Jon + Roy

1) a) $a_1 = 16$

$a_{n+1} = a_1 + nd$ in general. (d is the common difference)

$a_{21} = 16 + 20d = 24$

So $20d = 8$ $\quad d = \frac{8}{20} = \frac{4}{10}, \frac{2}{5}, 0.4.$

as required.

b) Expression for the sum of n terms, S_n, is:

$$S_n = \frac{1}{2}n\left(2a_1 + (n-1)d\right)$$

So $$S_{500} = \frac{500}{2}\left(2 \times 16 + 499 \times 0.4\right)$$

$$= \underline{\underline{57900}}$$

2) $f(x) = 7 - 2x^2$

$g(x) = \dfrac{3x}{5x+1}$

a) Since $2x^2$ is never <0 and has a single 0 when $x=0$, $f(x)$ is never >7. But it can be as low (negative) as we want, since $2x^2$ is unbounded.

So range$(f) = (-\infty, 7]$

b) $f(1.8) = 7 - 2(1.8)^2$

$= 7 - 2(3.24)$

$= 7 - 6.48$

$= \underline{\underline{0.52}}$

So $gf(1.8) = \dfrac{3 \times 0.52}{5 \times 0.52 - 1}$

$= \underline{\underline{0.975}}$

c) Let $y = g^{-1}(x)$

then $g(y) = x = \dfrac{3y}{5y-1}$

$x = \dfrac{3y}{5y-1}$

$(5y-1)x = 3y$

$5xy - 3y = x$

$y(5x-3) = x$

$y = \underline{\dfrac{x}{5x-3}} = g^{-1}(x)$ as required.

3) $\log_3(12y+5) - \log_3(1-3y) = 2$

so $\log_3 \dfrac{(12y+5)}{(1-3y)} = 2$

$$\frac{12y+5}{1-3y} = 3^2 = 9$$

$$12y+5 = 9(1-3y) = 9 - 27y$$

$$39y = 4$$

$$y = \frac{4}{39}$$

4) This is simply a case of plugging the approximations in:

$$4\sin\frac{\theta}{2} + 3\cos^2\theta$$

$$\approx 4\left(\frac{\theta}{2}\right) + 3\left(1 - \frac{\theta^2}{2}\right)^2$$

$$= 2\theta + 3 - 3\times\frac{\cancel{2}\theta^2}{\cancel{2}} + \cancel{3\left(\frac{\theta^2}{2}\right)^2}$$

$$= 3 + 2\theta - 3\theta^2$$

as required, with $a=3$, $b=2$, $c=-3$.

5) $y = 5x^4 - 24x^3 + 42x^2 - 32x + 11$

a) i) $\frac{dy}{dx} = 20x^3 - 72x^2 + 84x - 32$

ii) $\frac{d^2y}{dx^2} = 60x^2 - 144x + 84$

b) i) when $x = 1$,

$$\frac{dy}{dx} = 20\times 1^3 - 72\times 1^2 + 84\times 1 - 32$$

$= 0$, so this is confirmed as a turning-point.

ii) when d $x = 1$,

$$\frac{d^2y}{dx^2} = 60\times 1^2 - 144\times 1 + 84$$

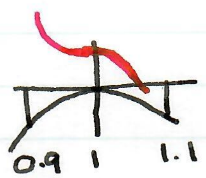
$= 0.$

If the second derivative is 0 we know this is a point of inflection. We can confirm this by calculating

$\frac{dy}{dx}$ at $x = 0.9$ $= -0.14$

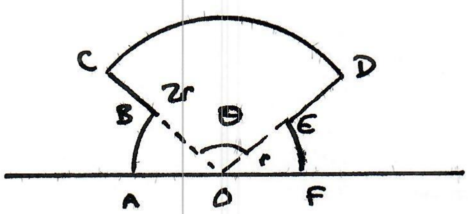
$\frac{dy}{dx}$ at $x = 1.1$ $= -0.1$

So the values of $\frac{dy}{dx}$ are:



which indicates y is shaped as red. — i.e. with a point of inflection.

