1) $f(x) = \frac{4x^3}{3} - 16x$

$$\frac{df}{dx} = \frac{3 \times 4x^2}{3} - 16 = 4x^2 - 16$$

At a local min, this is zero:

$$4x^2 - 16 = 0$$

$$4x^2 = 16$$

$$x^2 = 4$$

$$x = \pm 2$$

We know $p > 0$ so $\underline{\underline{p = 2}}$ and $q = \frac{4(2^3)}{3} - 16 \times 2$

$$= \frac{32}{3} - 32$$

$$= \frac{32 - 96}{3}$$

$$\underline{\underline{q = -\frac{64}{3}}}$$

2) This depends slightly on your understanding of 'periodic interest rates', but here goes:

Nominal annual interest 4%

Quarterly (periodic) rate = 4%/4 = 1%.

a) In terms of the expression $1000(1+k)^4$

this means $k = 0.01$ or $\frac{1}{100}$.

b) $(1+x)^4 =$ (using Pascal's Δ / Binomial exp)

$$1 + 4x + 6x^2 + 4x^3 + x^4$$

c) If $x = 0.01$ then

$$(1+0.01)^4 = 1 + 4(0.01) + 6(0.0001) + 4(0.000001) + 0.00000001$$

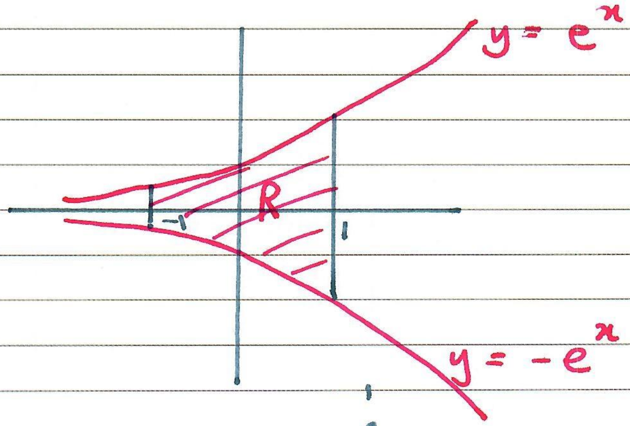
Multiplying this by 1000 dinar gives:

$$1000 + 40 + 0.6 + 0.004 + 0.00001$$

$$= 1040.60401$$

or 1041 (dinar) to the nearest dinar.

3)



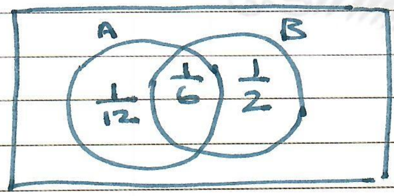
$$\text{Area required} = R = \int_{-1}^{1} e^x - (-e^x)\,dx$$

$$= 2\int_{-1}^{1} e^x\,dx$$

$$= 2\left[e^x\right]_{-1}^{1}$$

$$= 2\left(e - \frac{1}{e}\right)$$

4)



$P(A') = \frac{3}{4}$ so $P(A) = \frac{1}{4}$

$P(A \cup B) = \frac{3}{4}$

$P(B|A) = \frac{2}{3}$

$P(A) = \frac{1}{4}$ and $P(B|A)$ is $\frac{2}{3}$ of this.

so $P(A \cap B) = \frac{1}{4} \times \frac{2}{3} = \frac{1}{6}$

~~$P(B) = P(A \cup B) + P(A \cap B') - P(A)$~~

~~$= \frac{3}{4} - \frac{1}{12} = \frac{2}{3}$.~~

~~so $P(A) \cdot P(B) = \frac{1}{6} \times \frac{2}{3} = \frac{1}{9}$~~

$$\left.\begin{array}{l} P(A) = \frac{1}{4} \\ P(B) = \frac{2}{3} \end{array}\right\} \text{ So } P(A) \cdot P(B) = \frac{1}{4} \cdot \frac{2}{3} = \frac{1}{6}.$$

This is equal to $P(A \cap B)$ already found, so the events are independent.

5) a) i) Let $F_n = w_n \times h_n$

we know $w_n = 150\% \times w_{n-1} = \frac{3}{2} w_{n-1}$

$h_n = \frac{3}{2} h_{n-1}$

So $F_n = \left(\frac{3}{2}\right)\left(\frac{3}{2}\right) w_{n-1} h_{n-1} = \frac{9}{4} F_{n-1}$.

