1) $f(x) = x^3 + 2x^2 - 8x + 5$

$f'(x) = 3x^2 + 4x - 8$

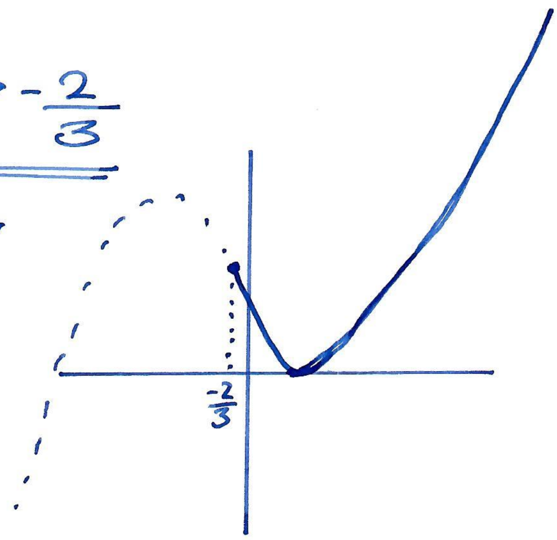
$f''(x) = 6x + 4.$

$f(x)$ is concave when $f'(x)$ is increasing from negative to positive, which is when $f''(x)$ is positive.

$6x + 4 > 0$ when $6x > -4$

i.e. $\underline{\underline{x > -\frac{2}{3}}}$

It actually looks like...



2) $u_1 = 35$

$u_{n+1} = u_n + 7\cos\left(\frac{n\pi}{2}\right) - 5(-1)^n$

a) $u_2 = 35 + 7\cos\left(\frac{\pi}{2}\right) - 5(-1)^1$

$= 35 + 7\times 0 + 5 \quad = \underline{40}$ as required.

b) If the sequence is periodic with order 4 then $u_5 = u_1 = 35$.

(though it might be good to check there isn't a growth from u_1 to u_5.

So $u_2 = 40$

$$u_3 = 40 + 7\cos\left(\frac{2\pi}{2}\right) - 5(-1)^2$$

$$= 40 + 7\times(-1) - 5$$

$$= 28.$$

$$u_4 = 28 + 7\cos\left(\frac{3\pi}{2}\right) - 5(-1)^3$$

$$= 28 + 7\times 0 + 5$$

$$= 33$$

$$u_5 = 33 + 7\left(\cos\frac{4\pi}{2}\right) - 5(-1)^4$$

$$= 33 + 7\times 1 - 5$$

$$= 35.$$

So we've confirmed $u_5 = 35$ and the sequence is truly periodic.

So $\sum_{1}^{25} u_r = \sum_{1}^{4} + \sum_{5}^{8} \cdots + \sum_{21}^{24} + u_{25}$

$= 6\times(35+40+28+33)+35 = 6\times136+35 = 816+35 = \underline{851}$

3) $\log_2(x+3) + \log_2(x+10) = 2 + 2\log_2 x$

$(\log_2 4 = 2)$ $\qquad = \log_2 4 + 2\log_2 x$

So undoing the logarithms,

$$(x+3)(x+10) = 4x^2$$

$$x^2 + 13x + 30 = 4x^2$$

$$-3x^2 + 13x + 30 = 0$$

or $\underline{\underline{3x^2 - 13x - 30 = 0}}$ as required.

The equation factorises:

$$(3x+5)(x-6) = 0$$

This gives $x = \frac{-5}{3}$ or $x = 6$.

But since $\log_2\left(\frac{-5}{3}\right)$ is nonsense
(log of a -ve number)

we accept only $x = 6$ as a solution and

reject $x = \frac{-5}{3}$.

4) $H = Ae^{-Bt} + 30$

a) when $t = 0$, $H = 85$:

$$85 = Ae^{0} + 30 = A + 30 \quad \text{So} \quad \underline{\underline{A = 55}}$$

b) we know $H = 55e^{-Bt} + 30$

so $$\frac{dH}{dt} = -B \times 55e^{-Bt}$$

and when $t = 0$, $\frac{dH}{dt} = -7.5$:

$$-B \times 55e^{0} = -7.5$$

$$55B = 7.5$$

$$B = \frac{7.5}{55} = 0.136$$

So $\underline{\underline{H = 65e^{-0.136t} + 30}}$ is the complete equation.

