Pearson ~~Edexcel~~ Pure Maths 2 2024 9MA0/02

1) a) $y = 4x^3 - 7x^2 + 5x - 10$

$\frac{dy}{dx} = 12x^2 - 14x + 5$

$\frac{d^2y}{dx^2} = 24x - 14.$

b) when $\frac{d^2y}{dx^2} = 24x - 14 = 0,$

$x = \frac{14}{24} = \frac{7}{12}$

2) $P_0 = 8100$ P = Principal outstanding

p_n = payment in month n.

$P_1 = 8100 - 400 = P_0 - 400$ $p_1 - 400$

$P_2 = P_1 - 390$ $p_2 = 390$

$p_3 = 380$

a) $p_1, p_2, p_3 \dots$ is an arithmetic sequence

$p_{n+1} = p_n + d = p_1 + nd$

$p_1 = 400$

$p_2 = p_1 + d$: $390 = 400 + d$

so $d = -10$

and $p_{n+1} = p_1 - 10n.$

So $p_{12} = p_1 - 10 \times 11 = 400 - 110$

b) Let P_n (capital P) be the amount remaining after month n.

Then $P_0 = 8100$

$$P_1 = P_0 - p_1 = P_0 - 400$$

$$P_2 = P_0 - p_1 - p_2 = P_0 - 400 - 400 + 10$$

$$P_3 = P_0 - p_1 - p_2 - p_3 = P_0 - 400 - 400 + 10 - 400 + 2\times 10$$

$$P_n = P_0 - \underbrace{400 - 400 - \cdots - 400}_{n \text{ terms}} \underbrace{+ 10 + 2\times 10 + \cdots + (n-1)\times 10}_{\substack{n-1 \text{ terms} \\ \text{making } \Delta \text{ num.}}}$$

$$= P_0 - 400n + \frac{10(n-1)n}{2}$$

$$= \underset{8100}{P_0} - 400n + 5n(n-1)$$

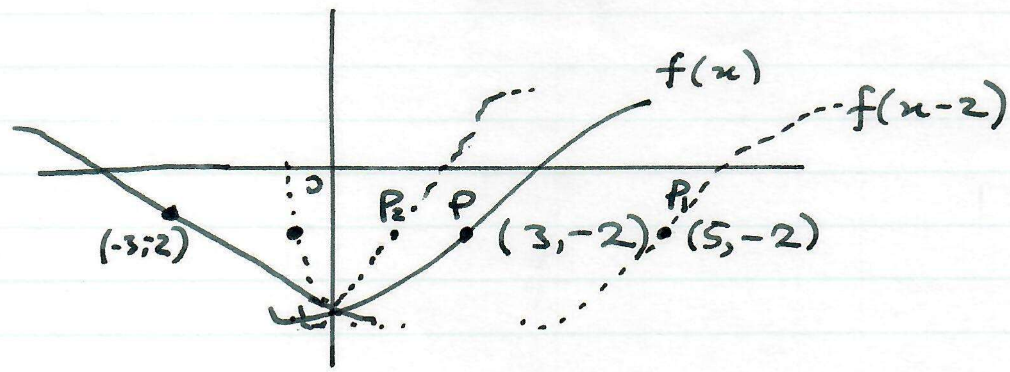
$$= \cancel{P_0} - 400n + 5n^2 - 5n$$

So if $P_N = 0$, $8100 - 405N + 5N^2 = 0$

i.e. $1620 - 81N + N^2 = 0$ as required.

c) We can factorise $(N-36)(N-45) = 0$

3)



i) $y = f(x-2)$ moves P to $\underline{\underline{(5,-2)}}$.

ii) $y = f(2x)$ maps P to $\underline{\underline{(\frac{3}{2},-2)}}$

iii) $y = 3f(-x)+5$

$f(-x)$ maps P to $(-3,-2)$

$3f(-x)$ - - - $(-3,-6)$

$3f(-x)+5$ - - - - - $\underline{\underline{(-3,-1)}}$.

(Changing x in $f(x)$ <u>reverses</u> what it seems to be)

(Changing y around $f(x)$ <u>does</u> what it seems to be.)

4) $u_1 = 6$

$u_{n+1} = ku_n - 5$

a) $u_3 = -1$

So $u_2 = ku_1 - 5 = 6k - 5$

$u_3 = ku_2 - 5 = k(6k-5) - 5 = -1$ (given)

b) i) $6k^2 - 5k - 4 = 0$

Factorises as $(3k-4)(2k+1) = 0$

So $\underline{\underline{k = \frac{4}{3}}}$ or (discount) $k = -\frac{1}{2}$.

ii) We could do this with the summation formulae but it's hardly worth it...