6)



a) we know AOF is a straight line,

so $2\hat{AOB} + \theta = \pi$:

$$\hat{AOB} = \frac{\pi - \theta}{2}$$

b) Using the formula Area $= \frac{\theta r^2}{2}$,

Area AOB $= \left(\frac{\pi-\theta}{2}\right)\frac{r^2}{2}$

Area EOF $=$ ^

Area COD $= \frac{\theta(2r)^2}{2} = 2\theta r^2$

So overall area $=$

$$\cancel{2}\left(\frac{\pi-\theta}{2}\right)\frac{r^2}{\cancel{2}} + 2\theta r^2$$

$$= \left(\frac{\pi}{2} - \frac{\theta}{2} + 2\theta\right)r^2$$

$$= \frac{1}{2}(\pi + 3\theta)r^2 \quad \text{as required.}$$

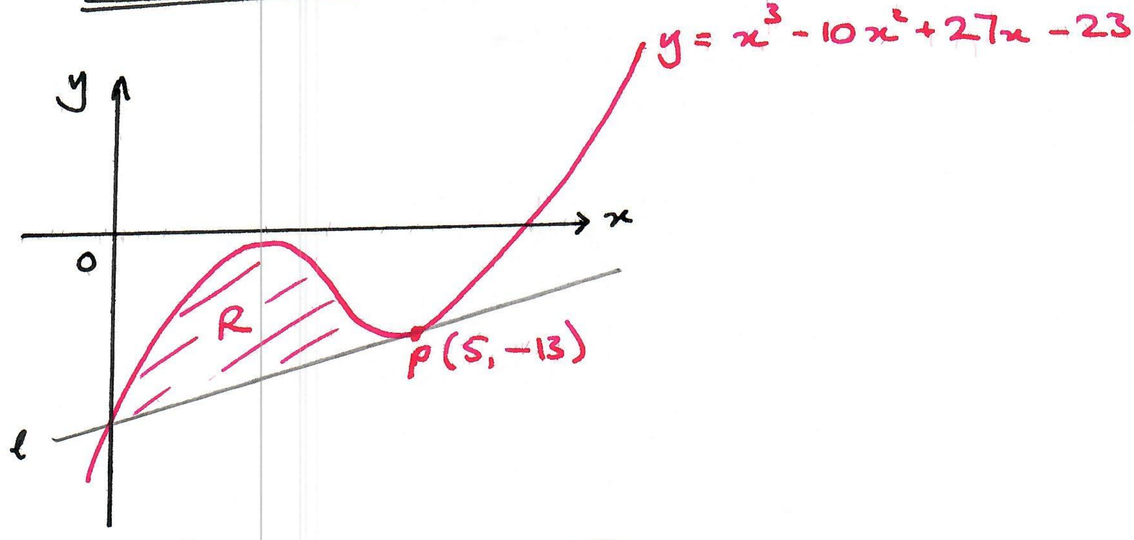
c) Using the formula Perimeter $= \theta r$:

$$\underset{AO}{r} + \underset{AB}{\left(\frac{\pi-\theta}{2}\right)r} + \underset{BC}{r} + \underset{CD}{2\theta r} + \underset{DE}{r} + \underset{EF}{\left(\frac{\pi-\theta}{2}\right)r} + \underset{FO}{r}$$

$= 4r + (\pi - \theta)r + 2\theta r$

$= (4 + \pi + \theta)r$

7.)



$y = x^3 - 10x^2 + 27x - 23$

l is the tangent at $(5, -13)$, so we calculate:

$\frac{dy}{dx} = 3x^2 - 20x + 27$

$= 3 \times 25 - 20 \times 5 + 27$

$= 75 - 100 + 27$

$= 2$ at $P(5, 13)$.

So the tangent is $y = 2x + c$ for some c.

Subs. for P: $-13 = y = 2 \times 5 + c = 10 + c$

So $c = -23$.

So $y = 2x - 23$. is l.

when this intercepts the y axis $x = 0$,

So $y = -23$.

