Se^{t} + T$ $\qquad R+S+T = 0$

$\frac{dx}{dt} = -3Re^{-3t} + Se^{t}$ $\qquad -3R+S = k$

$\frac{d^2x}{dt^2} = 9Re^{-3t} + Se^{t}$ $\qquad 9R+S = 6$

This leads to $R = \frac{6-k}{12}$

$S = \frac{3k+6}{12}$

$T = -\frac{2k-6}{3}$

So the Particular Solution is

$$\underline{\underline{x = \frac{6-k}{12}e^{-3t} + \frac{3k+6}{12}e^{t} - \frac{2k+6}{3}}}$$

(v) Lastly ... as t increases, the e^t term will dominate and be unbounded: unless $\frac{3k+6}{12} = 0$. But we know $k > 0$, so this cannot happen – so x is always unbounded.

2) (a) $\dfrac{dv}{dt} = -0.25\,(v^2+2v)$

We can do this by (1) Separation of Variables, (2) Separation into Fractions:

$$\frac{dv}{v^2+2v} = -0.25\,dt$$

$$\frac{1}{v^2+2v} = \frac{A}{v} + \frac{B}{v+2} = \frac{A(v+2)+Bv}{v(v+2)}$$

Comparing coefficients:

$$Av + Bv = 0 \qquad \text{so} \qquad A = \tfrac{1}{2}$$
$$2A = 1 \qquad\qquad\qquad B = -\tfrac{1}{2}$$

So $\quad \dfrac{1}{2}\left(\dfrac{1}{v} - \dfrac{1}{v+2}\right) dv = -0.25dt$

$$\left(\frac{1}{v+2} - \frac{1}{v}\right) dv = 0.5\,dt$$

Integrating:

$$\ln(v+2) - \ln(v) = 0.5t + C$$

or $\quad \dfrac{v+2}{v} = Ae^{0.5t} \qquad v = \dfrac{2}{Ae^{0.5t}-1}$

At $(0, 20)$:

$$\frac{22}{20} = A$$

So $\quad v = \dfrac{2}{\frac{22}{20}e^{0.5t}-1}$

$$v = \frac{20}{11e^{0.5t}-10}$$

26) (i) $x\dfrac{dy}{dx} - 4y = x^3\sqrt{x} = x^{\frac{7}{2}}$

SF: $\dfrac{dy}{dx} - \dfrac{4}{x}y = x^{\frac{5}{2}}$

IF: $e^{\int -\frac{4}{x}dx} = e^{-4\int \frac{dx}{x}} = e^{-4\ln x} = e^{\ln\left(\frac{1}{x^4}\right)} = \dfrac{1}{x^4}$

So $\dfrac{1}{x^4}\dfrac{dy}{dx} - \dfrac{4}{x^5}y = x^{-\frac{3}{2}}$

$\dfrac{d}{dx}\left(\dfrac{y}{x^4}\right) = x^{-\frac{3}{2}}$

$\dfrac{y}{x^4} = \int x^{-\frac{3}{2}}\,dx = -2x^{-\frac{1}{2}} + C$

So GS: $\underline{\underline{y = -2x^{\frac{7}{2}} + Cx^4}}$

(you can do this by separating the original equation into LHS - solve by SOV, and RHS - find a PI $-2x^{7/2}$, but in fact it's solvable all in one piece)

For the PS: use $(1, 0)$:

$0 = -2 + C \; : \; C = 2$

So $\underline{\underline{y = 2x^4 - 2x^{\frac{7}{2}} = 2x^3(x - \sqrt{x})}}$

(ii) $\dfrac{dy}{dx} = 8x^3 - 2\times\dfrac{7}{2}x^{\frac{5}{2}} = 8x^3 - 7x^{\frac{5}{2}}$

$= x^{\frac{5}{2}}(8\sqrt{x} - 7)$

Given $x > 0$, this can only be 0 when $8\sqrt{x} - 7 = 0$:

$\sqrt{x} = \dfrac{7}{8} \qquad x = \dfrac{49}{64}$

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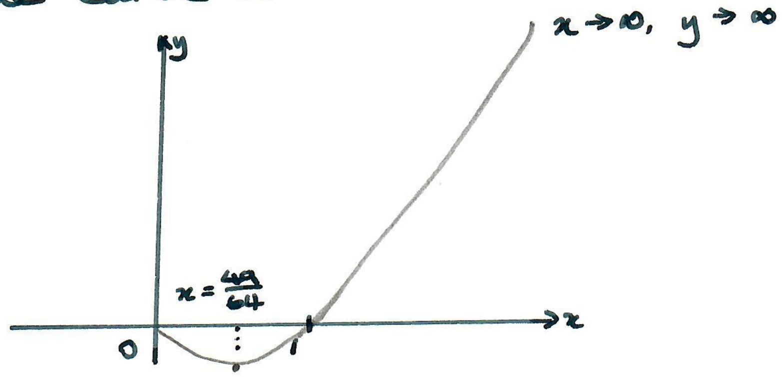
2) b) (iii) when $x = 1$, $\frac{dy}{dx} = 1(8-7) = 1$.

(Also $y = 0$)

As $x \to \infty$ the $2x^4$ dominates so $y \to \infty$.

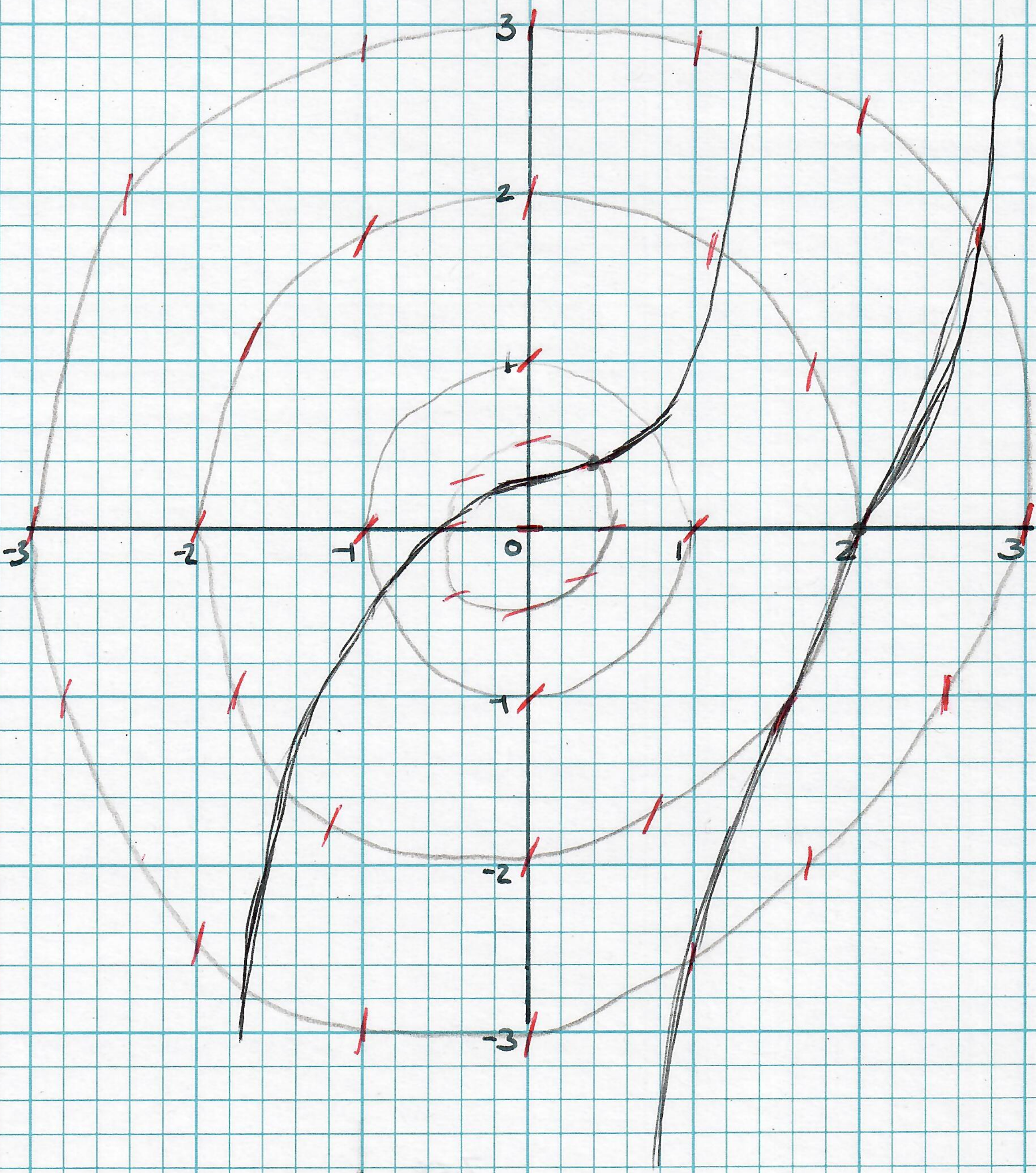
So the curve is:



3) a) $\frac{dy}{dx} = \sqrt{x^2+y^2}$

(i) The isoclines are lines where $\frac{dy}{dx}$ is constant - and clearly lines of the form $m = \sqrt{x^2+y^2}$ are circles about the origin of radius m. So each isocline is a circle where $\frac{dy}{dx}$ of the solution curve (though not of the circle) is constant.

(ii) + (iii) See next sheet. The solution curves aren't great, but without further (difficult) analysis we can't be more accurate about x and y intercepts etc.



(iv) $x_{r+1} = x_r + h$

$y_{r+1} = y_r + h y_r'$

n	x_n	y_n	y_n'
0	0.5	0.5	0.7071
1	0.55	0.5354	0.7675
2	0.6	0.5738	0.8302
3	0.65	0.6153	

Estimate for y when $x = 0.65$ is

$y = 0.6153$.

(v) Improve the accuracy by using a smaller h.

3) (b)(i) $\dfrac{dy}{dx} + y = x^2 - 1.$

AE: $\lambda + 1 = 0 \quad \lambda = -1$

CF: $y = Ae^{-x}$

For the PF, $y = Bx^2 + Cx + D$

$y' = 2Bx + C$

$y' + y = Bx^2 + (2B+C)x + (C+D) = x^2 - 1$ (given)

Equating coeffs: x^2: $\underline{B = 1}$

x: $2B + C = 0 \quad 2 + C = 0 \quad \underline{C = -2}$

1: $C + D = -1 \quad -2 + D = -1 \quad \underline{D = 1}$

So the PI is $x^2 - 2x + 1$ and

the GS is: $Ae^{-x} + x^2 - 2x + 1$

$= \underline{\underline{Ae^{-x} + (x-1)^2}}$

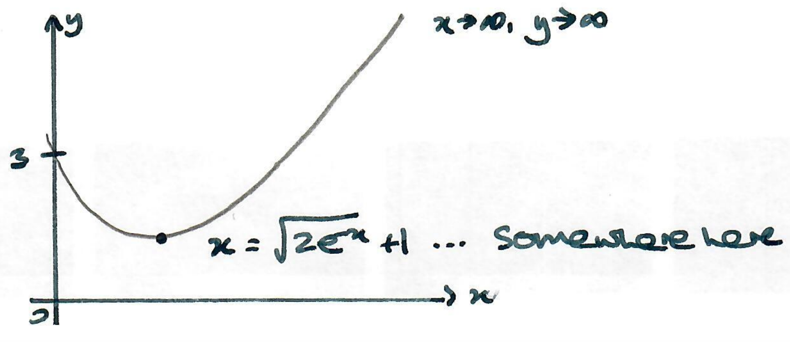
(ii) At $(0, 3)$:

$3 = A \times 1 + (-1)^2 = A + 1 \quad \underline{A = 2}$

So the PS is $\underline{\underline{y = 2e^{-x} + (x-1)^2}}$

The e^{-x} term is always +ve, and so is the $(x-1)^2$ term (being a square).