$u_1 = 6$

$u_2 = \frac{4}{3}.6 - 5 = 8 - 5 = 3$

$u_3 = \frac{4}{3}.3 - 5 = 4 - 5 = -1$.

So $\sum_{r=1}^{3} u_r = 6 + 3 - 1 = \underline{8}$

5.) $\phi = \dfrac{\theta \tan 2\theta}{1 - \cos 3\theta}$ 1

Assume θ, 2θ and 3θ are all small.

Also assume it's safe to divide by the $1 - \cos 3\theta$ term, which might be close to 0??

$\cancel{\phi = \theta \times \frac{\sin 2\theta}{\cos 2\theta} \times \frac{1}{1 - \cos 3\theta}}$

$\cancel{= \theta \times 2\theta \times \frac{1}{\ldots}} \quad \cancel{= 2 \times 2\theta^2 \times \frac{2}{\ldots}}$

using the approximations:

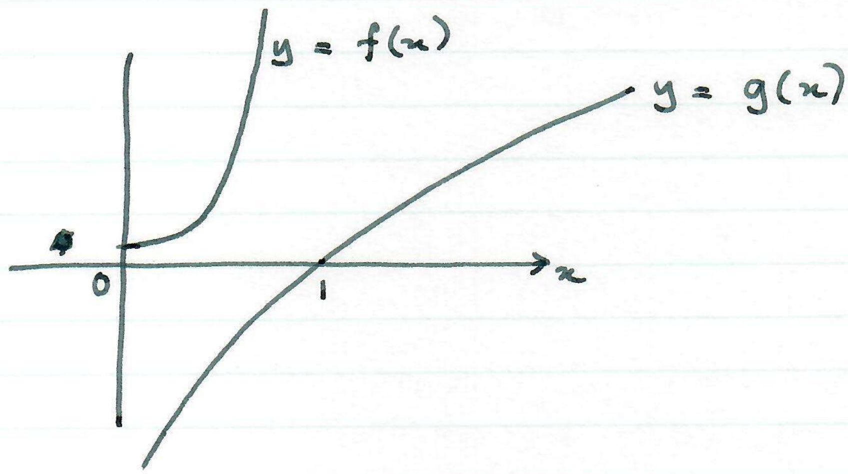
$$\tan 2\theta = 2\theta$$

$$1 - \cos 3\theta = 1 - \left(1 - \frac{(3\theta)^2}{2}\right)$$

$$= 1 - 1 + \frac{9\theta^2}{2} = \frac{9\theta^2}{2}$$

So in 1: $\phi \approx \dfrac{\theta\, 2\theta}{9\theta^2/2} = \underline{\underline{\dfrac{4}{9}}}$

6)



$$f(x) = e^{4x^2 - 1}$$

$$g(x) = 8 \ln(x)$$

$$f'(x) = e^{4x^2-1}\left(\frac{d}{dx}(4x^2 - 1)\right)$$

$$= e^{4x^2-1}(8x)$$

$$= \underline{8x\, e^{4x^2-1}}$$

$$g'(x) = \frac{8}{x}.$$

$$f'(a) = g'(a): \quad \cancel{8}a e^{4a^2-1} = \frac{\cancel{8}}{a}$$

$$e^{4a^2-1} = \frac{1}{a^2}$$

$$4a^2 - 1 = \ln\left(\frac{1}{a^2}\right) = 2\ln\left(\frac{1}{a}\right) = -2\ln a$$

$$\text{So} \quad 4\alpha^2 + 2\ln\alpha - 1 = 0$$

as required.

(α satisfies $4x^2 + 2\ln x - 1 = 0$)

$$x_{n+1} = \sqrt{\frac{1 - 2\ln x_n}{4}}$$

$x_1 = 0.6$

$$x_2 = \sqrt{\frac{1 - 2\ln(0.6)}{4}} = 0.7109$$

$$x_3 = \sqrt{\frac{1 - 2\ln(0.7109)}{4}} = 0.6485$$

$x_4 = 0.6830$

$x_5 = 0.6638$

$x_6 = 0.6745$

$x_7 = 0.6685$

$x_8 = 0.6718$

$x_9 = 0.6700$

(we could make a guess by now...)