And when $x=0$ the curve C gives

$$y = 0 - 10\times 0 + 27\times 0 - 23 = -23$$

So the curve and the tangent meet on the y-axis at $\underline{\underline{(0,-23)}}$, as required.

c) $$R = \int_0^5 C - \ell \, dx$$

$$= \int_0^5 x^3 - 10x^2 + 27x - 23 - 2x + 23 \, dx$$

$$= \int_0^5 x^3 - 10x^2 + 25x \, dx$$

$$= \left[\frac{x^4}{4} - \frac{10x^3}{3} + \frac{25x^2}{2} \right]_0^5$$

$$= \frac{625}{4} - \frac{1250}{3} + \frac{625}{2} - 0 + 0 - 0$$

$$= \frac{1875}{4} - \frac{1250}{3}$$

$$= \frac{5625 - 5000}{12}$$

$$= \underline{\underline{\frac{625}{12}}}$$

8) $px^3 + qxy + 3y^2 = 26$ ——— ①

a) Differentiating:

$$3px^2 + q\left(y + x\frac{dy}{dx}\right) + 6y\frac{dy}{dx} = 0$$

$$\frac{dy}{dx}(qx + 6y) = -3px^2 - qy$$

$$\frac{dy}{dx} = \frac{-3px^2 - qy}{qx + 6y} \quad \text{——— ②}$$

which is as required with

$a = -3$

$b = -1$

$c = 6$

At $P(-1, -4)$:

1 from ①: $p(-1)^3 + q(-1)(-4) + 3(-4)^2 = 26$

$-p + 4q + 48 = 26$

$p - 4q = -22$ ③

2 from ② and because we know gradient of the tangent is $\frac{-1}{\text{gradient}}$ of the normal:

Normal is $19x + 26y + 123 = 0$

$$26y = -19x - 123$$

$$y = \frac{-19}{26}x - \frac{123}{26}$$

So grad of the normal is $\frac{-19}{26}$

So grad of C is $\frac{26}{19}$.

So $\frac{26}{19} = \frac{-3px^2 - qy}{qx + 6y}$

$$26(qx + 6y) = -19(3px^2 + qy)$$

Subs. for $P(-1, -4)$:

$$26(-q - 24) = -19(3p - 4q)$$

$$57p - 26q - 76q = 624$$

$$57p - 102q = 624. \text{ ——— } (4)$$

(3)×57: $57p - 228q = 1254$ ——— (5)

(4) − (5): $126q = -630$

$$\underline{\underline{q = -5}}$$

in (3): $p = 22 - 20 = \underline{\underline{2}}.$

(after about 8 attempts ☹)

9) $\sum_{n=2}^{\infty} \left(\frac{3}{4}\right)^n \cos(180n)^\circ$

$= \left(\frac{3}{4}\right)^2 \underbrace{\cancel{\cos 360^\circ}}_{1} + \left(\frac{3}{4}\right)^4 \underbrace{\cancel{\cos 720^\circ}}_{1} + \left(\frac{3}{4}\right)^6 \times 1 \ldots$

$+ \left(\frac{3}{4}\right)^3 \underbrace{\cancel{\cos 540^\circ}}_{-1} + \left(\frac{3}{4}\right)^5 \underbrace{\cancel{\cos 900^\circ}}_{-1} + \left(\frac{3}{4}\right)^7 \times -1 \ldots$

So these are two geometric series to ∞,
each with $r = \left(\frac{3}{4}\right)^2 = \frac{9}{16}$

one with first term $\left(\frac{3}{4}\right)^2 = \frac{9}{16}$

the other - - - $\left(\frac{3}{4}\right)^3 = \frac{27}{64}$.

Using the standard formula for sum to ∞
this gives:

$$\frac{\frac{9}{16}}{1-\frac{9}{16}} - \frac{\frac{27}{64}}{1-\frac{9}{16}} = \frac{\frac{9}{16} - \frac{27}{64}}{\frac{7}{16}}$$

$$= \frac{16}{7}\left(\frac{9}{16} - \frac{27}{64}\right) = \frac{1}{7}\left(9 - \frac{27}{4}\right)$$

$$= \frac{1}{7}\left(\frac{36-27}{4}\right) = \frac{1}{7}\left(\frac{9}{4}\right) = \underline{\underline{\frac{9}{28}}}$$

as required.

