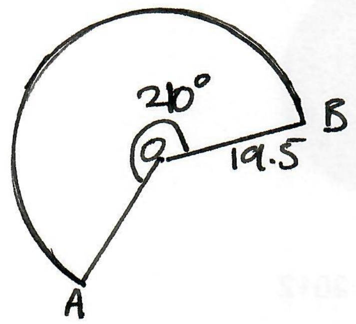
1)




a) Area of a sector $= \left(\frac{r^2\theta}{2}\right)$ $\qquad 210° = \frac{7}{6}\pi$ rad.

$$= \frac{19.5^2}{2} \times \frac{7}{6}\pi$$

$$= \underline{\underline{697\ \text{cm}^3}} \quad (3\ \text{sf}) \checkmark$$

b) Length of the AB arc $= (r\theta)$

$$= 19.5 \times \frac{7}{6}\pi$$

$$= 71.471\ \text{cm}$$

which is the circumference of the cone... $2\pi r$

So

$$r_{cone} = \frac{71.471}{2\pi} = \underline{\underline{11.375.}} \checkmark$$

(as a check ... this $= \frac{210}{360} \times 19.5$, which is as it should be.)

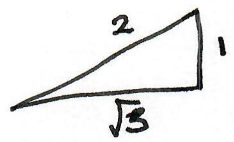
2) $f(x) = a\tan(2x) + b$

a) $\tan(x)$ repeats on a period of $\pi$ (not the usual $2\pi$) - so $\tan(2x)$ repeats on a period $\frac{\pi}{2}$.

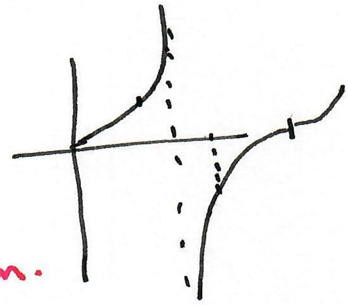
so the period of $f(x)$ is $\frac{\pi}{2}$. ✓

b) Using the 2 points given:

$$f\left(\frac{\pi}{12}\right) = a\tan\left(\frac{\pi}{6}\right) + b = 5$$




$$\Rightarrow \frac{a}{\sqrt{3}} + b = 5 \quad \text{—— ①}$$



$$f\left(\frac{\pi}{3}\right) = a\tan\left(\frac{2\pi}{3}\right) + b = 7.$$

idiot: this is cos, not tan.

$$\Rightarrow a\left(\cancel{-\frac{1}{2}}\ -\sqrt{3}\right) + b = 7$$

$$-\frac{a\sqrt{3}}{\cancel{2}} + b = 7. \quad \text{—— ②}$$

① - ②: $\frac{a}{\sqrt{3}} + \frac{+a\sqrt{3}}{\cancel{2}} = -2$

$$\cancel{\frac{2a + \sqrt{3}a}{2\sqrt{3}} = -2}$$

$a + 3a = -2\sqrt{3}$

$a = -\frac{\sqrt{3}}{2}$

$$\cancel{a\left(\frac{2+\sqrt{3}}{2\sqrt{3}}\right) = -2}$$

$$\cancel{a = \frac{-4\sqrt{3}}{2+\sqrt{3}}}$$

$$\cancel{= \frac{-4\sqrt{3}(2-\sqrt{3})}{(2+\sqrt{3})(2-\sqrt{3})}}$$

$$\cancel{= \frac{-8\sqrt{3}+12}{4-3}}$$

$$\cancel{= \underline{\underline{12-8\sqrt{3}}}.}$$

~~(This seems hugely over-complex)~~

from ②: $-\dfrac{12-8\sqrt{3}}{2} + b = 7$

①: $-\dfrac{\sqrt{3}}{2\sqrt{3}} + b = 5$

$b = 5\frac{1}{2}\left(\frac{11}{2}\right).$

$$b = \frac{14+12-8\sqrt{3}}{2}$$

$$= \underline{\underline{13+4\sqrt{3}}}.$$

3) $\dfrac{dP}{dt} = -104000\,e^{-0.0145t}$

$t$ = time from 0:00 1 Jan 2022, in years.

we want to know $P$ when $t=4$. (start of 2026)

And at $t=0$, $P = 6.78\times10^{6} = 6780000$.