$x_{10} = 0.6710$

$x_{11} = 0.6704$

$x_{12} = 0.6708$

$x_{13} = 0.6706$

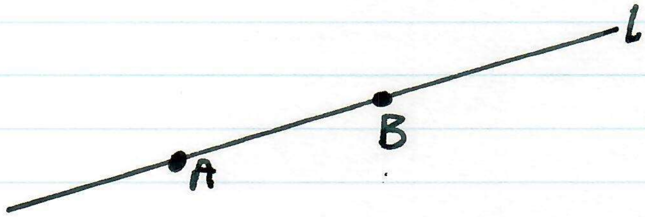
$x_{14} = 0.6707$

$x_{15} = 0.6706$

$x_{16} = 0.6706.$

So this is the solution for α to 4dp.

7)



$\overrightarrow{OA}$: $2\underline{i} - 3\underline{j} + 5\underline{k}$

$\overrightarrow{OB}$: $5\underline{i} + 6\underline{j} + 8\underline{k}$

a) $\overrightarrow{AB}$ is given by $\overrightarrow{OB} - \overrightarrow{OA}$

$$= (5-2)\underline{i} + (6+3)\underline{j} + (8-5)\underline{k}$$

$$= \underline{\underline{3\underline{i} + 9\underline{j} + 3\underline{k}}}$$

b) $\overrightarrow{AP}$ can be expressed as $\lambda(3\underline{i} + 9\underline{j} + 3\underline{k})$
and BP as $(1-\lambda)(3\underline{i} + 9\underline{j} + 3\underline{k})$
and since the geometry is linear we can work with only one of the components.

So $|\overrightarrow{AP}| = 2|\overrightarrow{BP}|$ gives, working with the $\underline{i}$ component:

$$3\lambda = \overset{①}{2(3(1-\lambda))} \text{ or } \overset{②}{(2(3(\lambda-1))}$$

since we are only interested in $|\ |$.

① gives $3\lambda = 6 - 6\lambda$: $9\lambda = 6$ $\lambda = \frac{2}{3}$

② gives $3\lambda = 6\lambda - 6$: $3\lambda = 6$ $\lambda = 2$.

So the position vector of P is ① $\frac{2}{3}(3\underline{i} + 9\underline{j} + 3\underline{k}) = 2\underline{i} + 6\underline{j} + 2\underline{k}$

8) a) $\dfrac{1}{\operatorname{cosec}\theta - 1} + \dfrac{1}{\operatorname{cosec}\theta + 1}$

$$= \frac{\operatorname{cosec}\theta + 1 + \operatorname{cosec}\theta - 1}{\operatorname{cosec}^2\theta - 1}$$

$= \left(\text{since } 1 + \cot^2\theta = \operatorname{cosec}^2\theta\right)$

$$\frac{2\operatorname{cosec}\theta}{\cot^2\theta}$$

$=$

$$= 2\,\frac{\sin^2\theta}{\sin\theta\,\cos^2\theta}$$

$$= 2\,\frac{\sin\theta}{\cos\theta}\cdot\frac{1}{\cos\theta}$$

$$= \underline{\underline{2\tan\theta\sec\theta}} \quad \text{as required.}$$

b) From (a) we can express the LHS as:

$$2\tan 2x \sec 2x = \cot 2x \sec 2x.$$

Dividing by $\sec 2x$ (which is never 0*) this gives:

$$2\tan 2x = \cot 2x$$

$$2\sin^2 2x = \cos^2 2x$$

$$\tan^2 2x = \frac{1}{2} \qquad \tan 2x = \pm\frac{1}{\sqrt{2}}$$

35.26°*

* There's actually an error in the question, because if $0 < x < 90°$ then $x = 45°$ gives

$\sin 2x = \sin 90° = 1$,

so $\operatorname{cosec} 2x = \frac{1}{1} = 1$

so in the LHS:

the $\frac{1}{1-1}$ term becomes $\frac{1}{0}$ which is meaningless.

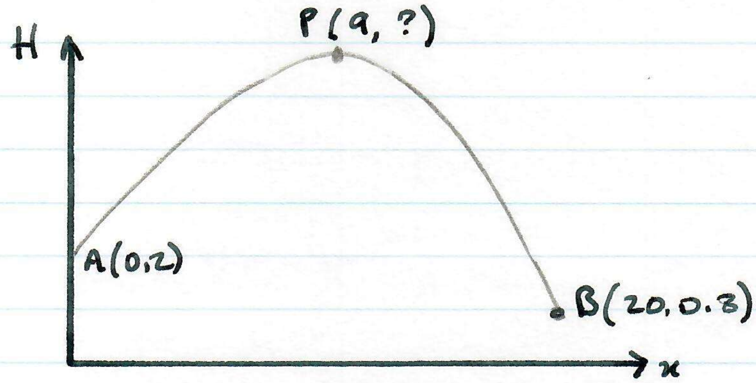
This doesn't produce an extra answer, but it is an error in the question.