Also the curve is always increasing from $x = \sqrt{2e^{-x}} + 1$, and approaches x^2 as $x \to \infty$.



4) $\frac{dx}{dt} = 2x + 2y$ ——— 1

$\frac{dy}{dt} = 6y - 4x$ ——— 2

(i) from 1: $\ddot{x} = 2\dot{x} + 2\dot{y}$

from 2: $= 2\dot{x} + 2(6y - 4x)$

$= 2\dot{x} + 12y - 8x$

from 1: $= 2\dot{x} + (6\dot{x} - 12x) - 8x$

$= 8\dot{x} - 20x$

Tidying-up: $\ddot{x} - 8\dot{x} + 20x = 0$ ——— 3

AE: $\lambda^2 - 8\lambda + 20 = 0$

$(\lambda = \frac{8 \pm \sqrt{64-80}}{2}$

$= 4 \pm 2i$

So the CF is

$x = e^{4t}(A\cos 2t + B\sin 2t)$

The equation 3 is homogeneous so there is no PI, and the GS is:

$$x = e^{4t}(A\cos 2t + B\sin 2t) \text{ — } 5$$

(ii) (Method 1) in 2: (Method 2 is better...)

$\dot{y} - 6y = -4x = -4e^{4t}(A\cos 2t + B\sin 2t)$

IF: $e^{\int -6dt} = e^{-6t}$

So $e^{-6t}\dot{y} - 6e^{-6t}y = -\cancel{2}4e^{-2t}(A\cos 2t + B\sin 2t)$

So $\frac{d}{dt}\left(e^{-6t}y\right) = -4e^{-2t}(A\cos 2t + B\sin 2t)$

$$e^{-6t}y = -4\left[A\int e^{-2t}\cos 2t\,dt + B\int e^{-2t}\sin 2t\,dt\right] - (4)$$

The two integrals are horrible to do 'from scratch', though they come right. If you're a physicist or an engineer they're featured in many standard tables, so although they're not given in the book for these maths exams we'll quote them here:

$$\int e^{bx}\sin(ax)\,dx = \frac{1}{a^2+b^2}e^{bx}\left(b\sin ax \cancel{+b} - a\cos ax\right)$$

$$\int e^{bx}\cos(ax)\,dx = \frac{1}{a^2+b^2}e^{bx}\left(a\sin ax + b\cos ax\right)$$

In our case $a = 2$ and $b = -2$, so this gives

$$\int e^{-2t}\sin 2t\,dt = \frac{1}{4}e^{-2t}\left(-\cancel{2}\sin 2t - \cancel{2}\cos 2t\right)$$

$$\int e^{-2t}\cos 2t\,dt = \frac{1}{4}e^{-2t}\left(\sin 2t \cancel{+} - \cos 2t\right)$$

(even here I'm making mistakes!)

So the RHS of (4) is:

$$\frac{-4}{4}e^{-2t}\left[A(\sin 2t - \cos 2t) + B(-\sin 2t - \cos 2t)\right]$$

$$= e^{-2t}\left[(B-A)\sin 2t + (A+B)\cos 2t\right]$$

Finally, multiplying out:

$$y = e^{4t}\left[(B-A)\sin 2t + (A+B)\cos 2t\right]$$

Method 2 (much easier):

Use (1) (there's still life in it):

$$2y = \dot{x} - 2x$$

$$y = \frac{\dot{x}}{2} = \frac{1}{2}\left[\begin{array}{l}4e^{4t}(A\cos 2t + B\sin 2t)\\ +2e^{4t}(-A\sin 2t + B\cos 2t)\end{array}\right]$$

$$-x = -e^{4t}(A\cos 2t + B\sin 2t)$$

$$= e^{4t}\left[(2A+B-A)\cos 2t + (2B-A-B)\sin 2t\right]$$

$$= e^{4t}\left[(A+B)\cos 2t + (B-A)\sin 2t\right] \quad \text{— (6)}$$

(and even this was hard enough.)