Solving the equation:

Separate variables:

$$dP = -104000\,e^{-0.0145t}\,dt$$

Integrate:

$$P = \frac{-104000}{-0.0145}\,e^{-0.0145t} + c$$

$$P = 7172414\, e^{-0.0145t} + c$$

using the initial condition:

$$6780000 = 7172414\, \underbrace{e^{0}}_{=1} + c$$

So $C = 6780000 - 7172414$

$= -392414.$

So $P = 7172414\, e^{-0.0145t} - 392414.$

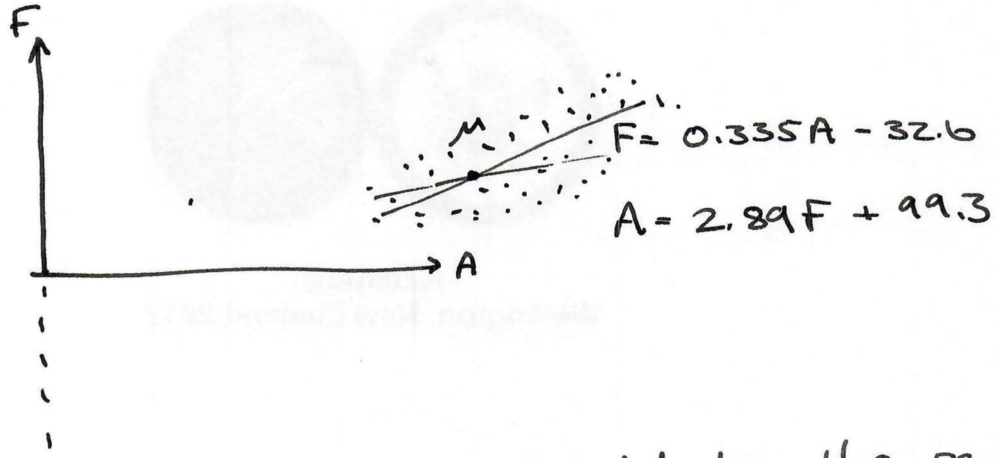
So when $t = 4$,

$$P = 7172414\, e^{-0.0145 \times 4} - 392414$$

$= \underline{\underline{6375834}}.$ ✓ but my working seems long-winded.

(which ties in with the -ve term in the original eq$^n$ - indicating the population is declining.)

4)




(you'd think the lines should be the same, but we're dealing with scattered data).

a) For this, use the A line:

$$A = 2.89F + 99.3$$

$$= 2.89 \times 19.8 + 99.3$$

$$= 156.5 \text{ cm} \quad (157 \text{ cm to } 3\text{ sf})$$

(which seems long)

b) If both lines pass through the mean, we solve the two equations as simultaneous...

$$F = 0.335A - 32.6 \quad \text{— ①}$$

$$A = 2.89F + 99.\cancel{6}3 \quad \text{— ②}$$

This makes all the difference - which suggests an ill-conditioned problem.

① × 2.89 : $2.89F = 0.9682A - 94.214$

In ② : $A = 0.9682A - 94.214 + 99.\cancel{6}3$

$0.0318A = \cancel{193.814}$ ~~5.386~~ $-5.086$

$A =$ ~~6095 cm (this is mad)~~

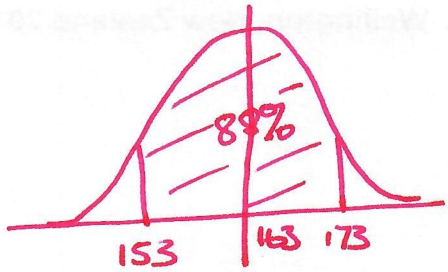
$\underline{169.4 \text{ cm}}$. 159.9 cm / 160cm

$F = 0.335 \times 169.4 - 32.6 = \underline{24.1 \text{ cm}}$

20.9cm

c) READ THE QUESTION!

And note this is about something entirely different: the height. IGNORE everything previous.

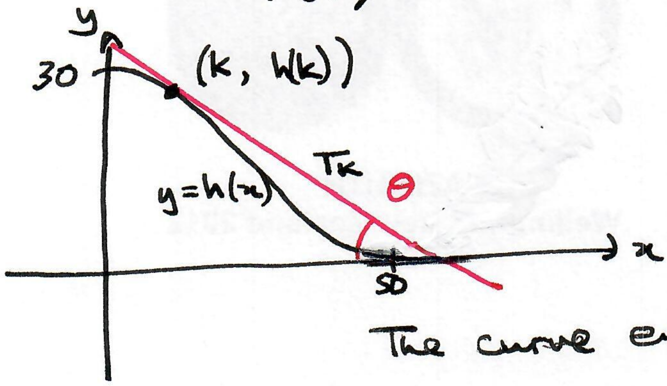



Happily the graph is symmetric, so we know $P(H<153) = P(H>173) = 6\%$ or $0.06$.

So we look for a $\sigma$ value where in Excel terms: norm.dist$(153, 163, \sigma, \text{TRUE}) = 0.06$.

and by iterating a bit we get $\sigma = 6.43$.

5) $h(x) = 15\cos\left(\frac{\pi x}{50}\right) + 15$




The curve ends at 50.

a) Gradient at $(k, h(k))$ is $\frac{dh}{dx}(k)$

$$= -15\left(\frac{\pi}{50}\right)\sin\left(\frac{\pi x}{50}\right)$$

$$= -\frac{3\pi}{10}\sin\left(\frac{\pi k}{50}\right)$$

b) $\theta = \frac{\pi}{8}$

In terms of relation to the gradient of $T_k$, this means actually the opposite angle to $\theta$ is $-\frac{\pi}{8}$, and we have:

$$-\tan\left(\frac{\pi}{8}\right) = -\frac{3\pi}{10}\sin\left(\frac{\pi k}{50}\right)$$

I can't think of any way to solve this without a calculator...

