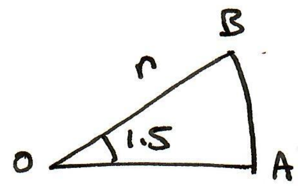
1)



a) Area of a sector $= \dfrac{\theta r^2}{2} = \dfrac{1.5r^2}{2} = 48$

So $r^2 = \dfrac{2\times 48}{1.5} = 64$ so $\underline{\underline{r = 8\,\text{cm}}}$

b) Arc length $= r\theta$

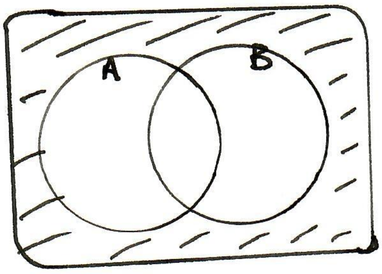
So the perimeter is $2r + 1.5r = 16+12$

$= \underline{\underline{28\,\text{cm}}}.$

2) We know $P(A\cup B) = P(A) + P(B) - P(A\cap B)$

$0.85 = 0.65 + 0.45 - P(A\cap B)$

$P(A\cap B) = 1.1 - 0.85 = \underline{\underline{0.25}}$





$P(B') = 1 - 0.45 = 0.55$

and $P((A\cup B)') = 1 - 0.85 = 0.15$

So $P(A'|B') = \dfrac{0.15}{0.55} = \underline{\underline{\dfrac{3}{11}}}$

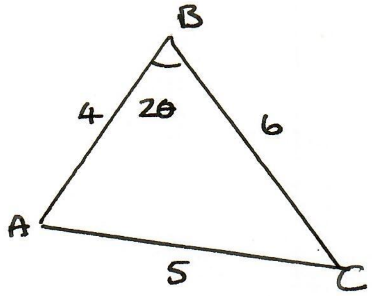
3) $(3n+2)^2 - (3n-2)^2$

$= \cancel{9n^2} + 12n + \cancel{4} - \cancel{9n^2} + 12n - \cancel{4}$

$= 2\times 12n$ which is (obviously) a multiple of 12.

So the expression is a multiple of 12 $\forall n \in \mathbb{Z}^+$.

4)




Using the cosine rule,

$$\cos 2\theta = \frac{6^2+4^2-5^2}{2.4.6}$$

$$= \frac{36+16-25}{48}$$

$$= \frac{27}{48} = \frac{9}{16}$$

From a standard identity,

$$\cos 2\theta = \cos^2\theta - \sin^2\theta$$

$$= 2\cos^2\theta - 1$$

$$= \frac{9}{16}$$

So $\quad 2\cos^2\theta = 1 + \frac{9}{16} = \frac{25}{16}$

$$\cos^2\theta = \frac{25}{32}$$

$$\cos\theta = \frac{5}{4\sqrt{2}} = \underline{\underline{\frac{5\sqrt{2}}{8}}}$$

which is in the form required

with $p = 5 \quad q = 8$.

5) Using standard formula for an arithmetic sequence:

$$u_n = u_1 + (n-1)d$$

$$S_n = \frac{n}{2}\left(2u_1 + (n-1)d\right)$$

we know: $u_{10} = u_1 + 9d = 16$ —— ①

$$S_{25} = \frac{25}{2}\left(2u_1 + 24d\right) = 100$$

$$= 25u_1 + 300d = 100 \text{ —— ②}$$

25 × ①: $25u_1 + 225d = 400$ —— ③

② − ③: $75d = -300$

$$\underline{d = -4.}$$

in ①: $u_1 = 16 + 4 \times 9 = 16 + 36 = \underline{\underline{52}}$

So in general, $u_K = 52 - (k-1)4 = 0$

$$52 = 4(k-1)$$

$$k - 1 = 13$$

$$\text{so } \underline{\underline{k = 14}}$$

6) a) $2x^2 - 15x + 18 < 0$

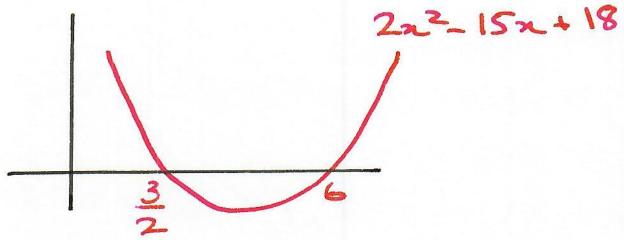
$= (2x-3)(x-6)$

So roots are $x = \frac{3}{2}$, $x = 6$.