We also note that since there's a $\tan^2$ term, the solution should be $\underline{\underline{\pm 17.6°}}$ (thanks for the hint in the question.)

Though the examiners interpret this (rightly, since $0 \leq x \leq \frac{\pi}{2}$)

as $17.6°$ or $90 - 17.6 = \underline{\underline{72.4°}}$.

9) a)



Let the quadratic function be

$$H = ax^2 + bx + c$$

Then substituting for A and B:

$$0a + 0b + c = 2 \qquad \text{so} \quad c = 2.$$

$$400a + 20b + 2 = 0.8 \qquad (1)$$

Also, we know when $x = 9$ H is a maximum

so $\frac{dH}{dx} = 2ax + b = 0$

i.e. $18a + b = 0 \qquad (3)$

$$360a + 20b = 0 \qquad (2)$$

(1) - (2) : $\cancel{360}\,40a + 2 = 0.8$

$$\cancel{36}40a = -1.2 \qquad a = -0.03 \quad \cancel{\tfrac{-3}{910}}$$

from (3): $b = -18a = 0.54 \quad \cancel{\tfrac{54}{910}}$

So $H(x) = -0.03x^2 + 0.54x + 2.$

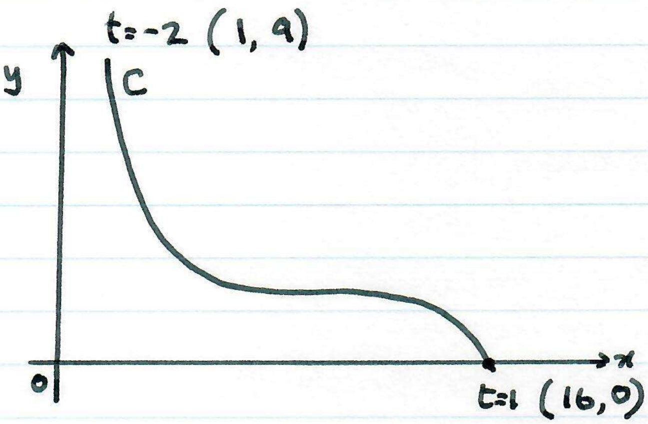
b) It doesn't allow for air resistance

c) When $x = 16$,

$$H = -0.03(16^2) + 0.54(16) + 2$$

$$= 2.96 \text{ m}$$

Since Chandra can only reach to 2.5m they <u>cannot</u> catch the ball.

10) 

$C: \quad x = (t+3)^2 \qquad y = 1 - t^3 \quad -2 \leqslant t \leqslant 1$

$P: \ (4, 2).$

$$\frac{dy}{dt} = -3t^2 \qquad \frac{dx}{dt} = 2(t+3)$$

$$\text{So } \frac{dy}{dx} = \frac{-3t^2}{2(t+3)}$$

At P, $(t+3)^2 = 4$ so $t+3 = 2$: $t = -1$

So at P, $\frac{dy}{dx} = \frac{-3(-1)^2}{2(-1+3)} = \frac{-3}{2\times 2} = -\frac{3}{4}$.

This means the tangent to C has gradient $-\frac{3}{4}$, and hence is a line of the form

$$y = -\frac{3}{4}x + c$$

or $4y = -3x + 4c$

or $3x + 4y = 4c$.

Substituting for $P(4,2)$:

$$3\times 4 + 4\times 2 = 4c$$

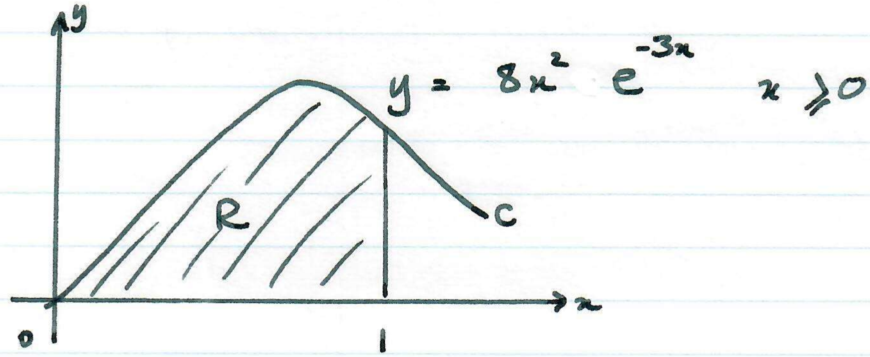
$$20 = 4c$$

So the tangent is, $\underline{\underline{3x + 4y = 20}}$ as required.