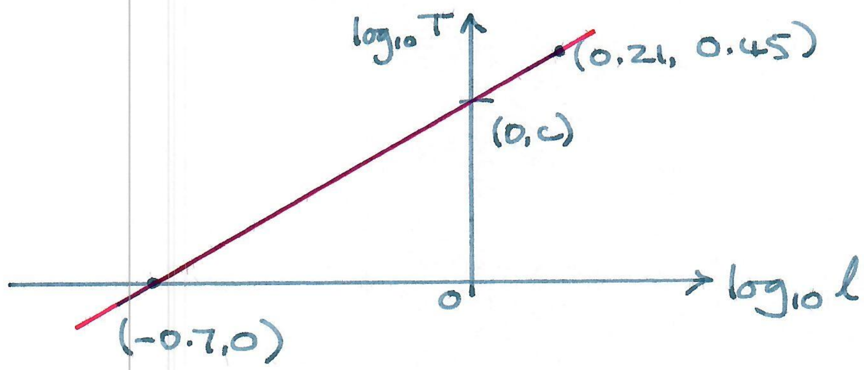
10) $T = al^b$

Taking logs to base 10:

$$\log_{10} T = \log_{10} a + \log_{10}(l^b)$$

$$= \log_{10} a + b \log_{10} l$$

$$= \underline{\underline{b \log_{10} l + \log_{10} a}} \quad \text{as required.}$$



The straight line has an equation of the form $\log_{10} T = m \log_{10} l + c$

where m is the gradient $= \dfrac{0.45 - 0}{0.21 - (-0.7)} = \dfrac{0.45}{0.91}$

$\underline{m = 0.4945}$

Subs. for the point $(-0.7, 0)$:

$$\log_{10} T = 0 = 0.4945(-0.7) + c$$

$$= -0.3462 + c$$

$\underline{c = 0.3462}$

So $\log_{10} T = 0.4945 \log_{10} l + 0.3462$

Inverting the logarithms:

$$T = l^{0.4945} \times \underbrace{10^{0.3462}}_{= 2.2190}$$

$$= 2.2190\, l^{0.4945}$$

Or to 3 sf:

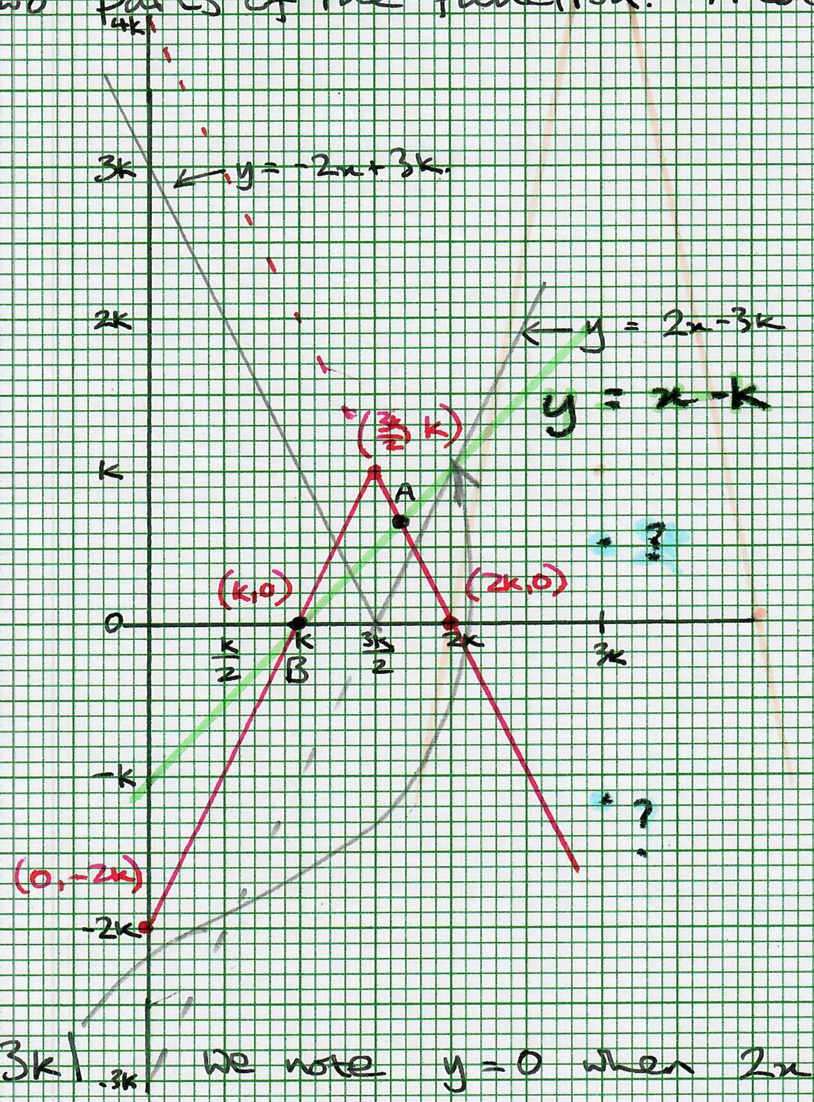
$$\underline{\underline{T = 2.22\, l^{0.495}}}$$