The curve is an upward parabola,

So the solution required is

$\frac{3}{2} < x < 6$




b) $f(x) = \sqrt{2x^2 - 15x + 18}$

We can obviously calculate this for $x$ on either side of the parabola - and the range of $f$ is all of $\mathbb{R}^+$ and $0$ - so it's hard to see what the $x \leq k$ condition means. But using the RHS would give 2 possible values of $f^{-1}$ (i.e. $x$) - and IB want only a single-valued function.

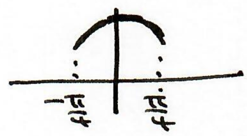
So by saying $k = \frac{3}{2}$ this limits $f^{-1}$ to a single set of values for $x$.

(Dubious, I think).

7) $f(x) = \sec\left(x - \frac{\pi}{4}\right)$

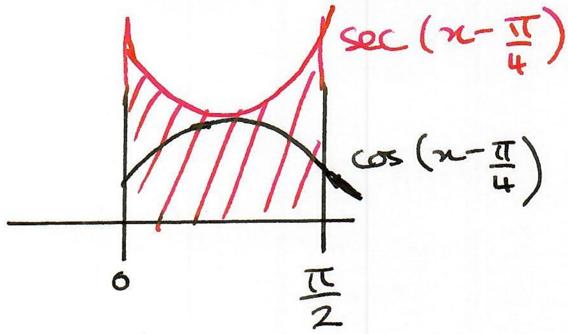
$= \dfrac{1}{\cos\left(x-\frac{\pi}{4}\right)} \quad 0 \leq x \leq \frac{\pi}{2}$

$= \dfrac{1}{\cos(t)} \quad -\frac{\pi}{4} \leq t \leq \frac{\pi}{4}$




a) $\cos(t)$ ranges between $\frac{1}{\sqrt{2}}$ and 1,

so the range of $f(x)$ is $[1, \sqrt{2}\,]$




b) The volume generated is

$$V = \int_0^{\frac{\pi}{2}} \pi\,(f(x))^2\,dx$$

$$= \pi \int_0^{\frac{\pi}{2}} \sec^2\left(x - \frac{\pi}{4}\right) dx.$$

Let $x - \frac{\pi}{4} = t$ (again)

then $dx = dt$

and the limits are : $x = 0 \quad t = -\frac{\pi}{4}$

$x = \frac{\pi}{2} \quad t = -\frac{\pi}{4}$

So $V = \pi \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \sec^2 t \, dt.$

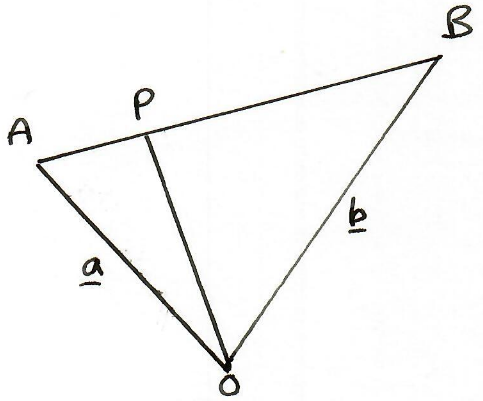
Using a standard result, this is equal to

$$\pi \left[ \tan t \right]_{-\frac{\pi}{4}}^{\frac{\pi}{4}}$$

$$= \pi \left( \tan \frac{\pi}{4} - \tan\left(-\frac{\pi}{4}\right)\right)$$

$$= \underline{\underline{2\pi.}}$$

8) a)