5) $y = f(x)$ $P(3;10)$

a) at P $f(x) = -10$ and $f'(x) = 0$

$$\frac{dy}{dx} = 2x^3 - 9x^2 + 5x + k \quad \text{———} ①$$

So at $P(3,10)$,

$$2\times3^3 - 9\times3^2 + 5\times3 + k = 0$$
$$2\times27 - 9\times9 + 15 + k = 0$$
$$54 - 81 + 15 + k = 0$$
$$-12 + k = 0$$

So $\underline{\underline{k = 12}}$ as required.

b) Integrating ①:

$$y = \frac{2}{4}x^4 - \frac{9}{3}x^3 + \frac{5}{2}x^2 + 12x + c$$

$$= \frac{1}{2}x^4 - 3x^3 + \frac{5}{2}x^2 + 12x + c$$

Substituting for $P(3;10)$:

$$-10 = \frac{1}{2}3^4 - 3\times3^3 + \frac{5}{2}\times3^2 + 12\times3 + c$$

$$= \frac{81}{2} - 81 + \frac{5}{2}\times9 + 36 + c$$

$$= -\frac{81}{2} + \frac{45}{2} + 36 + c = -\frac{36}{2} + 36 + c$$

$$= -18 + 36 + c = 18 + c.$$

So $\cancel{c} = -10 - 18 = -28$.

So $y = f(x) = \frac{1}{2}x^4 - 3x^3 + \frac{5}{2}x^2 + 12x - 28$. —— (2)

~~when the curve C crosses the y-axis (multiply by 2, since this has no effect:~~

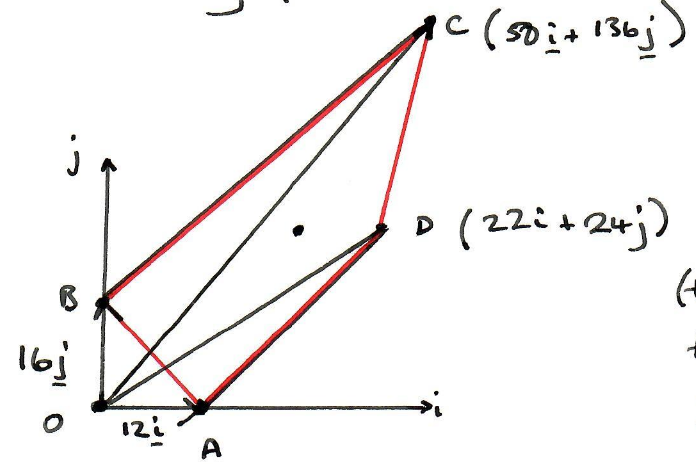
~~$x^4 - 6x^3 + 5x^2 + 24x - 56 = 0$~~

stoopid.

The curve crosses the y-axis when $x = 0$, which in (2) is clearly $y = -28$.

So the crossing point coords are $(0, -28)$

6)



(the diagram's just to get everything on paper)

a) we calculate

$$\vec{AD} = \vec{OD} - \vec{OA} = (22\underline{i} + 24\underline{j}) - (12\underline{i} + 0\underline{j})$$
$$= 10\underline{i} + 24\underline{j}.$$

$$\vec{BC} = \vec{OC} - \vec{OB} = (50\underline{i} + 136\underline{j}) - (0\underline{i} + 16\underline{j})$$
$$= 50\underline{i} + 120\underline{j}$$

To prove these are parallel:

a) The 'gradients' are equal $\left(\frac{24}{10} = \frac{120}{50} \right)$ so...

b) $\vec{BC} = 5\vec{AD}$ so...

c) $\vec{BC} \times \vec{AD} = \begin{pmatrix} 24 \\ 10 \end{pmatrix} \times \begin{pmatrix} 120 \\ 50 \end{pmatrix} = 24 \times 50 - 120 \times 10 = 0$

... but not convinced (c) would satisfy.

b) This is a tedius exercise in lengths...