which is in the required form.

c) a is the time taken for the swing of a pendulum length 1m.

(I suspect there's a cleverer answer involving g - particularly if we went to a different planet.)

11) This question feels horrible, and it is. It's best answered on graph paper with a large copy of the diagram, and separate equations for the two parts of the function. And a lot of colours.



$y = |2x - 3k|$ We note $y = 0$ when $2x = 3k$,

i.e. $x = \frac{3k}{2}$.

We also note the two 'arms' have gradients ± 2, giving scope for calculating the y-intercepts as ± 3, and the equations of the two 'arms' as

$y = 2x - 3k \quad (x \geq \frac{3}{2}k)$ ①

and $y = -2x + 3k \quad (x \leq \frac{3}{2}k)$ ②

a) Now we can really get to (a).

$$f(x) = k - |2x - 3k|$$

We can do this with the individual equations for the arms, giving

$$f(x) = k - (2x - 3k) = -2x + 4k \quad (x \geqslant \tfrac{3}{2}k) \quad ③$$

$$= k - (-2x + 3k) = 2x - 2k \quad (x \leqslant \tfrac{3}{2}k). \quad ④$$

These are easy to plot once we've marked up the axes, and they give the answers required:

max value is k, at $(\tfrac{3}{2}k, k)$
x - intercepts $(k, 0)$ and $(2k, 0)$

y - intercept $(0, -2k)$

A more efficient way, though also more 'courageous', would be to say $f(x)$ is just an inversion in the line $y = \frac{k}{2}$ ... and hence...

But this would still need some point-flipping to find e.g. $(2k, 0)$.

b) You might think (a) was enough, but there's more: now the line $y = x - k$ is involved.

This cuts the existing curves in a slightly unnerving way, but we note:

a) it does cut $f(x)$
b) we can work out the intersections with equations ③ and ④.

For $(x \geq \frac{3}{2}k)$ we use:

$$-2x + 4k \geq x - k$$

$$5k \geq 3x$$

So $x \leq \frac{5}{3}k$.

(i.e left of the point A, $(\frac{5}{3}k, ?)$)

For $(x \leq \frac{3}{2}k)$ we use:

$$2x - 2k \geq x - k$$

$$x \geq k$$

(i.e. right of the point B, $(k, 0)$)

So the condition is satisfied when

$$k \leq x \leq \frac{5}{3}k \qquad x \in [k, \tfrac{5}{3}k]$$

or in set notation: $\{x \mid k \leq x \leq \frac{5}{3}k\}$

c) And then...

$$y = 3 - 5f(\tfrac{1}{2}x)$$

Consider firstly, $y = -5f(\frac{1}{2}x)$

To plot this we stretch the $\overset{f(x)}{\text{curve}}$ $\times 2$ x-wise, and stretch ~~shrink~~ (oops) it by $5\times$ and invert it y-wise.

This gives a max. value of $5k$ at $(3k, 5k)$.

So turning to $3 -$, its minimum value is clearly also when $x = 3k$, and $y = 3 - 5k$: i.e the point $(3k, 3-5k)$

Curiously a) although the question's about max and min, there's not the usual differentiation (thankfully), and (b) because we don't know k we don't know where $(3k, 3-5k)$ is: it could be very +ve, slightly +ve or anywhere −ve.

