1) a) $E(M) = \sum_{1}^{4} m \cdot P(M=m)$

$= 1\times\frac{1}{16} + 2\times\frac{3}{16} + 3\times\frac{5}{16} + 4\times\frac{7}{16}$ (*)

$= \frac{1}{16}(1+6+15+28)$

$= \frac{50}{16} = \frac{25}{8} = 3\frac{1}{8}$

b) $G(t) = \frac{1}{16}t + \frac{3}{16}t^2 + \frac{5}{16}t^3 + \frac{7}{16}t^4$

$G(1) = \frac{1}{16} + \frac{3}{16} + \frac{5}{16} + \frac{7}{16} = \frac{16}{16} = 1$

c) $G'(t) = \frac{1}{16} + \frac{6}{16}t + \frac{15}{16}t^2 + \frac{28}{16}t^3$ (simple differentiation)

So $G'(1)$ follows the same structure as (*), and calculating it out,

$G'(1) = \frac{1}{16} + \frac{6}{16}1 + \frac{15}{16}1^2 + \frac{28}{16}1^2$

$= \frac{50}{16} = 3\frac{1}{8}$ again

$= E(M)$

d) $G_X(t) = \sum_{x=0}^{2} P(X=x)\,t^x$

$= P(X=0)t^0 + P(X=1)t^1 + P(X=2)t^2$ — (1)

Now given there are 2 red and 3 yellow balls,

there are the following Probabilities:

	1st ball	2nd ball
red	0.4	red 0.25 yellow 0.75
yellow	0.6	red 0.5 yellow 0.5

So $P(YY) = 0.6 \times 0.5 = 0.3$

$P(RY) = 0.4 \times 0.75 = 0.3$

$P(YR) = 0.6 \times 0.5 = 0.3$

$P(RR) = 0.4 \times 0.25 = 0.1$ (sum = 1 ✓)

So $P(X=0) = 0.3$

$P(X=1) = 0.3 + 0.3 = 0.6$

$P(X=2) = 0.1$

Expressing these as $\frac{1}{10}$s, in ①:

$$G_X(t) = \frac{3}{10} + \frac{6}{10}t + \frac{1}{10}t^2$$

$$= \frac{3}{10} + \frac{3}{5}t + \frac{1}{10}t^2$$ as required. ✓

e) In similar fashion to (d)

	Unbiased	Biassed
Head	0.5	$(1-p)$
Tail	0.5	p

We know the coeff. of t^2 is $P(\text{Tails} = 2)$

So we have $0.5p = \frac{1}{3}$

$$\underline{\underline{p = \frac{2}{3}.}} \quad \text{and } 1-p = \frac{1}{3}$$

Again in similar fashion

$$P(HH) = 0.5 \times \frac{1}{3} = \frac{1}{6}$$

$$P(HT) = 0.5 \times \frac{2}{3} = \frac{1}{3}$$

$$P(TH) = 0.5 \times \frac{1}{3} = \frac{1}{6}$$

$$P(TT) = 0.5 \times \frac{2}{3} = \frac{1}{3} \quad \text{(known)}$$

So $P(T=1) = P(HT) + P(TH) = \frac{1}{3} + \frac{1}{6} = \frac{1}{2}.$

As ~~before~~, $G_Y(t) = P(Y=0) + P(Y=1)t + P(Y=2)t^2$

$$= \underline{\underline{\frac{1}{6} + \frac{1}{2}t + \frac{1}{3}t^2}}$$

f) $G_Z(t) = G_X(t)\, G_Y(t)$

$$= \left(\frac{3}{10} + \frac{3}{5}t + \frac{1}{10}t^2\right)\left(\frac{1}{6} + \frac{1}{2}t + \frac{1}{3}t^2\right)$$

$$= \frac{3}{10}\left(\frac{1}{6} + \frac{1}{2}t + \frac{1}{3}t^2\right)$$

$$+ \frac{3}{5}t\left(\frac{1}{6} + \frac{1}{2}t + \frac{1}{3}t^2\right)$$

$$+ \frac{1}{10}t^2\left(\frac{1}{6} + \frac{1}{2}t + \frac{1}{3}t^2\right)$$

$$= \frac{1}{20} + \left(\frac{3}{20} + \frac{3}{30}\right)t + \left(\frac{1}{10} + \frac{3}{10} + \frac{1}{60}\right)t^2$$