So $F_1 \dots F_{10}$ form a geometric seq. with ratio $\frac{9}{4}$, and $F_1 = 4 \times 5 \text{ cm}^2$,

So $F_n = 20 \left(\frac{9}{4}\right)^{n-1}$ by usual geometric sequence formula.

ii) Using the sum formula,

$$S_{10} = \sum_{i=1}^{10} F_i = \frac{F_1 (r^{10} - 1)}{r - 1}$$

$$= \frac{20\left(\left(\frac{9}{4}\right)^{10} - 1\right)}{\frac{9}{4} - 1} = \frac{20}{5/4}\left(\left(\frac{9}{4}\right)^{10} - 1\right)$$

$$= 16\left(\left(\frac{9}{4}\right)^{10} - 1\right)$$

So the mean area is $\frac{16}{10}\left(\left(\frac{9}{4}\right)^{10}-1\right)$

$= \frac{8}{5}\left(\left(\frac{9}{4}\right)^{10}-1\right)$ (cm^2) as required with $p=\frac{8}{5}$, $a=10$.

c) Because there are 10 F values the median will be between F_5 and F_6. We take the arithmetic average, which is

$$\frac{1}{2}(F_5+F_6)$$

$$= \frac{1}{2}\left[20\left(\frac{9}{4}\right)^4 + 20\left(\frac{9}{4}\right)^5\right]$$

$$= 10\left(\frac{9}{4}\right)^4\left(1+\frac{9}{4}\right)$$

$$= 10 \times \frac{13}{4}\left(\frac{9}{4}\right)^4$$

$$= \frac{65}{2}\left(\frac{9}{4}\right)^4$$ as required, with $q = \frac{65}{2}$.

6) L_1: $\underline{r} = 4\underline{i} - \underline{k} + \lambda(a\underline{j} + \underline{k})$

L_2: $\underline{r} = \underline{i} - b\underline{k} + \mu(\underline{i} + 2\underline{j} + 3\underline{k})$

direction vectors.

The lines are perpendicular so the • product of the direction vectors is 0:

$$(a\underline{j} + \underline{k}) \cdot (\underline{i} + 2\underline{j} + 3\underline{k}) = 0$$

$$0 \times 1 + a \times 2 + 1 \times 3 = 0$$

$$2a + 3 = 0$$

$$a = -\frac{3}{2}$$

Also the lines intersect at a point where

$$4\underline{i} - \underline{k} + \lambda\left(-\frac{3}{2}\underline{j} + \underline{k}\right)$$

$$= \underline{i} - b\underline{k} + \mu(\underline{i} + 2\underline{j} + 3\underline{k})$$

Equating coeffs of $\underline{i}, \underline{j}, \underline{k}$:

$4 = 1 + \mu \quad \therefore \mu = 3$

$-\frac{3}{2}\lambda = 2\mu = 6 \quad \therefore \lambda = -4$

$-1 + \lambda = -b + 3\mu$

$-5 = -b + 3 \times 3 \qquad b = 14.$

7) $z = 3^{i-1}$

a) $\underline{3 = e^{\ln 3}}$ (since $e^{\ln x} = x$)

b) So $z = \left(e^{\ln 3}\right)^{i-1}$

$= e^{\ln 3(i-1)}$

$= e^{-\ln 3 + i\ln 3}$

$= e^{-\ln 3} \times e^{i\ln 3}$

$= \dfrac{1}{e^{\ln 3}} e^{i\ln 3}$

$= \underline{\dfrac{1}{3} \operatorname{cis}(\ln 3)}$

So i) $\operatorname{Re}(z) = \dfrac{1}{3}\cos(\ln 3)$

ii) $\dfrac{1}{z} = 3\operatorname{cis}(-\ln 3)$

So $\operatorname{Re}\left(\dfrac{1}{z}\right) = \underline{3\cos(-\ln 3)}$

8) a) LHS $= 1 + \log_2 n$

$= \log_2 2 + \log_2 n$

$= \log_2 (2n)$

RHS $= \log_2 (n+1)$

So since $2n \geqslant n+1 \quad \forall\ n \in \mathbb{Z}^+$

and $\log_2$ is a continuous increasing function,

we know that $\log_2(2n) \geqslant \log_2(n+1)\ \forall\ n \in \mathbb{Z}^+$,

so the result is proved.

b) Proof by induction:

i) Clearly when $n=1$, $1 > \log_2 1 = 0$:

so the result is true when $n=1$.

ii) Suppose the result is true for $k=n$:

$$n > \log_2 n \quad \text{——— ①}$$

Consider $\log_2 (n+1)$

By (a) we know

$$1 + \log_2 n \geqslant \log_2 (n+1) \quad \text{——— ②}$$

Putting ① into ②:

$$1 + n > 1 + \log_2 n \geqslant \log_2 (n+1)$$

So $1 + n > \log_2 (n+1)$

So the result is also true for $k = n+1$: (i) and (ii) together show by induction that it's true for all

a) $\dfrac{dy}{dx} = \dfrac{x-y}{x+y}$ $y = 0$ when $x=2$.