12) $I = \int_0^{16} \frac{x}{1+\sqrt{x}}\,dx$

a) Let $u = 1+\sqrt{x}$

then $\sqrt{x} = u-1$

$x = (u-1)^2$

$du = \frac{1}{2}\frac{1}{\sqrt{x}}\,dx$

$dx = 2\sqrt{x}\,du$

$= 2(u-1)\,du$

Limits are given by

$x = 16: \quad u = 1+\sqrt{16} = 5$

$x = 0: \quad u = 1+\sqrt{0} = 1$

Putting this together:

$$I = \int_1^5 \frac{(u-1)^2}{u}\,2(u-1)\,du$$

$$= \int_1^5 \frac{2(u-1)^3}{u}\,du \quad \text{as required.}$$

with $p=1$ and $q=5$.

b) $I = 2\int_1^5 \frac{(u^3-3u^2+3u-1)}{u}\,du$

$$= 2\int_1^5 \left(u^2-3u+3-\frac{1}{u}\right)du$$

$$= 2\left[\frac{1}{3}u^3 - \frac{3}{2}u^2 + 3u - \ln u\right]_1^5$$

$$= 2\left[\frac{125}{3} - \frac{75}{2} + 15 - \ln 5 - \frac{1}{3} + \frac{3}{2} - 3 + \cancel{\ln 1}^{\,0}\right]$$

$$= 2\left[\frac{124}{3} - \frac{72}{2} + 12 - \ln 5\right]$$

$$= 2\left[\frac{248 - 216 + 72}{6} - \ln 5\right]$$

$$= \frac{104}{3} - 2\ln 5$$

as required, with $A = \frac{104}{3}$

and $B = 2$

13) $x = \sin 2\theta$

$0 < \theta < \frac{\pi}{2}$

$y = \operatorname{cosec}^3\theta = \dfrac{1}{\sin^3\theta}$

a) $\dfrac{dx}{d\theta} = 2\cos 2\theta \quad \left(= 2(\cos^2\theta - \sin^2\theta) \right)$

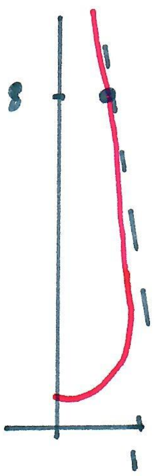
$\dfrac{dy}{d\theta} = 3\operatorname{cosec}^2\theta \dfrac{d}{d\theta}(\operatorname{cosec}\theta)$

$= 3\operatorname{cosec}^2\theta\left(-\operatorname{cosec}\theta \cot\theta\right)$

$= \dfrac{-3}{\sin^3\theta}\dfrac{\cos\theta}{\sin\theta} = -\dfrac{3\cos\theta}{\sin^4\theta}.$

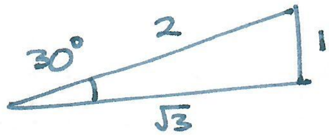
So $\dfrac{dy}{dx} = \dfrac{dy}{d\theta} \Big/ \dfrac{dx}{d\theta} = -\dfrac{3\cos\theta}{2\sin^4\theta\cos 2\theta}$

or $\dfrac{-3\cos\theta}{2\sin^4\theta(\cos^2\theta - \sin^2\theta)}$



b) when $y = 8$. $\operatorname{cosec}^3\theta = 8 = \dfrac{1}{\sin^3\theta}$

So $\sin\theta = \dfrac{1}{2}$



so $\theta = 30° = \dfrac{\pi}{6}.$

So the gradient of C is $-\dfrac{3}{2}\dfrac{\cos\frac{\pi}{6}}{\sin^4\frac{\pi}{6}\cos\frac{\pi}{3}}$

$= -\dfrac{3}{2}\dfrac{\sqrt{3}}{2} \Big/ \left(\dfrac{1}{2}\right)^4\left(\dfrac{1}{2}\right) = \dfrac{-3\sqrt{3}}{4}.32 = \underline{\underline{-24\sqrt{3}.}}$

14) a) $V = 24h$ $\qquad \frac{dV}{dt} = 24\frac{dh}{dt}$ —— ①

$\frac{dV}{dt} = 0.48 - 0.1h$ —— ②

① in ②: $24\frac{dh}{dt} = 0.48 - 0.1h$

$2400\frac{dh}{dt} = 48 - 10h$

$1200\frac{dh}{dt} = 24 - 5h$ as required.

b) At $t=0$, $h=2$.