$\vec{AP} = \lambda \vec{AB}$

and $\vec{AB} = \underline{b} - \underline{a}$ (properties of vectors)

so $\vec{AP} = \lambda(\underline{b} - \underline{a})$

and $\vec{OP} = \underline{a} + \vec{AP}$

$$= \underline{a} + \lambda \underline{b} - \lambda \underline{a}$$

$$= \underline{\underline{(1-\lambda)\underline{a} + \lambda \underline{b}}}$$ as required.

b) $|\underline{a}| = 1$ $\quad |\underline{b}| = 2$ $\quad \underline{a}.\underline{b} = \frac{1}{4}$

If $\vec{OP} \perp \vec{AB}$ their dot product is 0:

So $\vec{OP} . \vec{AB} = 0$

$$\left( (1-\lambda)\underline{a} + \lambda \underline{b} \right) \cdot (\underline{b} - \underline{a}) = 0$$

$$(1-\lambda)\underline{a}\cdot\underline{b} + \lambda\, \underline{b}.\underline{b} - (1-\lambda)\, a.\underline{a} - \lambda b.a = 0$$

$a.b = \frac{1}{4}$, $a.a = 1$ and $b.b = 2^2 = 4$

$$\text{So } (1-\lambda)\frac{1}{4} + 4\lambda - (1-\lambda) - \lambda \cdot \frac{1}{4} = 0$$

$$\frac{1}{4} - \frac{\lambda}{4} + 4\lambda - 1 + \lambda - \frac{\lambda}{4} = 0$$

$$\lambda\left(4 + 1 - \frac{1}{2}\right) - \frac{3}{4} = 0$$

$$\frac{9}{2}\lambda = \frac{3}{4} \qquad \lambda = \frac{\cancel{2}}{\cancel{9}_{3}} \cdot \frac{\cancel{3}}{\cancel{4}_{2}} = \underline{\underline{\frac{1}{6}}}$$

a) This looks like we need the standard trig identities:

$$\tan\left(\theta - \frac{\pi}{4}\right) = \frac{\sin\left(\theta - \frac{\pi}{4}\right)}{\cos\left(\theta - \frac{\pi}{4}\right)}$$

$$= \frac{\sin\theta \cancel{\cos\frac{\pi}{4}} - \cos\theta \cancel{\sin\frac{\pi}{4}}}{\cos\theta \cancel{\cos\frac{\pi}{4}} + \sin\theta \cancel{\sin\frac{\pi}{4}}}$$

(cancelled terms all $= \frac{1}{\sqrt{2}}$)

$$= \frac{\sin\theta - \cos\theta}{\cos\theta + \sin\theta} \quad (*)$$

Also $$\frac{\sin 2\theta - 1}{\cos 2\theta} = \frac{2\sin\theta\cos\theta - 1}{\cos^2\theta - \sin^2\theta}$$

$$= \frac{2\sin\theta\cos\theta - 1}{(\cos\theta + \sin\theta)(\cos\theta - \sin\theta)} \quad (*)$$

So - can we say ... ?

$$\sin\theta - \cos\theta = \frac{2\sin\theta\cos\theta - 1}{\cos\theta - \sin\theta}$$

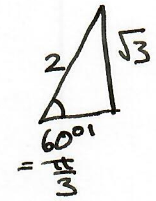
i.e. $-\sin^2\theta + 2\cos\theta\sin\theta - \cos^2\theta = 2\sin\theta\cos\theta - 1$

well, yes - since $(-\sin^2\theta - \cos^2\theta) = -1$.

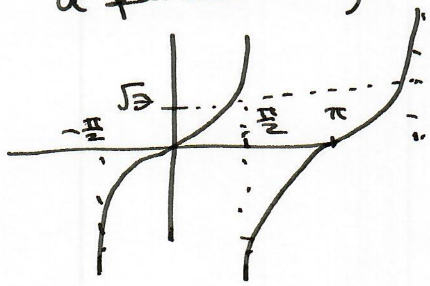
So the two expressions (*) are equal and the result is proved. ✓

b) Using (a), we have

$$\tan\left(\frac{x}{2}-\frac{\pi}{4}\right) = \sqrt{3}$$




So $\frac{x}{2} - \frac{\pi}{4} = \frac{\pi}{3}$ or $\frac{\pi}{3} \pm n\pi$ (tan repeats on a period $\pi$)