$|AB| = \sqrt{16^2 + 12^2} = \sqrt{256 + 144} = \sqrt{400} = 20$

$|BC| = \sqrt{120^2 + 50^2} = \sqrt{14400 + 2500} = \sqrt{16900} = 130$

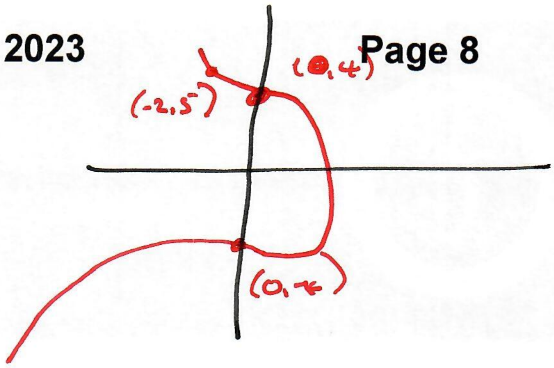
$|CD| = \sqrt{28^2 + 112^2} = \sqrt{1 + 16} \times 28 = 28\sqrt{17}$

$|DA| = \sqrt{10^2 + 24^2} = \sqrt{100 + 576} = \sqrt{676} = 26.$

So the length of the track is $20 + 130 + 26 + 28\sqrt{17}$

$= \underline{176 + 28\sqrt{17}}$ m.

So average speed $= \frac{2(176 + 28\sqrt{17})}{5} \times$ m/min $= \underline{\underline{6.945 \text{ km/h}}}.$



7) $x^3 + 2xy + 3y^2 = 47$

a) Differentiate:

$$3x^2 + 2x\frac{dy}{dx} + 2y + 6y\frac{dy}{dx} = 0$$

$$\frac{dy}{dx}(2x+6y) = -(3x^2+2y)$$

so $$\frac{dy}{dx} = -\frac{3x^2+2y}{2x+6y}$$

So at $P(-2,5)$:

$$\frac{dy}{dx} = -\frac{3(-2)^2+10}{-4+30}$$

$$= -\frac{12+10}{26} = -\frac{22}{26} = -\frac{11}{13}.$$

This is the gradient of the curve so the gradient of the normal is the -ve reciprocal:

i.e. $\frac{13}{11}$.

and the eq^n of the normal is $y = \frac{13}{11}x + c$ for some c.

Subs for $P(-2,5)$: $5 = \frac{-13.2}{11} + c = \frac{-26}{11} + c$

so $c = \frac{1}{11}(55+26) = \frac{81}{11}$ $\quad y = \frac{13}{11}x + \frac{81}{11}$: $11y = 13x + 81$

$13x - 11y + 81 = 0$ as required.

8) $2\cos\theta + 8\sin\theta$

a) Use the identity: $\cos(\theta - \alpha) = \cos\theta\cos\alpha + \sin\theta\sin\alpha$

and let $R\cos\alpha = 2$

$R\sin\alpha = 8$

so $\tan\alpha = 4$: $\alpha = 75.964°$

$= \underline{\underline{1.326 \text{ radians.}}}$

~~(and $R = \frac{2}{\cos(1.326)} = 8.254$)~~

and $(R^2\cos^2\alpha + R^2\sin^2\alpha) = R^2 = 2^2 + 8^2 = 68$

So $R = \sqrt{68} = \underline{\underline{2\sqrt{17}}}$

and we have $2\cos\theta + 8\sin\theta = 2\sqrt{17}\cos(\theta - 1.326)$

b) $u_1 = \cos x$

$u_2 = \cos x + \sin x$

$u_3 = \cos x + 2\sin x$

$u_n = \cos x + (n-1)\sin x.$

So $S_n = \sum_{i=1}^{n} u_n = n\cos x + \frac{n(n-1)}{2}\sin x$

$S_9 = 9\cos x + \frac{9 \times 8}{2}\sin x.$

(i) $S_9 = 9\cos x + 36\sin x$

$= 9(\cos x + 4\sin x)$

$= \frac{9}{2}(2\cos x + 8\sin x)$

from the expression in (a) this gives

$S_9 = \frac{9}{2} \times 2\sqrt{17}\left(\cos(x - 1.326)\right)$

$= 9\sqrt{17}\cos(x - 1.326)$

The maximum value of the cos term is 1 (when $x = 1.326$)

and this gives $S_9 = 9\sqrt{17}$.