$\tan\left(\frac{\pi}{8}\right) = \tan(22.5^\circ) = 0.4142$

$\sin\left(\frac{\pi k}{50}\right) = 0.4142 \times \frac{10}{3\pi} = 0.4395$

$\frac{\pi k}{50} = \cancel{19.24^\circ} = \cancel{0.3357\text{ rad}}$

$26.07^\circ = 0.4550$ rad.

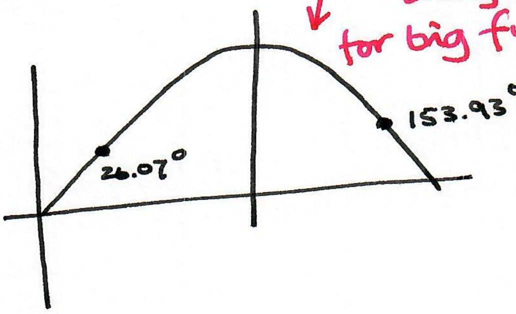
So $\underline{\underline{k = 7.24}}$ is the primary value.

But there might be others, albeit in the range up to $x = 50$. So look for

$$\sin\left(-\frac{\pi}{50}k + \pi\right) = 0.4395$$

$$\pi - \frac{\pi k}{50} = 0.4550$$

Always draw a diagram for big functions above $\frac{\pi}{2}$.



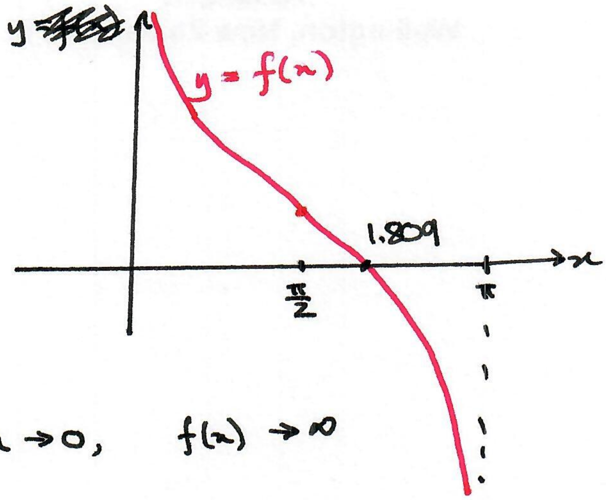

giving $\underline{\underline{k = 42.76}}$.

Given that the next solution for $k$ would be well beyond $x = 50$, we conclude these are the only two values of $k$.

(and it accords with the diagram.)

6) $f(x) = 4\cot x + \sin x.$

i) $= \dfrac{4\cos x}{\sin x} + \sin x$




a) when $x \to 0$, $f(x) \to \infty$

$x \to \pi$, $f(x) \to -\infty$

b) at $x = \frac{\pi}{2}$, $f(x) = 0 + \sin\frac{\pi}{2} = 1.$

c) on the $x$-axis (at intercept):

$$\frac{4\cos x}{\sin x} = -\sin x$$

$$4\cos x = -\sin^2 x$$

$$= -(1-\cos^2 x)$$

$$\cos^2 x - 4\cos x - 1 = 0$$

$$\cos x = \frac{4 \pm \sqrt{16+4}}{2} = 2 \pm \sqrt{5}$$

$$= -0.2361$$

$$x = 103.65^\circ = 1.809 \text{ rad.}$$

(which is $> \frac{\pi}{2}$)

Other than this I can't think what to say.

b) $\dfrac{4\cos x}{\sin x} + \sin x = 2$

$4\cos x + \sin^2 x = 2\sin x$

$4\cos x + (1-\cos^2 x) = 2\sin x$

$4\cos x + 1 - \cos^2 x = 2\sin x$

But I can't see how to go further without a numerical approach ... and the published markscheme has nothing at all, other than the answer 1.32 (which works).

7) $\underset{\sim}{r} = \begin{pmatrix}10\\3\\0.5\end{pmatrix} + 4t\begin{pmatrix}10\\-25\\0\end{pmatrix}$

$\begin{pmatrix}10\\3\\0.5\end{pmatrix}$ is a fixed point: $4\begin{pmatrix}10\\-25\\0\end{pmatrix}$ represents the speed of the helicopter with 1 unit of its 'size' traversed in every hour.