b) Assuming the graph is remotely correct, y is maximum when (as we've already determined) $t = -2$: i.e. at the point $(1, 9)$ – $\underline{\underline{y = 9}}$.

Further analysis indicates $y = 1 - t^3$ and on the given interval $[-2, 1]$ this has max. value when $t = -2$, as discussed.

11)



We want to find $R = \int_0^1 8x^2 e^{-3x}\,dx$

This will need at least one integration by parts:

$$\int u\,dv = uv - \int v\,du$$

or $u\,dv + v\,du = d(uv)$

Let $u = 8x^2$ and $dv = e^{-3x}$ $\left(v = \dfrac{e^{-3x}}{-3}\right)$
$du = 16x$

then $\displaystyle\int 8x^2 e^{-3x}\,dx = \frac{8x^2}{-3}e^{-3x} - \int \frac{16x\,e^{-3x}}{-3}\,dx$

$$= \frac{-8}{3}x^2 e^{-3x} + \frac{16}{3}\int x e^{-3x}\,dx \quad ①$$

Apply IBP again to get $\int x e^{-3x}\,dx$:

$u = x$ $\qquad dv = e^{-3x}$

$du = 1$ $\qquad v = \dfrac{e^{-3x}}{-3}$

So $\displaystyle\int x e^{-3x}\,dx = \frac{x e^{-3x}}{-3} - \int \frac{e^{-3x}}{-3}\,dx$

$$= \frac{-1}{3} x e^{-3x} + \frac{1}{3}\int e^{-3x}\,dx$$

$$= \frac{-1}{3} x e^{-3x} - \frac{1}{9} e^{-3x}$$

So in ①:

$$\int 8x^2 e^{-3x}\,dx = -\frac{8}{3}x^2 e^{-3x} + \frac{16}{3}\left(-\frac{1}{3}x e^{-3x} - \frac{1}{9}e^{-3x}\right)$$

$$= \left(-\frac{8}{3}x^2 - \frac{16x}{9} - \frac{16}{27}\right)e^{-3x}$$

$$= \left(-72x^2 - 48x - 16\right)\frac{e^{-3x}}{27}$$

So between the limits 0 and 1 this equals

$$\frac{-72-48-16}{27}e^{-3} + \frac{16}{27}$$

$$= -\frac{136}{27}e^{-3} + \frac{16}{27}$$

which is in the form required, with

$$A = \frac{16}{27} \qquad B = -\frac{136}{27}$$

(12) a) Let $\dfrac{1}{v(25-v)} = \dfrac{A}{v} + \dfrac{B}{25-v}$

$$= \frac{A(25-v) + Bv}{v(25-v)}$$

Equating coeffs of v: $B - A = 0$

$25A = 1$

So $A = \dfrac{1}{25}$, $B = \dfrac{+1}{25}$

So the expression is $\dfrac{1}{25V} + \dfrac{1}{25(25-V)}$

b) $\dfrac{dV}{dt} = \dfrac{1}{10} V(25-V)$ ——— ①

Separating the variables:

$$\frac{dV}{V(25-V)} = \frac{dt}{10}$$

or using (a):

$$\frac{dV}{25V} + \frac{dV}{25(25-V)} = \frac{dt}{10}$$

Integrating:

$$\frac{1}{25}\ln V - \frac{1}{25}\ln(25-V) = \frac{1}{10}t + C$$

So $\ln V - \ln(25 - v) = 2.5t + c$ (new c)

$$\frac{V}{25-V} = Ae^{2.5t} \qquad (A = e^{c})$$

Using the initial condition:

$$\frac{20}{5} = Ae^{0} = A \quad \text{so} \quad A = 4.$$

$$\frac{V}{25-V} = 4e^{2.5t}$$

So $V = 4e^{2.5t}(25 - v)$

$$= 100e^{2.5t} - 4e^{2.5t}V$$

$$V + 4e^{2.5t}V = 100e^{2.5t}$$

$$V = \frac{100e^{2.5t}}{1 + 4e^{2.5t}}$$

$$= \frac{100}{e^{-2.5t} + 4}$$ as required,

with
$A = 100$
$B = 4$
$k = 2.5$

So when $V = 24$,

$$\frac{V}{4(25-V)} = e^{2.5t} = \frac{24}{4} = 6.$$

So $2.5t = \ln 6$

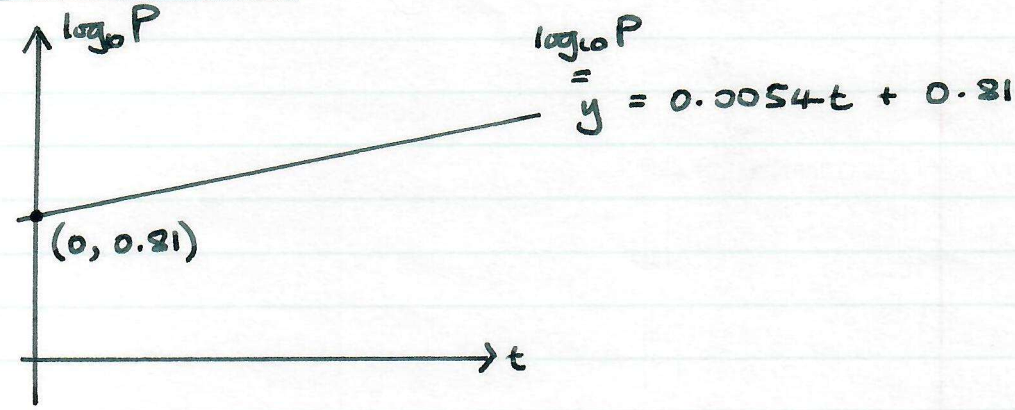
$t = 0.717$ hrs

$= 43$ min.

c) As $t \to \infty$ the $e^{-2.5t}$ term $\to 0$,

so $V \to \frac{100}{0+4} = 25$ as the upper limit L

13) $\underline{P = ab^t}$ (1)



a) From the data given we know the straight line plot has a gradient 0.0054 and intercept on the $\log_{10} P$ axis, so its equation is

$$\log_{10} P = 0.0054t + 0.81$$

(the usual $y = mx + c$ form)

Taking $\log_{10}$ of (1):

$$\log_{10} P = \log_{10} a + t \log_{10} b$$

So matching the coeffs:

$$\log_{10} a = 0.81 : \quad a = 10^{0.81} = \underline{\underline{6.457}}$$

$$\log_{10} b = 0.0054 : \quad b = 10^{0.0054} = \underline{1.013}$$

So the model is $\quad P = \underline{\underline{6.457 \times 1.013^t}}$

(which seems reasonable)

b) i) a is the initial size of the population in 2004, i.e. 6.457 Billion.

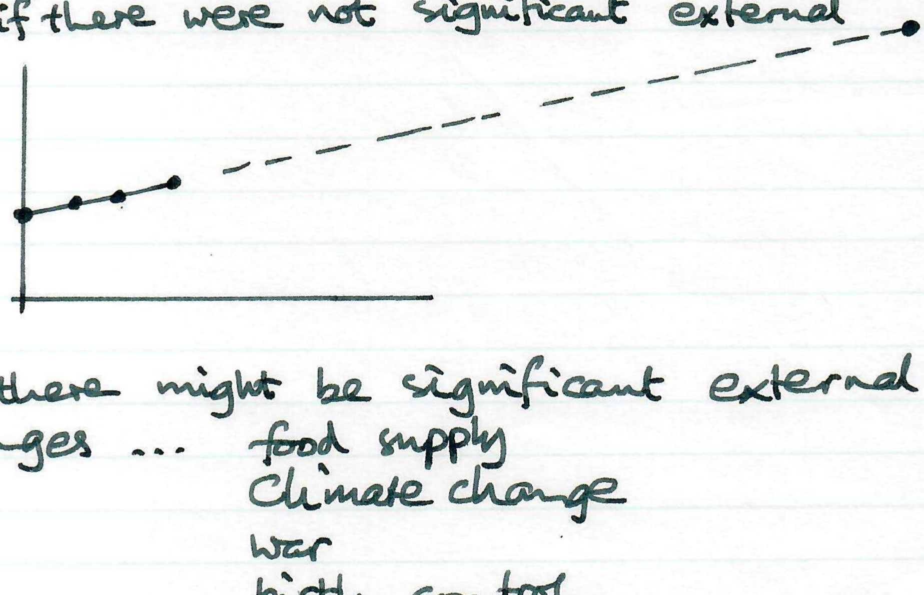
ii) b is the multiple by which the population ~~ex~~ changes each year, i.e. 1.013 or an <u>increase</u> of 1.3%.