(ii) As noted, the max. value is reached when $x = 1.326$.

And since the period of the cos term is 2π, this is the smallest positive value of x which gives the maximum value of S_9.

9) C: $x = t^2 + 6t - 16 \quad = (t-2)(t+8)$

$y = 6\ln(t+3)$

a) Consider $(t+3)^2 = t^2 + 6t + 9$

Then $x = (t+3)^2 - 25$

$$x + 25 = (t+3)^2$$

$$t + 3 = (x+25)^{\frac{1}{2}}$$

So $y = 6\ln(x+25)^{\frac{1}{2}}$

$$= \frac{6}{2}\ln(x+25)$$

$$= 3\ln(x+25) \quad \text{———} \ ①$$

which matches $y = A\ln(x+B)$ with $A = 3$, $B = 25$

b) At P, $x = 0$ so $y = 3\ln(25)$

And from ①

$$\frac{dy}{dx} = \frac{3}{x+25} = \frac{3}{25}.$$

So the tangent at P has equation

$y = \frac{3}{25}x + c$ for some c.

Subs. for P: $3\ln(25) = \frac{3 \times 0}{25} + c \quad : c = 3(\ln(25))$

So $y = \frac{3}{25}x + 3\ln(25) \quad : \quad 25y = 3x + 75\ln(25)$

$$= 3x + 150\ln 5$$

10.) $f(x) = \dfrac{3kx-18}{(x+4)(x-2)}$

a) Split into fractions:

$$\frac{A}{x+4} + \frac{B}{x-2} = \frac{A(x-2)+B(x+4)}{(x+4)(x-2)} = \frac{(A+B)x + (4B-2A)}{(\quad)(\quad)}$$

Equating coefficients:

x: $3k = A+B$

$-18 = 4B-2A$

$12k = 4A+4B$

$12k+18 = 4A+2A$: $A = 2k+3$.

$B = 3k - A = k-3$.

So $f(x) = \dfrac{2k+3}{x+4} + \dfrac{k-3}{x-2}$

b) $\displaystyle\int_{-3}^{1} f(x)\,dx = \int_{-3}^{1} \frac{2k+3}{x+4}\,dx + \int_{-3}^{1} \frac{k-3}{x-2}\,dx$

$= \Big[(2k+3)\ln|(x+4)| + (k-3)\ln|(x-2)|\Big]_{-3}^{1}$

$= (2k+3)\ln 5 + \cancel{(k-3)\ln 1}$

$- \cancel{(2k+3)\ln 1} + -(k-3)\ln 5$

$= (2k+3)\ln 5 + (-k+3)\ln 5$

$= (k+6)\ln 5 \qquad = 21$ (given)

So $k = \dfrac{21}{\ln 5} - 6$

11) a) $V = 200h$ —— ①

$\dfrac{dV}{dt} = \dfrac{k}{\sqrt{h}}$ for some k —— ②

(this is what the second • means).

From ① $\dfrac{dV}{dt} = 200\dfrac{dh}{dt}$

So $\dfrac{dh}{dt} = \dfrac{1}{100}\dfrac{dV}{dt}$

$= \dfrac{k}{100\sqrt{h}}$ in ②.

Let $\dfrac{k}{100} = \lambda$:

$\dfrac{dh}{dt} = \dfrac{\lambda}{\sqrt{h}}$ as required. ③

b) when $t=0 \quad h = 1.44$

$t = 8 \quad h = 3.24$.

Solve ③ with ~~integration by parts~~ separation of variables: $\sqrt{h}\,dh = \lambda\,dt$

so $\frac{2}{3}h^{\frac{3}{2}} = \lambda t + c$ for some c.