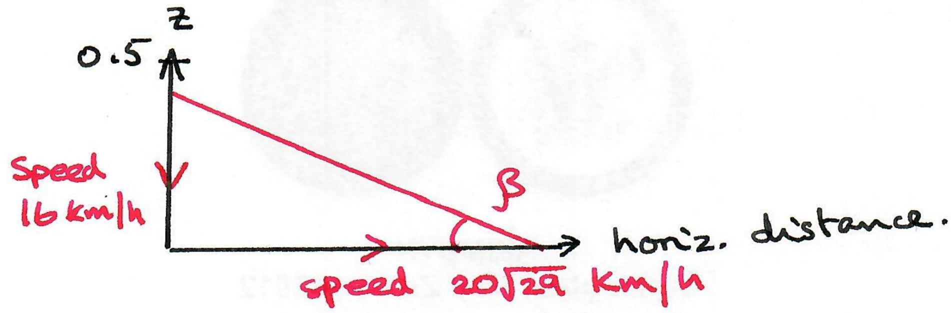
So $\left|4\begin{pmatrix}10\\-25\\0\end{pmatrix}\right| = 4\sqrt{10^2+25^2+0^2}$

$= 4\sqrt{725}$

$= 4\times 5\sqrt{29}$

$= \underline{\underline{20\sqrt{29}}}$ km/h. ✓

b)




$\beta$ is as shown in the diagram,

with $\tan\beta = \dfrac{16}{20\sqrt{29}} = 0.1486$

So $\beta = 8.450^\circ$ ✓

8) $f(t) = \sin(2t+1)$

$g(t) = \sin(2t+3)$

$h(t) = f(t) + g(t) = \sin(2t+1) + \sin(2t+3)$

It's not quite clear what we're allowed to assume here (and the marking scheme doesn't help) so we'll note from the formula book:

we know $e^{i\theta} = \text{cis}\,\theta = \cos\theta + i\sin\theta$

$e^{-i\theta} = \text{cis}(-\theta) = \cos\theta - \sin\theta$

So $\sin\theta = \text{Im}(e^{i\theta})$ as we'd expect.

I think we're actually supposed to take this as given.

and also an identity like $\text{Im}(w) + \text{Im}(z) = \text{Im}(w+z)$

- and also $\text{cis}(w) \cdot \text{cis}(z) = \text{cis}(w+z)$ etc

though not $\text{Im}(w) \cdot \text{Im}(z) = \text{Im}(w+z)$ or $\text{Im}(wz)$ etc.

So in the expressions given:

$$f(t) = \sin(2t+1) = \text{Im}\left(e^{(2t+1)i}\right)$$

$$g(t) = \sin(2t+3) = \text{Im}\left(e^{(2t+3)i}\right)$$

$$h(t) = \text{Im}\left(e^{(2t+1)i}\right) + \text{Im}\left(e^{(2t+3)i}\right)$$

$$= \text{Im}\left(e^{(2t+1)i} + e^{(2t+3)i}\right)$$ (OK to add Im parts)

$$= \text{Im}\left(e^{2ti}\left(e^{i} + e^{3i}\right)\right)$$ (ordinary factorisation of 'ordinary' numbers)

as required.

b) Find $e^{i} + e^{3i}$.

This can be baffling but comes out easily if you go back to basics and if you use compound angle formulas which are not given in the formula book but should be.

So: $e^{i} + e^{3i}$

$= e^{1i} + e^{3i}$

$= \cos(1) + i\sin(1) + \cos(3) + i\sin(3)$

$= (\cos(1) + \cos(3)) + i(\sin(1) + \sin(3))$

using the identities $\cos A + \cos B = 2\cos\frac{A+B}{2}\cos\frac{A-B}{2}$

and $\sin A + \sin B = 2\sin\frac{A+B}{2}\cos\frac{A-B}{2}$

this gives (with $A=3$, $B=1$):

$2\cos\frac{4}{2}\cos\frac{2}{2} + 2i\left(\sin\frac{4}{2}\cos\frac{2}{2}\right)$

$= 2\cos(1)\left(\cos 2 + i\sin 2\right)$

$= 2\cos(1)\, e^{2i}$

which is in the required form $re^{i\theta}$

with $r = 2\cos(1) = \underline{\underline{1.08}}$ ✓

and $\underline{\underline{\theta = 2}}$ (rad). ✓

c) From (a) and (b),

$h(t) = \operatorname{Im}\left(e^{2ti}\left(e^{i} + e^{3i}\right)\right)$

$$= \operatorname{Im}\left(e^{2i} \cdot 2\cos(1) e^{2i}\right)$$

$$= 2\cos(1) \operatorname{Im}\left(e^{2i} \cdot e^{2i}\right)$$

$$= 2\cos(1) \operatorname{Im}\left(e^{4i}\right)$$

$$= 2\cos(1) \sin(4)$$

$$= 1.0806 \times -0.7568$$

$$= \underline{\underline{-0.818}}$$

a) $\dfrac{dy}{dx} = \dfrac{2x}{x^2+y}$

(1,0) is on the solution curve.

a) Take $h = 0.25$ and use the Euler formulation $y_{n+1} = y_n + h\dfrac{dy}{dx}(n)$.