We can solve the equation with an integrating factor, or more simply with separation of variables:

Let $24-5h = u$. Then $-5\frac{dh}{dt} = \frac{du}{dt}$.

$\frac{dh}{dt} = -\frac{1}{5}\frac{du}{dt}$

So $1200\left(-\frac{1}{5}\frac{du}{dt}\right) = u$

$-240\frac{du}{dt} = u$

$\frac{du}{u} = \frac{1}{-240}dt$

$\ln u = \frac{1}{-240}t + c$ (constant c)

or $u = Ae^{\frac{1}{-240}t}$ (constant A)

$$24 - 5h = Ae^{-\frac{1}{240}t}$$

$$5h = 24 - Ae^{-\frac{1}{240}t}$$

$$h = 4.8 - Be^{-\frac{1}{240}t} \text{ (constant } B\text{)}$$

Using the initial value:

$$2 = 4.8 - Be^{0}$$
$$= 4.8 - B$$

So $B = 2.8$

and $h = 4.8 - 2.8e^{-\frac{1}{240}t}$

which is as required with
$A = 4.8$
$B = -2.8$
$k = 240.$

c) The $-2.8e^{-\frac{1}{240}t}$ term is always positive so the level of the water will never exceed 4.8: it'll never fill the tank.

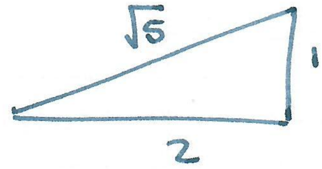
15) a) Use the identity

$$R\cos(A+B) = R\cos A\cos B - R\sin A\sin B$$

$$\theta+\alpha \qquad (2)\cos\theta - (+1)\sin\theta$$

So $R\cos\alpha = 2$

$R\sin\alpha = 1.$



$\tan\alpha = \frac{1}{2}$

$\sin\alpha = \frac{1}{\sqrt{5}}$

$\cos\alpha = \frac{2}{\sqrt{5}}$

$$R = \frac{1}{\sin\alpha} = \frac{1}{1/\sqrt{5}} = \sqrt{5}.$$

So $2\cos\theta - \sin\theta = \sqrt{5}\cos(\theta+\alpha)$

where $\alpha = \tan^{-1}\left(\frac{1}{2}\right) = 26.57^\circ$

$= \underline{\underline{0.464}}$ radians

b) $H = 3 + 4\cos(0.5t) - 2\sin(0.5t)$

using (a):

$H = 3 + 2\sqrt{5}\cos(0.5t + 0.464)$

i) The cos term has max. value 1, so the max. value of H is $3 + 2\sqrt{5}$

ii) At this time, $\cos(0.5t + 0.464) = 1$

$0.5t + 0.464 = 2\pi$: $t = 2(2\pi - 0.464)$

(can't be 0) $t = 11.6$ m (to 1 dp)

c) Without a clear figure for the water height this looks impossible, but we note it's when $H = 0$.

So we want
$$H = 3 + 4\cos(0.5t) - 2\sin(0.5t) = 0$$

then it goes negative: then it comes back through 0.

So $3 + 2\sqrt{5}\cos(0.5t + 0.464) = 0$

$$2\sqrt{5}\cos(0.5t + 0.464) = -3$$

$$\cos(0.5t + 0.464) = \frac{-3}{2\sqrt{5}} = -0.6708$$

$$0.5t + 0.464 = 132.130^\circ$$

$$= 2.306$$

$$0.5t = 2.306 - 0.464$$

$$= 1.842$$

$$t = 3.684 \text{ m}$$

This is when P goes <u>under</u> the water. P comes back up when $\cos(\sim) = -0.6708$ for the 2nd time,

i.e $2\pi - 2.306 = 3.977$

At this point, $0.5t = 3.977 - 0.464$: $t = 2 \times 3.513$ m
$= 7.026$ m

So the total time T under water $= 7.026 - 3.684$
$= 3.[illegible]$ m (3 s.f.)