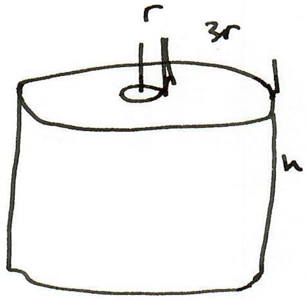

so $\frac{x}{2} - \frac{\pi}{4} = -\frac{2}{3}\pi \Rightarrow \frac{x}{2} = -\frac{5}{12}\pi$ too low

or $\frac{\pi}{3} \Rightarrow \frac{x}{2} = \frac{7}{12}\pi \Rightarrow x = \frac{14}{12}\pi$
$= \frac{7}{6}\pi$

or $\frac{4}{3}\pi \Rightarrow \frac{x}{2} = \frac{19\pi}{12}$ too high

So the (one) solution is $\underline{\underline{x = \frac{7}{6}\pi}}$ ✓

10)




(Like a toilet roll!)

a) Surface area $S$ presumably includes both ends and both inside and outside.

Ends $= 2\times \pi(4^2r^2 - r^2) = 30\pi r^2$

Sides $= h(2\pi r + 2\pi(4r)) = 10\pi r h$

So the total area $\underline{\underline{S = 30\pi r^2 + 10\pi r h}}$ as required.

b) It's not clear whether 'volume' means 'empty space in the middle' or 'volume of material' – but since the first would be simply $\pi r^2 h$ we'll assume the second.

This means $V = \pi h(4r^2 - r^2) = 15\pi r^2 h$. — ①

But we know $30\pi r^2 + 10\pi r h = 240\pi$,

so $30r^2 + 10rh = 240$

$10rh = 240 - 30r^2$

$h = \frac{1}{r}(24 - 3r^2)$

So in ①, $V = 15\pi r^2 h = 15\frac{\pi r^2}{r}(24 - 3r^2)$

$= \underline{\underline{360\pi r - 45\pi r^3}}$ as required.

c) $V = 360\pi r - 45\pi r^3$

$$\frac{dV}{dr} = 360\pi - 3 \times 45\pi r^2$$

$$= 360\pi - 135\pi r^2$$

d) $\frac{dV}{dr} = 0$ at the maximum, i.e. when $r = p\sqrt{\frac{2}{3}}$

So $360\pi - 135\pi\left(p^2 \times \frac{2}{3}\right) = 0$

$$\frac{\cancel{135}^{45} \times 2}{\cancel{3}} \cancel{\pi} p^2 = 360\cancel{\pi}$$

$$90p^2 = 360$$

$$p^2 = 4$$

$$p = 2.$$

e) This means the maximum volume $V$ is:

$$V = 360\pi \times 2\sqrt{\frac{2}{3}} - 45\pi \times \left(2\sqrt{\frac{2}{3}}\right)^3$$

$$= 720\pi\sqrt{\frac{2}{3}} - \cancel{45}^{15}\pi \times 8 \times \frac{2}{\cancel{3}}\sqrt{\frac{2}{3}}$$

$$= (720 - 240)\pi\sqrt{\frac{2}{3}} = 480\pi\sqrt{\frac{2}{3}}$$

which is in the form required with $q = 480.$

11) $y = \dfrac{e^{2x}-1}{e^{2x}+1}$ (This looks like a cosh/sinh/tanh question ... except IB doesn't do them.)

a) To show $\lim\limits_{x \to \infty}(y) = 1$ ... I'd say it's obvious since the $e^{2x}$ terms dominate and it's essentially $\dfrac{e^{2x}}{e^{2x}}$, ... but:

$$\lim_{x\to\infty} \frac{f(x)}{g(x)} = \lim_{x\to\infty} \frac{f'(x)}{g'(x)}$$

(usually de L'Hopital's rule is expressed in terms of a constant $c$, but its OK to use $\infty$ if everything is well-behaved, as in this case. And in fact we could take $x = c$ as arbitrarily large and it all still holds)