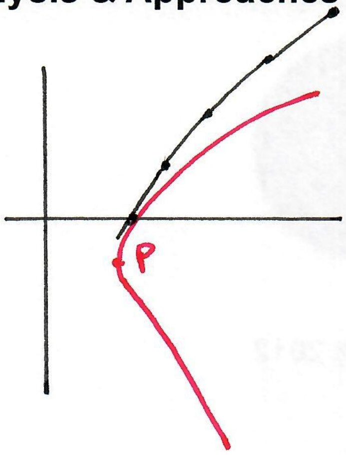
$x_{n+1} = x_n + h$

Setting this out:

| $n$ | $x_n$ | $y_n$ | $\dfrac{dy}{dx}(x_n)$ |
|---|---|---|---|
| 0 | 1.0 | 0.0 | $\dfrac{2}{1+0} = 2$ |
| 1 | 1.25 | $0.0 + 0.25 \times 2 = 0.5$ | $\dfrac{2.5}{1.25^2+0.5} = 1.2121$ |
| 2 | 1.5 | $0.5 + 0.25 \times 1.2121 = 0.8030$ | $\dfrac{3}{1.5^2+0.8030} = 0.9826$ |
| 3 | 1.75 | $0.8030 + 0.25 \times 0.9826 = 1.0487$ | $\dfrac{3.5}{1.75^2+1.0487} = 0.8573$ |
| 4 | 2.00 | $1.0487 + 0.25 \times 0.8573 = 1.2615.$ | |

So the estimate for $y$ when $x = 2$ is 1.2615 ✓

b) i)




The curve is concave down,
so the tangents all lie above the curve -
i.e. over-estimate what the gradient is. So
they over-estimate the values of $y$. ✓

(I worry this makes certain assumptions about the 'value field' away from the curve - but that seems to be beyond us.)

ii) The method here starts with a +ve gradient at $(1,0)$, and as it moves upwards the gradients remain +ve: so the method will head up the curve.

If we wanted to find the −ve value of $y$, we should start below the point P. ✓

10) a) $t_1 = 2$

$t_2 = 7$

$t_3 =$ 6 on the bottom + 2 bottom of middle

4 middle layer 1 - - - top

2 top

$2 \times (3+2+1) + 2+1$

$12 + 3$

$= \underline{\underline{15}}$ ✓

b) $t_4$ 'adds another layer on the bottom'

which means

+ 8 at the bottom + 3 'top of bottom'

$= 15 + 8 + 3$

$= \underline{\underline{26}}$. ✓

c) By now the pattern's becoming familiar

and we can <u>suggest</u>:

$$t_n = 2\sum_{i=1}^{n} i + \sum_{i=1}^{n-1} i$$

which using standard formulae

$$= 2\frac{n(n+1)}{2} + \frac{(n-1)n}{2}$$

$$= \frac{2n^2+2n+n^2-n}{2} = \frac{3n^2+n}{2} = \underline{\underline{\frac{n(3n+1)}{2}}}$$ ✓

 — no — in fact they just accept it.

We ought to prove this by induction, though in truth we'll only prove the maths is correct, not that we're analysed the problem correctly.

Suppose for $k = n$, $t_k = t_n = \dfrac{n(3n+1)}{2}$

Then 'adding another $(n+1)^{th}$ row to the bottom'

adds $(n+1)$ pairs of $\wedge$

and $n$ — elements.

so we have $t_{n+1} = \dfrac{n(3n+1)}{2} + 2(n+1) + n$

$$= \frac{3n^2 + n + 4n + 2 + 2n}{2}$$

$$= \frac{3n^2 + 7n + 2}{2}$$

$$= \frac{(n+1)(3n+4)}{2}$$

$$= \frac{(n+1)(3(n+1)+1)}{2}$$

which is in the form required for $t_{n+1}$.

Also we know $t_1 = 2$ which equals $\dfrac{1(3+1)}{2} = 2$,

so the result holds for $k=1$.

So by induction the result is proved.

(Phew!)

d) 14 packs of cards is $52 \times 14 = 728$ cards.

So we need to know the maximum $n$

such that $t_n = \frac{n(3n+1)}{2} \leqslant 728$.

Find $n$ by solving

$$\frac{n(3n+1)}{2} = 728$$

$$3n^2 + n = 1456$$

$$3n^2 + n - 1456 = 0$$

$$n = \frac{-1 \pm \sqrt{1 + 4 \times 3 \times 1456}}{2 \times 3}$$

$$= \frac{-1 \pm 132.19}{6}$$

$$= \frac{131.19}{6} = 21.86$$ (discarding -ve sol$^n$)

So we take $\underline{\underline{n = 21.}}$ ✓

(As a check: $t_{21} = \frac{21(64)}{2} = 672$.

The next (22nd) row would take $22 \times 2 + 21 = 65$ cards - but this would take total count to $672 + 65 = 737$ - which would be $> 728$.)

e) In this case,

$$t_n = \frac{n(3n+1)}{2} = 52N \text{ for some } N.$$

So wouldn't it be nice if 52 had two factors $n$ and $3n+1$ ....like 4 and $12+1=13$.