$$+ \left(\frac{1}{5} + \frac{1}{20}\right)t^3 + \frac{3}{10}t^4$$

(coeffs should all sum to 1)

$$= \frac{1}{20} + \frac{15}{60}t + \frac{25}{60}t^2 + \frac{5}{20}t^3 + \frac{1}{30}t^4$$

$$= \frac{1}{60}\left(3 + 15t + 25t^2 + 15t^3 + 2t^4\right)$$

(the coeffs sum to 1)

So $G'_z(t) = \frac{1}{\cancel{15}\,60}\left(15 + 50t + 45t^2 + 8t\right)$

idiot

And since in general $E(z) = G'_z(1)$,

$$E(z) = \frac{1}{60}\left(15 + 50 + 45 + 8\right)$$

$$= \frac{118}{60} = \frac{59}{30} = 1\frac{29}{30}$$

or 1.966

2) $$k(x) = \frac{|f''(x)|}{(1+(f'(x)))^{3/2}}$$

a) $g(x) = mx + c.$

here $g'(x) = m$, $g''(x) = 0$: $|g''(x)| = 0$

So since $k(x)$ has $|g''(x)|$ as the numerator, the curvature of any of these functions is $\underline{\underline{0.}}$

b) $h(x) = ax^2 + bx + c$

i) Given $k'(x)$ here, max k is reached when $k'(x) = 0$. The denominator is always +ve, so we need only consider the numerator:

$$12a|a|(2ax+b) = 0$$

and this gives $2ax + b = 0$ ——— ①

or $x = \dfrac{-b}{2a}.$

So k_{max} occurs at $\underline{\underline{x = \dfrac{-b}{2a}}}$

ii) Using the given formula for $k(x)$, and using ①:

$$k_{max} = \frac{|2a|}{(1+0^2)^{3/2}} = \underline{\underline{|2a|}}$$

iii) $$k(x) = \frac{2|a|}{\left(1 + (2ax+b)^2\right)^{3/2}}$$

As $x \to \infty$ the $(2ax+b)$ term will grow large and dominate the denominator, while the numerator will stay at $2|a|$. So $k(x)$ will tend to 0.

This indicates $h(x)$ will become more and more 'like a straight line' - which makes sense because it's a parabola, (even though for it the ax^2 term will dominate).
- so it'll never be a straight line.

iv) $p(x) = -2x^2 + 2x - 10$

$q(x) = 2x^2 + 5x + 25$

using the result in (bii):

k_{max} of $p = |2 \times -2| = 4$

'times', not 'x'

k_{max} of $q = |2 \times 2| = 4$

So $\underline{\underline{k_{max} \text{ of } p = k_{max} \text{ of } q}}$: answer $\underline{\underline{C}}$ as given.

c) $v(x) = \ln x$

i) As before, k_{max} occurs when $k'(x) = 0$

i.e $1 - 2x^2 = 0$

$$2x^2 = 1$$

$$x^2 = \frac{1}{2}$$

$$x = \cancel{\pm}\frac{1}{\sqrt{2}} \quad \text{(discard -ve)}$$

ii) using the formula given,

$$k_{max} = \frac{x}{(1+x^2)^{3/2}} = \frac{1}{\sqrt{2}} \frac{1}{(1+\frac{1}{2})^{3/2}}$$

$$= \frac{1}{\sqrt{2}} \frac{1}{(\frac{3}{2})^{3/2}} = \frac{1}{\sqrt{2}} \frac{2^{3/2}}{3^{3/2}}$$

$$= \frac{2}{3\sqrt{3}} = \frac{2\sqrt{3}}{9} \text{ as required.}$$

d) i) As before,

$k'(x) = 0$ at k_{max}

so $e^x(1 - 2e^{2x}) = 0$

$$1 - 2e^{2x} = 0 \quad (e^x \neq 0)$$

$$2e^{2x} = 1$$

$$e^{2x} = \frac{1}{2}$$

$$2x = \ln\left(\frac{1}{2}\right)$$

$$x = \frac{1}{2}\ln\left(\frac{1}{2}\right).$$

So $k_{max} = \dfrac{e^{\frac{1}{2}\ln\left(\frac{1}{2}\right)}}{\left(1 + e^{2\times\frac{1}{2}\ln\left(\frac{1}{2}\right)}\right)^{5/2}}$