So if $f(x) = e^{2x} - 1$, $f'(x) = 2e^{2x}$

$g(x) = e^{2x} + 1$, $g'(x) = 2e^{2x}$

So $\dfrac{f'(x)}{g'(x)} = \dfrac{2e^{2x}}{2e^{2x}} = 1$; so $\lim\limits_{x\to\infty}\left(\dfrac{e^{2x}-1}{e^{2x}+1}\right) = 1$ everywhere as required. ✓

b) i) $y = \dfrac{e^{2x}-1}{e^{2x}+1}$

Using the quotient rule:

$$\frac{dy}{dx} = \frac{(\cancel{e^{2x}}+1)\cdot 2e^{2x} - (\cancel{e^{2x}}-1)\cdot 2e^{2x}}{(e^{2x}+1)^2}$$

$$= \underline{\underline{\frac{4e^{2x}}{(e^{2x}+1)^2}}} \quad \text{as required.}$$

ii) $1-y^2 = 1-\left(\dfrac{e^{2x}-1}{e^{2x}+1}\right)^2$

$$= \frac{(e^{2x}+1)^2 - (e^{2x}-1)^2}{(e^{2x}+1)^2}$$

$$= \frac{(\cancel{e^{4x}} + 2e^{2x} \cancel{+1}) - (\cancel{e^{4x}} - 2e^{2x} \cancel{+1})}{(e^{2x}+1)^2}$$

$$= \frac{4e^{2x}}{(e^{2x}+1)^2}$$ which equals the earlier expression.

So $\underline{\underline{1-y^2 = \dfrac{dy}{dx}}}$ as required.

c) i) $1 - y^2 = \dfrac{dy}{dx}$ —— ①

So $\dfrac{d^2y}{dx^2} = -2y\dfrac{dy}{dx}$ (differentiating)

Subs. from ①:

$$\frac{d^2y}{dx^2} = -2y\left(1-y^2\right) = \underline{\underline{2y^3 - 2y}} \text{ as required.}$$

ii) Differentiating again:

$$\frac{d^3y}{dx^3} = \left(6y^2-2\right)\frac{dy}{dx}$$

$$= \left(6y^2-2\right)\left(1-y^2\right) = 2\left(3y^2-1\right)\left(1-y^2\right)$$

$$= \underline{\underline{2\left(3y^2-1\right)\left(1-y^2\right)}}$$

d) The Maclaurin series gives

$$y(x) = y(0) + x\,y'(0) + \frac{x^2}{2}\,y''(0) + \frac{x^3}{6}\,y'''(0) + \ldots$$

$$y(0) = \frac{e^0 - 1}{e^0 + 1} = 0$$

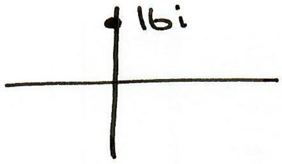
$$y'(0) = 1 - y^2 = 1 - 0^2 = 1$$

$$y''(0) = -2y\left(1-y^2\right) = -2 \times 0 \times 1 = 0$$

$$y'''(0) = -2\left(y^2-1\right)^2 = -2(-1)^2 = -2$$

So $y(x) = \underline{\underline{x - \dfrac{x^3}{3} + o\left(x^4\right)}}$ which looks familiar ...

12)




$$z^4 = 16i = 16\,\text{cis}\left(\frac{\pi}{2}\right) = 2^4\,\text{cis}\left(\frac{\pi}{2}\right)$$

So $z = 2\,\text{cis}\left(\frac{\pi}{8}\right)$ or $\frac{5\pi}{8}, \frac{9\pi}{8}, \frac{13\pi}{8}$. ✓

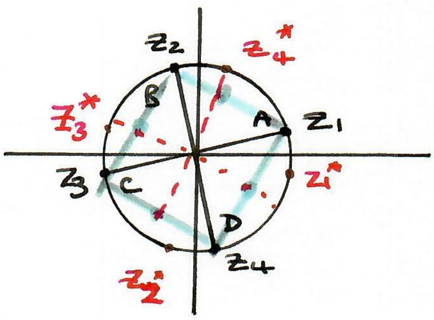
$z_1$ $z_2$ $z_3$ $z_4$.