Let $y = vx$. Then $\dfrac{dy}{dx} = v + x\dfrac{dv}{dx}$

and $\dfrac{x-y}{x+y} = \dfrac{x-vx}{x+vx} = \dfrac{1-v}{1+v}$.

So $x\dfrac{dv}{dx} + v = \dfrac{1-v}{1+v}$

$$x(1+v)\frac{dv}{dx} + v + v^2 = 1-v$$

$$x(1+v)\frac{dv}{dx} = 1-2v-v^2$$

$$\frac{(1+v)\,dv}{1-2v-v^2} = \frac{dx}{x}$$ (separating variables)

Aha. Let $v^2+2v-1 = p$ (in hindsight, should have left p as $1-2v-v^2$)

Then $\dfrac{dp}{dv} = 2v+2 = 2(v+1)$.

So $(v+1)\,dv = \dfrac{dp}{2}$.

So we have $\dfrac{dp}{-2p} = \dfrac{dx}{x}$

Integrating: $-\frac{1}{2}\ln p = \ln x + c$

or $p^{-\frac{1}{2}} = Ax$

$$P = \frac{1}{A^2x^2}$$

$$\left(\frac{y}{x}\right)^2 + 2\left(\frac{y}{x}\right) - 1 = \frac{1}{A^2x^2}$$

$$y^2 + 2xy - x^2 = \frac{1}{A^2}$$

Subs for $(2,0)$:

$$0 + 0 - 4 = \frac{1}{A^2}$$

so $\frac{1}{A^2} = -4$ (so A may be imaginary, but we actually want A^2).

So $y^2 + 2xy - x^2 = -4$

$$x^2 - 2xy + y^2 = 4.$$

10) $f(x) = 5(x+1)(x+3)$

a) $f(x) = 5(x^2 + 4x + 3)$

$= 5\left[(x+2)^2 - 1\right]$

$(x^2 + 4x + 4 - 1 \quad ✓)$

$= 5(x+2)^2 - 5$

$= 5(x-(-2))^2 - 5$

in the form required, with $a = 5$, $h = -2$, $k = -5$

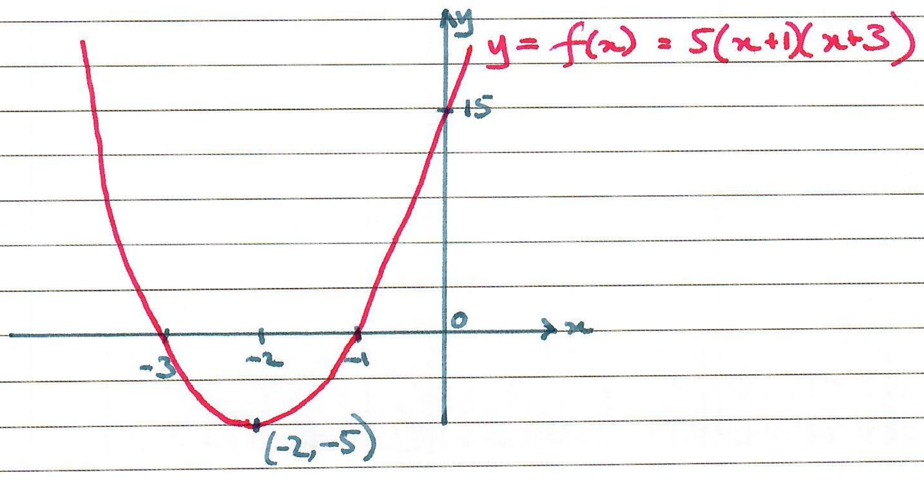
(seems rather odd)

b) ⓐ The curve is a parabola, ⓑ axis of symmetry $x = -2$,

ⓒ $\cup$, ⓓ lowest point $5(-2-(-2))^2 - 5 = -5$

ⓔ y-intercept $5(0-(-2))^2 - 5 = 20 - 5 = 15$

ⓕ x-intercepts when $x = -3$ and -1.