$$= \frac{e^{\ln\frac{1}{\sqrt{2}}}}{\left(1+e^{\ln\frac{1}{2}}\right)^{3/2}}$$

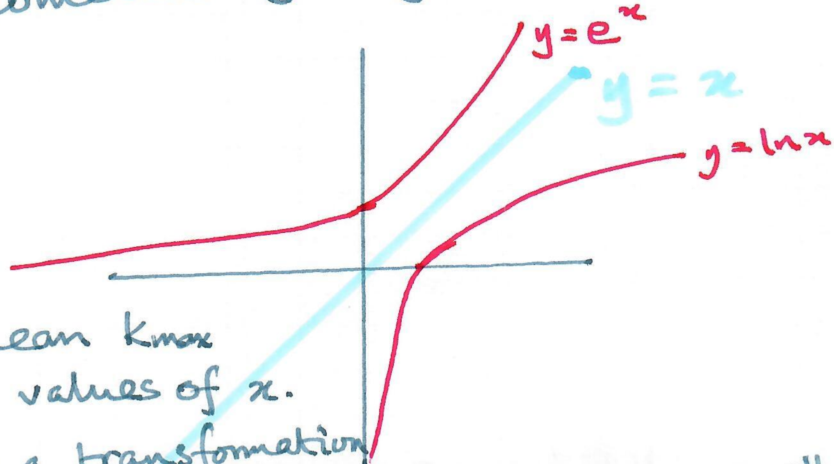
$$= \frac{1}{\sqrt{2}} \cdot \frac{1}{\left(1+\frac{1}{2}\right)^{3/2}}$$

$$= \frac{1}{\sqrt{2}} \frac{1}{\left(\frac{3}{2}\right)^{3/2}}$$

$$= \frac{1}{\sqrt{2}} \frac{2^{3/2}}{3^{3/2}}$$

$$= \frac{2}{3\sqrt{3}} = \frac{2\sqrt{3}}{9} \text{ as required}$$

ii) The values of k_{max} for $v(x)$ and $w(x)$ are equal – probably because the two curves are inverses or mirror images in the line $x = y$
– so geometrically they're the same shape.



Note this doesn't mean k_{max} occurs at the same values of x.

I suspect if we did a transformation $s = x+y$ $t = x-y$ we'd get them matching, with something so $t = \frac{1}{2}(e^s + \ln s)$, $t = \frac{1}{2}(e^s - \ln s)$ --- but lets not go there.

e) $y = \sqrt{r^2 - x^2}$ —— ①

(These are just circles, centre O, radius r)

i) from ①,

$$y^2 = r^2 - x^2$$

$$2y\frac{dy}{dx} = -2x$$

$$\frac{dy}{dx} = -\frac{x}{y}$$

$$\frac{d^2y}{dx^2} = -\frac{y \cdot 1 - x\frac{dy}{dx}}{y^2}$$

$$= -\frac{y + \frac{x^2}{y}}{y^2}$$

$$= -\frac{\frac{y^2+x^2}{y}}{y^2}$$

$$= -\frac{r^2}{y^3}$$ as required.

⊛ question may be slightly misleading – it's not 'constant' for the whole family.

+ve because || used.

ii) using the original formula for curvature:

$$k(x) = \frac{|y''|}{\left(1+(y')^2\right)^{3/2}} = \frac{r^2}{y^3}\frac{1}{\left(1+\frac{x^2}{y^2}\right)^{3/2}}$$

$$= \frac{r^2}{y^3\left(\frac{y^2+x^2}{y^2}\right)^{3/2}} = \frac{r^2}{y^3\left(\frac{r^2}{y^2}\right)^{3/2}} = \frac{r^2}{\cancel{y^3}\frac{r^3}{\cancel{y^3}}} = \frac{1}{r}.$$

This is constant for any one curve, though not the entire [illegible] ⊛