This gives $t_4 = \frac{4(13)}{2}$ but factor 2 messes things up.

If we look for another $3n+1$ which is a multiple of 13 this looks difficult...

... but if we let $n=13$, $3n+1=40$

and $t_{13} = \frac{13 \times 40}{2} = 13 \times 20 = 260$

And $260 = 5 \times 52$ - i.e. 5 packs of cards.

So $n=13$ works, but is it the minimum?

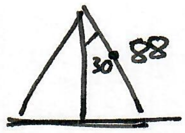
Consider $n$ and $3n+1$ and look for factors 13 and 4:

| $n$ | $3n+1$ | factors 13 and 4 |
|---|---|---|
| 1 | 4 | × |
| 2 | 7 | × |
| 3 | 10 | × |
| 4 | 13 | ✓ but not enough. |
| 5 | 16 | × |
| 6 | 19 | × |
| 7 | 22 | × |
| 8 | 25 | × |
| 9 | 28 | × |
| 10 | 31 | × |

This demonstrates $n=13$ is the lowest possible solⁿ. ✓

f) Each $\wedge$ or $\Delta$ is an equilateral $\Delta$ (debatable in real life)

so we know the height is




$$88 \cos 30^\circ = \frac{88\sqrt{3}}{2} \text{ mm}$$

$$= \underline{76.21 \text{ mm}}$$

So we want to know (min) N such that

$$t_N = \frac{N(3N+1)}{2} \geq 2\text{m}$$

when each layer is 76.21 mm

$$\frac{2000}{76.21} = 26.24$$

So 27 layers will be needed, giving height 2.058 m.

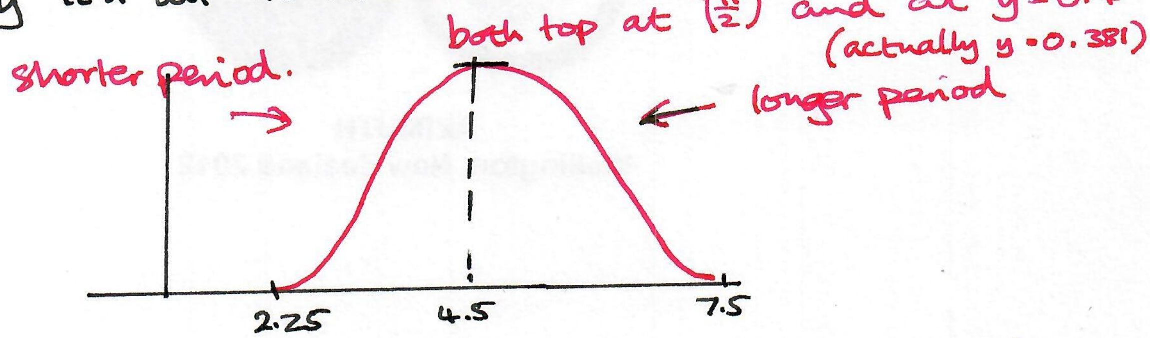
$$\text{And } t_{27} = \frac{27(3 \times 27 + 1)}{2}$$

$$= \frac{27 \times 82}{2}$$

$$= 1107$$

So $\underline{\underline{1107}}$ cards will be needed.

11) The equations here look rather alarming, but presumably it'll all work out.




Because the RHS of the curve has a longer period the 'weight' of the PDF is more to the right... so the median will be $> 4.5$... but we'll come to that.

a) ii) $\displaystyle\int_{2.25}^{4.5} f(t)\,dt = \frac{4}{21}\int_{2.25}^{4.5}\left(1 - \cos\frac{4\pi}{9}(t - 2.25)\right)dt$

$$= \frac{4}{21}\left[t - \frac{9}{4\pi}\sin\frac{4\pi}{9}(t - 2.25)\right]_{2.25}^{4.5}$$

$$= \frac{4}{21}\left[(4.5 - 2.25) - \frac{9}{4\pi}\left(\sin\frac{4\pi}{9}(2.25) - \sin(0)\right)\right]$$

$$= \frac{4}{21}\left[2.25 - \frac{9}{4\pi}\left(\cancel{\sin\pi} - \cancel{\sin 0}\right)\right]$$

$$= \frac{4}{21} \times 2.25 = \underline{\underline{0.429}} \quad \text{(actually } \tfrac{3}{7}\text{)}.$$

(This indicates the RHS will be $0.571$, $\frac{4}{7}$, confirming our view on the 'weight'.)

ii) The curve has been constructed so that each part is a sine curve with peak at $t = 4.5$ (both $(1-\cos)$ terms are 2 there) so the mode (highest value, most common run time) is $\underline{\underline{T = 4.5}}$.