c) $f(x) \leqslant 40$ requires

$$5(x+2)^2 - 5 \leqslant 40$$

$$(x+2)^2 \leqslant 9$$

$$|x+2| \leqslant 3$$

So x lies between -5 and 1
or equals

$$\underline{\underline{x \in [-5, 1] \qquad -5 \leqslant x \leqslant 1.}}$$

d) $g(x) = \ln x$

i) $(f \circ g)x = f(g(x))$

$= f(\ln x)$

$= \underline{\underline{5(\ln x + 2)^2 - 5}} \quad (= 5\ln^2 x + 20\ln x + 15)$

ii) if $f(\ln x) \leqslant 40$ then by (c),

$$\ln x \in [-5, 1]$$

$$\underline{\underline{-5 \leqslant \ln x \leqslant 1}}$$

Taking exponentials,

$$e^{-5} \leqslant x \leqslant e^1 = e$$

$$\underline{\underline{x \in [e^{-5}, e]}}$$

e) (we note g and f are now in the opposite order)

$g(x) = \ln x$, which covers every value in $\mathbb{R}^+$

So limits are imposed by where f is +ve:

i.e. by x being outside $[-3, -1]$

So the domain of $g \circ f$ is $\underline{\underline{(-\infty, -3) \cup (-1, \infty)}}$

11) a) Since we know the equations for Π_1 and Π_2 we know their normal vectors: (apart from scaling).

$$\underline{n}_1 = \begin{pmatrix}1\\2\\1\end{pmatrix} \qquad \underline{n}_2 = \begin{pmatrix}1\\-1\\-2\end{pmatrix}$$

The angle between the planes is equal to the angle between the normals, which is given by:

$$\cos\phi = \frac{\underline{n}_1 \cdot \underline{n}_2}{|\underline{n}_1||\underline{n}_2|}$$

$$= \frac{1\times 1 - 2\times 1 - 1\times 2}{\sqrt{1+4+1}\sqrt{1+4+1}}$$

$$= \frac{-3}{6} = -\frac{1}{2}$$

So $\cos\phi = -\frac{1}{2}$

$\phi = 120°. \quad \left(\frac{2\pi}{3}\right)$

So if ϕ is $120°$, the <u>acute</u> angle between the planes is $180° - \phi = \theta = 60° \quad \left(\frac{\pi}{3}\right)$

b) Π_3 is $\perp$ to Π_1 and Π_2, so its normal is given by the vector product $\underline{n}_1 \times \underline{n}_2$:

$$\begin{pmatrix}1\\2\\1\end{pmatrix} \times \begin{pmatrix}1\\-1\\-2\end{pmatrix} = \begin{pmatrix}2\times -2 - 1\times -1\\1\times 1 - -2\times 1\\1\times -1 - 2\times 1\end{pmatrix} = \begin{pmatrix}-4+1\\1+2\\-1-2\end{pmatrix}$$

$$= \begin{pmatrix}-3\\3\\-3\end{pmatrix} = -3\begin{pmatrix}1\\-1\\1\end{pmatrix}$$

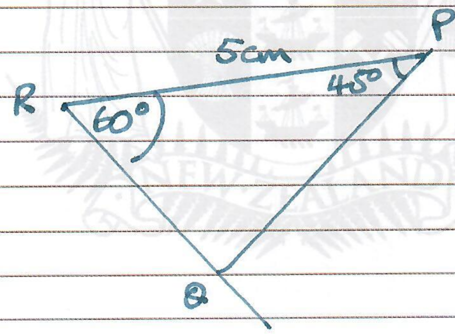
So Π_3 can be expressed in Cartesian form as $x - y + z = a$ for some a.

And we know $\begin{pmatrix} 5 \\ -5 \\ 5 \end{pmatrix}$ is in Π_3,

so $5 + 5 + 5 = a \therefore a = 15.$

So Π_3 is $\underline{\underline{x - y + z = 15}}$.

c) At this point we lose interest in 3D vector geometry and go back to good old stuff:



We note angle RQP $= 180° - 60° - 45° = 75°$.

i) $\sin 75° = \sin(30° + 45°)$

which, using a standard identity is:

$$\sin(30° + 45°) = \sin 30° \cos 45° + \sin 45° \cos 30°$$

$$= \frac{1}{2}\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}\frac{\sqrt{3}}{2}$$

$$= \underline{\underline{\frac{\sqrt{2} + \sqrt{6}}{4}}} \text{ as required.}$$

ii) To find QR use the sin rule:

$$\frac{QR}{\sin 45^\circ} = \frac{RP}{\sin 75^\circ} = \frac{5}{\sin 75^\circ}$$

$$\text{so } QR = \frac{5 \times 4}{\sqrt{2}+\sqrt{6}} \times \frac{\sqrt{2}}{2}$$

$$= \frac{10}{1+\sqrt{3}}$$

$$= \frac{10(1-\sqrt{3})}{(1+\sqrt{3})(1-\sqrt{3})}$$

$$= \frac{10(1-\sqrt{3})}{1-3} = \underline{\underline{5(\sqrt{3}-1)}}$$

which is as required with $p = 5$, $q = 3$.