The roots form a geometric sequence

with $r = \dfrac{z_2}{z_1} = \dfrac{2\,\text{cis}\left(\frac{5\pi}{8}\right)}{2\,\text{cis}\left(\frac{\pi}{8}\right)} = \text{cis}\left(\frac{4\pi}{8}\right) = \text{cis}\left(\frac{\pi}{2}\right)$

$$= \underline{\underline{0 + 1i}}$$ ✓

As above:



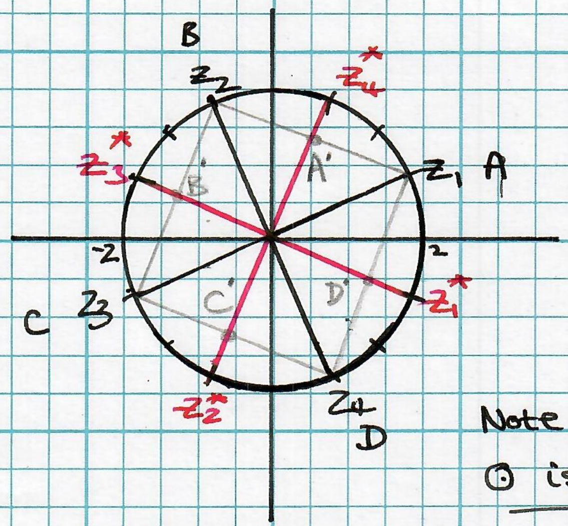

d) Assuming $z^*$ means conjugate of $z$;

clearly by inversion in the Real axis these are

all roots of the equation $v^2 = 16i$

So $\underline{\underline{a + bi = -16i}}$: $\underline{\underline{a = 0 \text{ and } b = -16}}$

e)




Note the radius of the ⊙ is 2, not 1.

By simple geometry we observe the point A' is:

$$w_1 = \sqrt{2}\,\text{cis}\left(\frac{3\pi}{8}\right)$$

or we could say, since it's the midpoint of AB:

$$w_1 = \frac{z_1 + z_2}{2} = \left\{\cos\left(\frac{\pi}{8}\right) + \cos\left(\frac{5\pi}{8}\right) + i\left(\sin\frac{\pi}{8} + \sin\frac{5\pi}{8}\right)\right\}$$

$$= \left\{2\cos\frac{6\pi}{2\cdot 8}\cos\frac{4\pi}{2\cdot 8} + 2i\left(\sin\frac{6\pi}{2\cdot 8}\sin\frac{4\pi}{2\cdot 8}\right)\right\}$$

note cos!

$$= 2\left\{\cos\frac{3\pi}{8}\cos\frac{\pi}{4} + i\sin\frac{3\pi}{8}\cos\frac{\pi}{4}\right\}$$

$$= 2\left\{\cos\frac{3\pi}{8} + i\sin\frac{3\pi}{8}\right\} \times \frac{1}{\sqrt{2}}$$

$$= \sqrt{2}\,\text{cis}\left(3\frac{\pi}{8}\right) \text{ as before.}$$

Since the arguments are $\frac{3\pi}{8} + n\frac{\pi}{2}$, we look for an eq$^n$ where $r\,\text{cis}\left(\frac{3\pi}{8}\right)$ is real - i.e. $n\left(\frac{3\pi}{8}\right)$ is a multiple of $\pi$. We can't do this with $\pi$ or $2\pi$,

but we can with $3\pi$:

$$\left[\text{cis}\left(\frac{3\pi}{8}\right)\right]^8 = -1$$

So we're looking for $w^8 = 2^q = -1$ for some $q$.

But we cannot achieve $-1$ this way.

So it'll have to be $w^{16} = 1$.

Now we check the radius (modulus):

$$w_1 = \frac{\sqrt{2}}{\sqrt{2}} \text{cis}\left(\frac{3\pi}{8}\right)$$

So $$w_1^{16} = \sqrt{2}^{16} \text{cis}(6\pi)$$

$$= 2^8.$$

So after all that,

$$w_1 = \sqrt{2} \text{cis}\left(\frac{3\pi}{8}\right) \text{ etc}$$

and $w^{16} = 2^8$ so $p = 16$

$q = 8$