(iii) when $t=0$, $\dot{x} = 10$ and $y = kx$

From (5) (again): $\dot{x} = 4e^{4t}(A\cos 2t + B\sin 2t) + 2e^{4t}(-A\sin 2t + B\cos 2t)$

$$= e^{4t}\left((4A+2B)\cos 2t + (4B-2A)\sin 2t\right)$$

$$= 1\,(4A+2B)$$

$$= 10 \text{ (given)}$$

So $\underline{2A+B=5}$

Also $y = kx$:

From 6 and 5:

$$(A+B) = k\dot{A}$$

$$A(1-k) + B = 0 \quad \text{——— } 7$$

$$2A + B = 5 \quad \text{——— } 8$$

8 - 7: $A(2-1+k) = 5$ $\qquad A = \dfrac{5}{1+k}$

In 7: $\dfrac{5(1-k)}{(1+k)} = -B$ $\qquad B = -5\dfrac{(1-k)}{(1+k)} = \dfrac{5(k-1)}{k+1}$

So $$x = e^{4t}\left(\frac{5}{k+1}\cos 2t + \frac{5(k-1)}{k+1}\sin 2t\right)$$

$$y = e^{4t}\left(\frac{5k}{k+1}\cos 2t + \frac{5(k-2)}{k+1}\sin 2t\right)$$

(iv) $k = 6$:

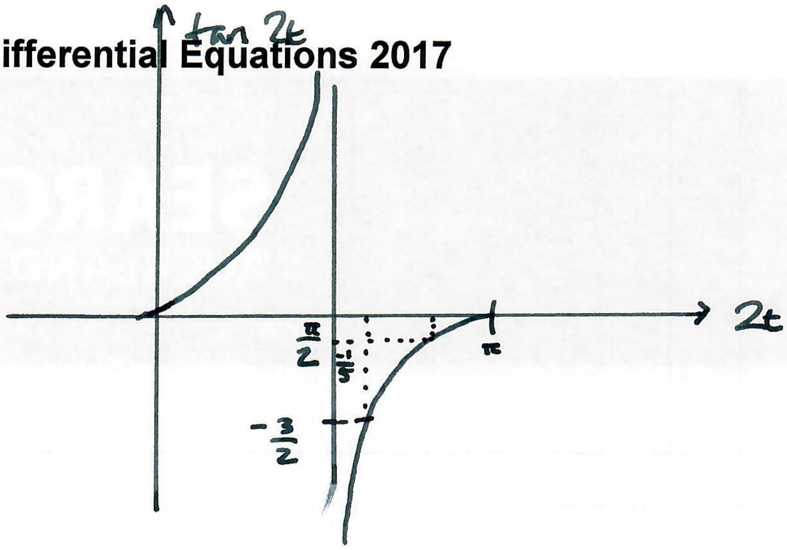
$$x = \frac{5e^{4t}}{\cancel{6}7}(\cos 2t + 5\sin 2t)$$

$$y = \frac{5e^{4t}}{\cancel{6}7}(6\cos 2t + 4\sin 2t)$$

This boils down to a question of which trig. expression becomes 0 first:

$\cos 2t + 5\sin 2t = 0 \quad \Rightarrow \tan 2t = -\frac{1}{5}$

$6\cos 2t + 4\sin 2t = 0 \quad \Rightarrow \tan 2t = \frac{-3}{2}$



Clearly $\tan 2t = -\frac{3}{2}$ occurs first, so Y dies out first, at $2t = 2.16$

or $\underline{\underline{t = 1.08}}$

(v) The model doesn't (claim to) hold when there's only one species left.