12) $f_n(x) = \cos^n x = \cos^{n-1}x \cos x$

a) use integration by parts:

$+ \quad \cos^{n-1}x \qquad \cos x$

$- \quad -(n-1)\cos^{n-2}x \sin x \qquad \sin x$

$$\int \cos^n x\,dx = \cos^{n-1}x \sin x + (n-1)\int \cos^{n-2}x \sin^2 x\,dx$$

$$= \cos^{n-1}x \sin x + (n-1)\int \cos^{n-2}x(1-\cos^2 x)\,dx$$

$$= \cos^{n-1}x \sin x + (n-1)\int \cos^{n-2}x\,dx$$

$$- (n-1)\int \cos^n x\,dx. \circledast$$

as required.

b) So taking the $\circledast$ term to the left,

$$(1+n-1)\int \cos^n x\,dx = n\int \cos^n x\,dx$$

$$= \cos^{n-1}x \sin x + (n-1)\int \cos^{n-2}x\,dx$$

Replacing by $f_n(x)$ and dividing by n:

$$\int f_n(x) = \frac{1}{n}\cos^{n-1}x \sin x + \frac{(n-1)}{n}\int f_{n-2}(x)\,dx$$

as required.

c) Find $\int \cos^4 x \, dx$.

Using (b):

$$\int \cos^4 x \, dx = \frac{1}{4}\cos^3 x \sin x + \frac{3}{4}\int \cos^2 x \, dx$$

and again $= \frac{1}{4}\cos^3 x \sin x$

$$+ \frac{3}{4}\left\{ \frac{1}{2}\cos x \sin x + \frac{1}{2}\int \underbrace{\cos^0 x}_{=1} \, dx \right\}$$

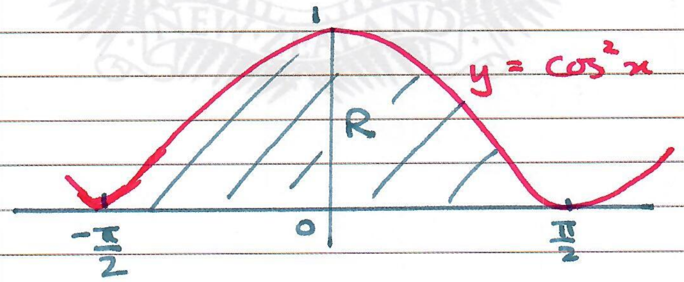
$$= \underline{\underline{\frac{1}{4}\cos^3 x \sin x + \frac{3}{8}\cos x \sin x + \frac{3}{8}x + c}}$$

which is in the required form with $p = \frac{1}{4}$

$q = \frac{3}{8}$

$r = \frac{3}{8}$.

d)



Volume of the solid of revolution

$$= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \pi y^2 \, dx$$

$$= \pi \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos^4 x \, dx$$

using (c), this is

$$\pi\left[\frac{1}{4}\cos^3 x\sin x+\frac{3}{8}\cos x\sin x+\frac{3}{8}x\right]_{-\frac{\pi}{2}}^{\frac{\pi}{2}}$$

$$=\pi\left(0+0+\frac{3}{8}\frac{\pi}{2}\right)-\pi\left(0+0-\frac{3}{8}\frac{\pi}{2}\right)$$

$$=\pi^2\left(\frac{3}{16}+\frac{3}{16}\right)=\underline{\frac{3\pi^2}{8}}$$

e) i) Maclaurin series in standard form is

$$f_n(x)=f_n(0)+x f_n'(0)+\frac{x^2}{2!}f_n''(0)+\cdots$$

$$=\cos^n(0)+x\left(n\cos^{n-1}(0)(\sin(0))\right)$$

$$+\frac{x^2}{2}n\left(-(n-1)\cos^{n-2}(0)\sin(0)\sin(0)+\cos^{n-1}(0)\cos(0)\right)+\cdots$$

$$=1+0x+\left(-\frac{n}{2}x^2\right)$$

$$=\underline{\underline{1-\frac{n}{2}x^2+\cdots o(x^3)}}$$

or prob $o(x^4)$

ii) $$\lim_{x\to 0}\frac{f_n(x)-1}{x^2}=\lim_{x\to 0}\frac{-\frac{n}{2}x^2+o(x^3)}{x^2}$$

$$=\lim_{x\to 0}\left(\frac{-n}{2}+o(x)\right)=\underline{\underline{\frac{-n}{2}}}$$