$t = 0$ gives $\frac{2}{3}(1.44)^{\frac{3}{2}} = c$

$= \frac{2}{3}(1.2)^3 = \frac{2}{3} \times 1.728 = 2 \times .576$

$= \underline{\underline{1.152}}$

$\frac{2}{3}h^{\frac{3}{2}} = \lambda t + 1.152$

$t = 8$ gives $\frac{2}{3}(3.24)^{\frac{3}{2}} = 8\lambda + 1.152$

$\frac{2}{3}(1.8)^3 = 3.888 = 8\lambda + 1.152$

$\lambda = 0.342$

So $\frac{2}{3}h^{\frac{3}{2}} = 0.342t + 1.152$

$\underline{\underline{h^{\frac{3}{2}} = 0.513t + 1.728}}$

where $A = 0.513$ and $B = 1.728$.

c) when $h = 5$,

$11.180 = 5^{\frac{3}{2}} = 0.513t + 1.728$

$\underline{\underline{t = 18.426 \text{ min.}}}$

12. a) Initially ($t=0$) $N_A = |-3|+4 = 7$

$N_B = 8-|-6| = 2$

So the initial difference is $\underline{\underline{5}}$

b) The min. value of N_A is when $|t-3|=0$, i.e. when $t=3$. The company's recovery dates from this point so $T=3$.

c) The initial crossover when $N_A = N_B$ is given by

$$N_A = \underbrace{3-t}_{\text{we know this is +ve}}+4 = 8-\underbrace{(6-2t)}_{\text{and this}} = N_B$$

so $7-t = 2+2t$

$5 = 3t$ $t = \frac{5}{3}$

The second crossover is given by reflection in the line $t=3$, since the graph is symmetrical (we know this because the gradients are basically ± 1 and ± 2.)

So this point is $t = 3+\left(3-\frac{5}{3}\right) = \frac{13}{3}$

So the condition for $N_A > N_B$ is

$$0 \leqslant t < \frac{5}{3} \quad \text{or} \quad \frac{13}{3} < t \leqslant 5$$

i.e. $$\left\{t : t \in \left[0, \tfrac{5}{3}\right)\right\} \cup \left\{t : t \in \left(\tfrac{13}{3}, 5\right]\right\}$$

(or equivalent)

d) Presumably it hits 0 at some point ...

when $2t - 6 = 8$ i.e. $t = \frac{14}{2} = 7$.

So the company will ...
- collapse
- go into bankruptcy
- become an offshore shell company in the Caymans.
- never repay the \$10 start-up grant it got from the government
- get bought up by A, for a very low price.
- default on its debts
- still somehow pay its shareholders a huge dividend
- default on its [illegible]

13) a) $(3+x)^{-2} = 3^{-2}\left(1+\frac{x}{3}\right)^{-2}$

(must take the factor out first)

$$= \frac{1}{9}\left[1 + (-2)\frac{x}{3} + \frac{(-2)(-3)}{2}\frac{x^2}{9} + o(x^3) \cdots\right]$$

$$= \frac{1}{9} - \frac{2}{27}x + \frac{3}{81}x^2 + o(x^3)$$

$$\approx \underline{\underline{\frac{1}{9} - \frac{2}{27}x + \frac{1}{27}x^2}}$$

b) Multiplying by $6x$, we want

$$\int_{0.2}^{0.4} \frac{6x}{9} - \frac{12}{27}x^2 + \frac{6}{27}x^3\,dx$$

$$= \int_{0.2}^{0.4} \frac{2}{3}x - \frac{4}{9}x^2 + \frac{2}{9}x^3\,dx$$

$$= \left[\frac{2}{3}\frac{x^2}{2} - \frac{4}{9}\frac{x^3}{3} + \frac{2}{9}\frac{x^4}{4}\right]_{0.2}^{0.4}$$

$$= \left[\frac{x^2}{3} - \frac{4}{27}x^3 + \frac{1}{18}x^4\right]_{0.2}^{0.4}$$

$$= \frac{1}{3}(0.16-0.04) - \frac{4}{27}(0.064-0.008) + \frac{1}{18}(0.0256-0.0016)$$

$$= \frac{1}{3}(0.12) - \frac{4}{27}(0.056) + \frac{1}{18}(0.0240)$$

$$= 0.03304 \quad (4\text{ sf})$$

c) Let $u = 3 + x$ then $du = dx$, $6x = 6u - 18$

and the integral required is

$$\int_{3.2}^{3.4} \frac{6u-18}{u^2}\, du$$

$$= \int_{3.2}^{3.4} \frac{6}{u} - \frac{18}{u^2}\, du \qquad \int u^{-2} du = -\int u^{-1}$$

$$= \left[6 \ln u + \frac{18}{u} \right]_{3.2}^{3.4}$$

$$= 6\left(\ln 3.4 - \ln 3.2\right) + 18\left(\frac{1}{3.4} - \frac{1}{3.2}\right)$$

$$= 6 \ln \frac{3.4}{3.2} + 18\left(\frac{1}{3.4} - \frac{1}{3.2}\right)$$

$$= 6 \ln 1.0625 + (-0.3309)$$

$= a \ln b + c$ where $a = 6$

$b = 1.0625$

$c =$ (strictly) -0.3309.