c) In 2030 the model predicts a population, after 26 years from 2004, of

$$6.457 \times 1.013^{26}$$

$$= 9.034 \text{ billion.}$$

d) Given that the model was derived from only 4 data points and 2030 is $6\frac{1}{2}$ times that far away, it is hard to believe the model would still apply, even if there were not significant external changes.



And there might be significant external changes ... food supply
Climate change
war
birth control
poverty
⋮

14) C: $x^2 + y^2 - 6x + 14y + 23 = 0$ — ①

(well-known fact): if a ⊙ is $x^2 + y^2 + 2px + 2qy + r = 0$

then the circle centre is $(-p, -q)$

and the radius is $\sqrt{p^2 + q^2 - r}$

So for C: centre is $(3, -7)$

radius is $\sqrt{3^2 + 7^2 - 23}$

$= \sqrt{9 + 49 - 23} = \sqrt{25} = 5$

(we can write the equation as

C_1: $(x-3) + (y+7)^2 = 25$)

C_2 can be written

$$(x+6)^2 + (y+8)^2 = k^2$$

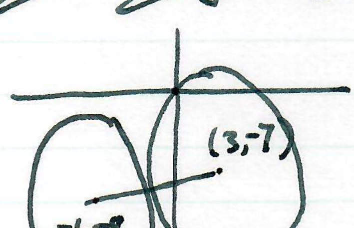
or $x^2 + y^2 + 12x + 16y + 36 + 64 = k^2$

$x^2 + y^2 + 12x + 16y + (100 - k^2) = 0$ — ②

To find the intersections:

~~②−① gives~~

$\cancel{18x + 2y + (100 - k^2) - 23 = 0}$



~~but we don't really need this.~~

For the ⊙s to overlap, the distance between the centres must be less than the sum of the radii:

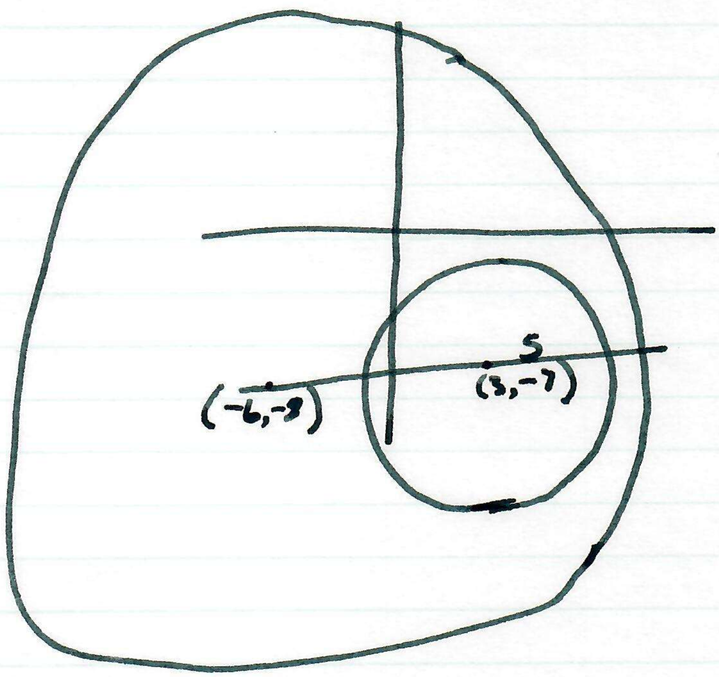
$$\text{So} \quad \sqrt{(-6-3)^2 + (-8-(-7))^2} < k+5$$

$$\sqrt{9^2+(-1)^2} < k+5$$

$$\sqrt{82} < k+5$$

$$k > \sqrt{82}-5 = 4.055$$

But also k must not be too large:



here $k-5 < \sqrt{82}$ so $k < \sqrt{82}+5 = 14.055$

So the overall conditions on k are

$$\sqrt{82}-5 < k < \sqrt{82}+5 \qquad 4.055 < k < 14.055)$$

15) $\quad C: \ (x+y)^3 = 3x^2 - 3y - 2 \qquad ①$

Differentiate:

$$3(x+y)^2\left(1+\frac{dy}{dx}\right) = 6x - 3\frac{dy}{dx}$$

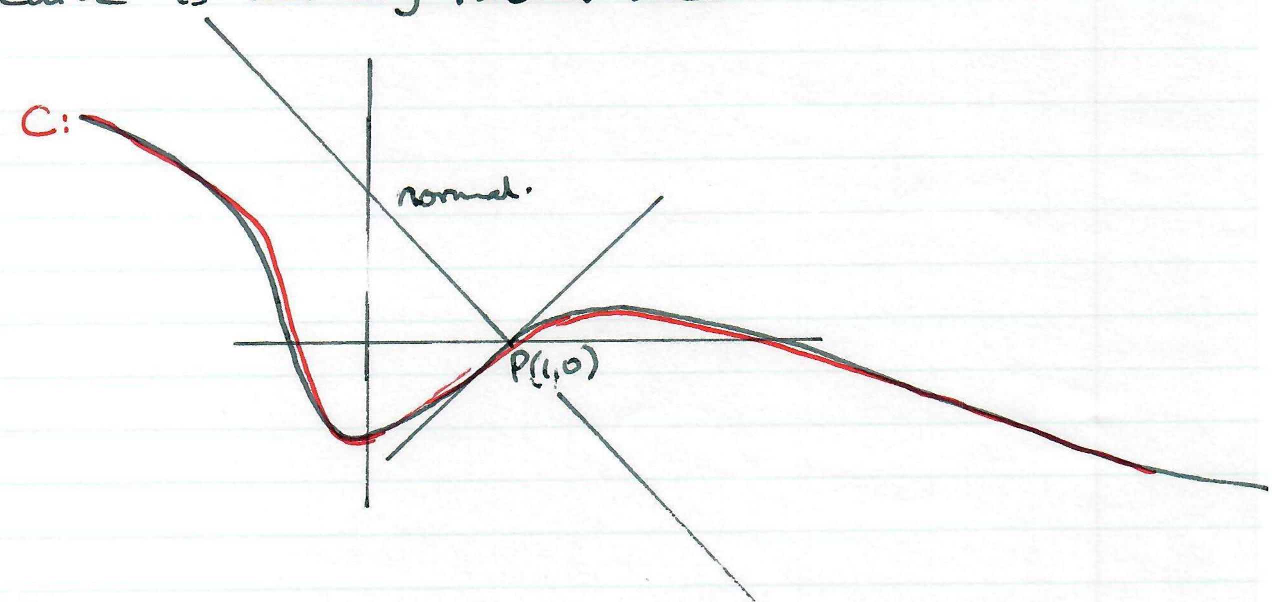
$$3(x+y)^2 + 3(x+y)^2\frac{dy}{dx} = 6x - 3\frac{dy}{dx}$$

$$\frac{dy}{dx}\left(3(x+y)^2+3\right) = 6x - 3(x+y)^2$$

$$\frac{dy}{dx} = \frac{6x-3(x+y)^2}{3(x+y)^2+3}$$

$$= \frac{2x-(x+y)^2}{1+(x+y)^2}$$

The curve is actually rather nice:



b) At P the gradient of the curve and tangent is

$$\frac{dy}{dx} = \frac{2x - (x+y)^2}{1+(x+y)^2} = \frac{2-1^2}{1+1^2} = \frac{1}{2}$$

So the gradient of the normal (negative reciprocal) is -2

and the eq^n is $y = -2x + c$

Subs. in for $P(1,0)$:

$$0 = -2 \cdot 1 + c$$

$$c = 2$$

So the normal is

$$\underline{y = -2x + 2}$$ as required. 2

c) If the normal meets C, then subst. 2 into 1:

$$(x - 2x + 2)^3 = 3x^2 - 3(-2x+2) - 2$$

$$(2-x)^3 = 3x^2 + 6x - 6 - 2$$

$$8 - 12x + 6x^2 - x^3 = 3x^2 + 6x - 8$$

$$-18x + 3x^2 = x^3 - 16$$

$$x^3 - 3x^2 + 18x - 16 = 0$$

we know this has a root $x=1$, so a factor $(x-1)$:

but the quadratic factor has ~~determinant~~ discriminant

(b^2-4ac) $\quad 4-4\times16 = -60,$

so there are no real roots.

Hence, as required, this normal only meets C at the point P.