iii) As commented earlier, the 'weight' or 'bulk' of the curve is on the RHS - confirmed by the a(i) result. Since median is the exact mid-way value of the weight (50% either side) this means the median is $> 4.5$. i.e. the median is greater than the mode.

b) The probability here is $\displaystyle\int_{2.25}^{3.5} f(t)\,dt$, which we know is given by

$$\frac{4}{21}\left[t - \frac{9}{4\pi}\sin\frac{4\pi}{9}(t-2.25)\right]_{2.25}^{3.5}$$

$$= \frac{4}{21}\left[(3.5-2.25) - \frac{9}{4\pi}\left(\sin\frac{4}{9}\times 1.25\pi\right.\right.$$
$$\left.\left. - \cancel{\sin\frac{4}{9}\times 0\pi}\right)\right]$$

$$= \frac{4}{21}\left[1.25 - 0.705\right] = \underline{\underline{0.104}}$$

c) we now have to do the same calculation but with upper limit of 3:

$$= \frac{4}{21}\left[(3-2.25) - \frac{9}{4\pi}\left(\sin\frac{4}{9}\times 0.75\pi\right) - 0\right]$$

$$= \frac{4}{21}\left[0.75 - \frac{9}{4\pi}\frac{\sqrt{3}}{2}\right]$$

$$= 0.0247.$$

So P (finish in $\leq 3$ hrs | fast runner)

$$= \frac{0.0247}{0.104} = \underline{\underline{0.238.}}$$

d) The lower quartile of T occurs where

$$\int_{2.25}^{t^*} f(t)\,dt = 0.25.$$

So $$\frac{4}{21}\left[t^* - \frac{9}{4\pi}\sin\frac{4\pi}{9}(t-2.25)\right]_{2.25}^{t^*} = 0.25$$

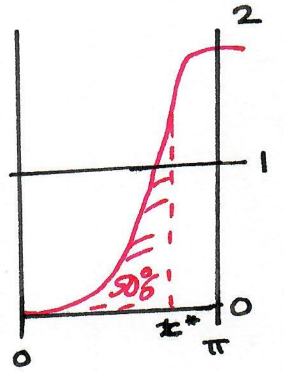
$$\frac{4}{21}\left[(t^*-2.25) - \frac{9}{4\pi}\sin\frac{4\pi}{9}(t^*-2.25)\right] = 0.25$$

I don't know how we're supposed to solve this one... but apparently the answer is $t^* = \underline{\underline{4.01.}}$

Actually .....→

I suppose we could say… and this might cut through a lot of faff…

what we really have is a cosine curve and we want to know the mid-weight point:




$y = 1 - \cos x$

Area under the curve to $x^*$ =

$$\int_0^{x^*} (1 - \cos x)\,dx$$

$$= \Big[x - \sin x\Big]_0^{x^*} = x^* - \sin x^* = \tfrac{1}{2}\text{ (req}^{d}\text{.)}$$

So find $x^*$ …. still not very easy.

e) Anyway…

$P = a - bt$

$E(P) = 100.$ Max. possible is 150, which means:

~~$a - 1.75b = 150$~~

$a - 2.25b = 150$ —— ①

Since $P$ is a linear function we know:

$E(P) = a - bE(T)$

So $100 = a - 4.723b$ —— ②

①−②: $2.473b = 50$

$b = 20.22$

in ①: $a = 150 + 2.25 \times 20.22 = 195.49$

to the nearest integer,

$a = 195$

$b = 20$

f) $Var(T) = 0.906$

Again we have $Var(a+bT) = b^2 Var(T)$

So $Var(P) = 20^2 \times 0.906 = 362$

(except that IB expect us to go back to the accurate value 20.22 and get 370)

12.) This question looks scary from the start, but we'll see...

$$f_n(x) = \sum_{r=0}^{n} (-2x^2)^r$$

$$= (-2x^2)^0 + (-2x^2)^1 + (-2x^2)^2 + \dots$$

$$= 1 - 2x^2 + 4x^4 \dots + (-2)^n x^{2n}$$

but we're getting ahead...

a) Clearly since the only reference to $x$ is $x^2$, replacing $x$ by $-x$ with give the same result. So $f_n(x)$ is even.

b) i) As written earlier,

$$f_3(x) = \sum_{r=0}^{3} (-2x^2)^r$$

$$= (-2x^2)^0 + (-2x^2)^1 + (-2x^2)^2 + (-2x^2)^3$$

$$= \underline{\underline{1 - 2x^2 + 4x^4 - 8x^6}}$$ as required.

ii) And $f_4(x)$ introduces just one more term, $(-2x^2)^4$ – giving:

$$f_4(x) = 1 - 2x^2 + 4x^4 - 8x^6 + 16x^8$$

c) i) Clearly we note the difference between

$f_{n+1}(x)$ and $f_n(x)$ is just one term:

$$(-2x^2)^{n+1} = (-2)^{n+1} x^{2(n+1)}$$

If this term is unbounded then there is no limit as $n \to \infty$: to make it bounded we need its modulus to decrease....

and this requires $2x^2 < 1$

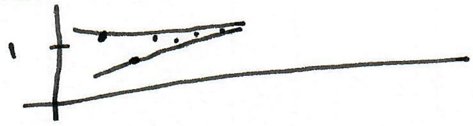
i.e. $x^2 < \frac{1}{2}$

$$|x| < \frac{1}{\sqrt{2}}$$

But is this enough?