14) $2\tan\theta\,(8\cos\theta + 23\sin^2\theta) = 8\sin 2\theta\,(1+\tan^2\theta)$

$= 8\sin 2\theta\,\sec^2\theta$

$= \dfrac{8\sin 2\theta}{\cos^2\theta}$

So $2\tan\theta\cos^2\theta\,(8\cos\theta + 23\sin^2\theta) = 8\sin 2\theta$ —— ①

$2\,\dfrac{\sin\theta}{\cancel{\cos\theta}}\,\cos^{\cancel{2}}\theta\,(8\cos\theta + 23\sin^2\theta)$

$= 2\sin\theta\,(8\cos\theta + 23(1-\cos^2\theta))\cos\theta$

$= 2\sin\theta\cos\theta\,(23 + 8\cos\theta - 23\cos^2\theta)$

$= \sin 2\theta\,(23 + 8\cos\theta - 23\cos^2\theta)$

$= 8\sin 2\theta$ from ①.

So $\sin 2\theta\,(-23\cos^2\theta + 8\cos\theta + 15) = 0$

or $\sin 2\theta\,(23\cos^2\theta - 8\cos\theta - 15) = 0$

Solutions are given by $\sin 2\theta = 0$

$2\theta = n\pi \quad n \in \mathbb{N}$

$\theta = \dfrac{n}{2}\pi = 90n°$ (a)

or $(23\cos^2\theta - 8\cos\theta - 15) = 0$

$(23\cos\theta + 15)(\cos\theta + -1)$

$\cos\theta = -\frac{15}{23}$ or $\cos\theta = +1$

$\theta = (2n)\pi \quad n \in \mathbb{N}$ (b)

$= 360n^\circ$

$\theta = (2n+1)\pi \pm 49.3^\circ = (2n+1)180 \pm 49.3^\circ$

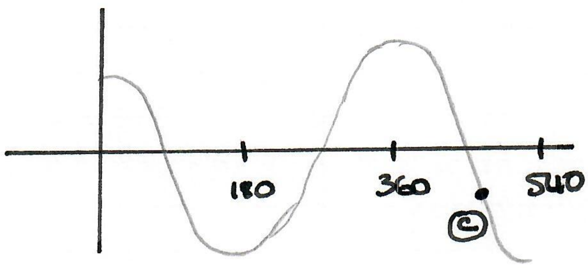
To satisfy $360^\circ \leqslant x \leqslant 540^\circ$ this means

(a) $x = 360^\circ, 450^\circ, 540^\circ$

disagrees with Pearson.

(b) $x = 360^\circ$

(c) $x = 490.7^\circ$



15) Prove by contradiction: $\frac{\pi}{2} < x < \pi$

$\sin x - \cos x \geqslant 1$

Assume that $\sin x < 1$

$\sin x - \cos x < 1$

then $(\sin x - \cos x)^2 < 1$ (actually this just isn't true: $x = 135°$ is an easy counterexample)

So $\sin^2 x - 2\sin x \cos x + \cos^2 x < 1$

$-2\sin x \cos x + 1 < 1$

$-2\sin x \cos x < 0$

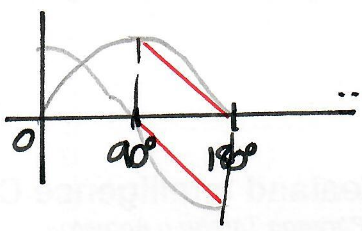
but $2\sin x \cos x$ is always $-$ve for obtuse angles, so this says

$-2(-\text{ve}) < 0$ which is a contradiction.

So the result is proved:

$\sin x - \cos x \geqslant 1.$

Actually a much better proof would be to consider ...



... And note a bit of geometry... or something.