It <u>is</u> because the terms are alternating in sign ... it's not like $1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} \dots$

So if n is even, $f_n(x)$ is always an upper bound: and it's decreasing: and the lower bound is increasing, with the two as close as we like for large n.



So we can say $\underline{\underline{K = \frac{1}{\sqrt{2}}}}$.

ii) by now we notice $f_n(x)$ is actually a geometric sequence with $u_1 = 1$ and $r = -2x^2$

So the sum to $\infty$, $f(x)$, is

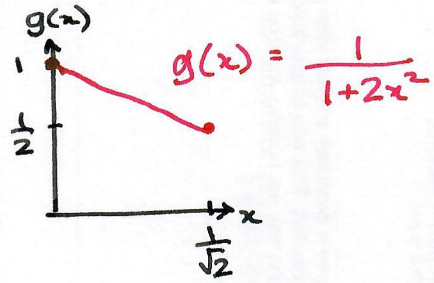
$$f(x) = \frac{1}{1-(-2x^2)} = \underline{\underline{\frac{1}{1+2x^2}}}$$

which is the form required, with $\underline{\underline{a=1 \quad b=2}}$.

This works if $|2x^2| < 1$, and we also note that this starts to look like a 'binomial expansion question' in reverse - where again $|2x^2| < 1$ would be a condition.

d) $g(x) = f(x) \quad 0 \leq x < K$ (i.e. only +ve $x$, and only within the limit.)

i) 'Justify' seems a strange word, particularly since we're not told here to find $g^{-1}$: what IB actually wants is:




We don't know the exact shape of $g(x)$ but

a) we know its endpoints in both $x$ and $y$

b) we observe it's continuous

c) we observe its decreasing everywhere.

So we conclude for every relevant value of $x$ ($x \in [0, \frac{1}{\sqrt{2}})$) there is one and only one value of $g(x)$: and the reverse is also true: for every $y \in (\frac{1}{2}, 1]$ there is a unique value $g^{-1}(y)$.

(Note we could talk about $[\frac{1}{2}$ now, but that would violate the original formulation. Strictly $f(x)$ as defined does not exist at $x = K$.)

ii) To find $g^{-1}$, let

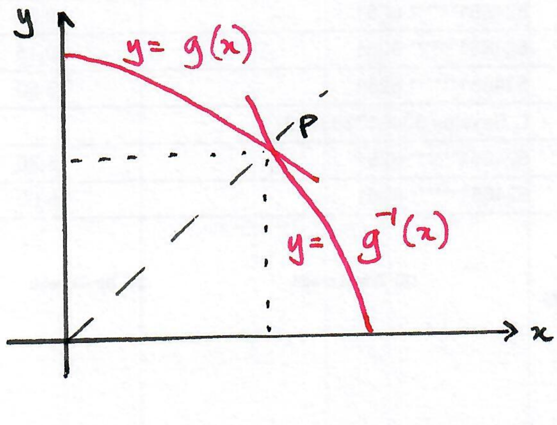
$$y = f(x) = g(x) = \frac{1}{1+2x^2}$$

Then $$y + 2x^2 y = 1$$

$$2x^2 = \frac{1-y}{y}$$

$$x = \sqrt{\frac{1-y}{2y}}$$ and this is $g^{-1}(y)$.

e) Here it gets nasty.



Clearly at the crossover P $g(x) = g^{-1}(x)$

which means $x = \dfrac{1}{1+2x^2}$ (or $x = \sqrt{\dfrac{1-x}{2x}}$, but this is the same.)

This means $x + 2x^3 = 1$

or $\underline{2x^3 + x - 1 = 0}$.

There's no easy solution here but happily we're allowed equation-solving calculators - which gives $x = 0.589754\ldots$

(IB gives it to 10dp!)

So now all we have to do is calculate

$$\int_0^{0.5897\ldots} \frac{1}{1+2x^2}\,dx = \frac{1}{2}\int \frac{1}{\frac{1}{2}+x^2}\,dx$$

$$= \frac{1}{2} \times \frac{1}{1/\sqrt{2}} \arctan\left(\frac{x}{1/\sqrt{2}}\right)\Bigg]_0^{0.5897\ldots}$$

$$= \frac{1}{2}\sqrt{2}\arctan\left(\sqrt{2}\cancel{x}) \times 0.5897\right) - 0\ldots$$

$$= 0.4915.$$

This includes the square (dotted line)

So the overall integral R required is

$$2\times(0.4915 - 0.5897^2) + 0.5897^2$$

$$= \underline{\underline{0.6353}}$$