1) $f(x) = ax^3 + 10x^2 - 3ax - 4$

Since $(x-1)$ is a factor, $f(1) = 0$:

$$a + 10 - 3a - 4 = 0$$

$$6 = 2a$$

$$\underline{\underline{a = 3.}}$$

2) a) $f(x) = x^2 - 4x + 5$

$= (x-2)^2 + 1$ (completing the square)

$(\quad = x^2 - 4x + 4 + 1 \checkmark)$

b) i) At the y-axis $x = 0$:

$y = f(0) = (-2)^2 + 1 = 5$

So $\underline{\underline{P \text{ is } (0,5)}}$

ii) For a turning point,

$$\frac{df}{dx} = 0 = 2x - 4$$

$$2x = 4$$

$$x = 2$$

when $x = 2$, $y = f(2) = 0^2 + 1 = 1$

So $\underline{\underline{Q \text{ is } (2,1)}}$

3) a) $u_{n+1} = k - \frac{24}{u_n}$

$u_1 = 2$

$u_1 + 2u_2 + u_3 = 0$ ——— ①

we calculate : $u_2 = k - \frac{24}{u_1} = k - \frac{24}{2} = k-12$

$u_3 = k - \frac{24}{u_2} = k - \frac{24}{k-12}$

using ①: $2 + 2(k-12) + k - \frac{24}{k-12} = 0$

$2(k-12) + 2(k-12)^2 + k(k-12) - 24 = 0$

$2k - 24 + 2k^2 - 48k + 288 + k^2 - 12k - 24 = 0$

$\underline{\underline{3k^2 - 58k + 240 = 0}}$ as required.

b) Factorising:

$(3k-40)(k-6) = 0$

So $k = \frac{40}{3}$ or 6.

$k \in \mathbb{N}$ so we discard $\frac{40}{3}$

$\underline{\underline{k = 6}}$.

c) $u_3 = 6 - \frac{24}{6-12} = 6 + \frac{24}{6} = \underline{\underline{10}}$

4) $y = f(x) = x^2 + \ln(2x^2 - 4x + 5)$

$$\frac{dy}{dx} = 2x + \frac{1}{2x^2 - 4x + 5}(4x - 4)$$

$$= 2x + \frac{4(x-4)}{2x^2 - 4x + 5}$$

$$= \frac{2x(2x^2 - 4x + 5) + 4(x-4)}{2x^2 - 4x + 5}$$

$$= \frac{4x^3 - 8x^2 + 10x + 4x - 16}{2x^2 - 4x + 5}$$

$$= \frac{4x^3 - 8x^2 + 14x - 16}{2x^2 - 4x + 5}$$

At the turning point $\frac{dy}{dx} = 0$, so the numerator must be 0:

So $4\alpha^3 - 8\alpha^2 + 14\alpha - 16 = 0$

$2\alpha^3 - 4\alpha^2 + 7\alpha - 8 = 0$ as required.

(α is a solution of $2x^3 - 4x^2 + 7x - 8 = 0$)

b) This would be much easier with a spreadsheet...

$x_1 = 0.3$

$x_2 = \frac{1}{7}\left(2 + 4 \times 0.3^2 - 2 \times 0.3^3\right) = 0.3294$

$x_3 = 0.3375$

$x_4 = 0.3398$

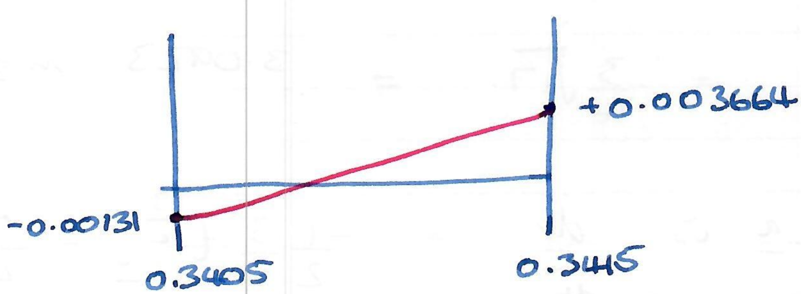
c) What this actually wants us to say is:

Picking up on the '3 dp' point,

consider the interval $[0.3405, 0.3415]$.

On this interval the derivative equation

$2x^3 - 4x^2 + 7x - 2$ changes sign:



And since the functions are continuous we can say the root falls between the values, probably at a ratio of $\frac{131}{131+366}$ along

$$= 0.3405 + \frac{131}{497} \times 0.001 = 0.34076$$

or 0.341 to 3dp.

as required.

5) $u_1 = 20000$

$u_{n+1} = u_n \times (1.08)^n$ $(r = 1.08)$

a) $u_2 = 20000 \times 1.08 = £21600$

$u_3 = 21600 \times 1.08 = £23328$

(or $u_3 = 20000 \times (1.08)^2$)

b) we need to know n' so that

$20000\, r^{n'} = 65000$

$(1.08)^{n'} = \frac{65}{20} = 3.25$

$n' \log 1.08 = \log 3.25$

$n' \times 0.0334 = 0.5119$

$n' = 15.326$

This tells us profit in year $n' = 16$ is $> £65000$

but u_n has a factor of $(1.08)^{n-1}$,

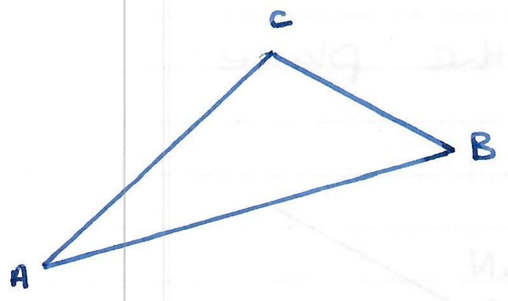
so it's actually u_{17} (17 years) where the profit exceeds £65000.

c) Using the standard formula,

$S_{20} = \frac{20000(1 - 1.08^{20})}{1 - 1.08} = £915239$

or £915000 to the nearest £1000.

6)



$$\overrightarrow{AB} = -3\underline{i} - 4\underline{j} - 5\underline{k}$$

$$\overrightarrow{BC} = \underline{i} + \underline{j} + 4\underline{k}$$

a) $\overrightarrow{AC} = \overrightarrow{AB} + \overrightarrow{BC}$

$$= (-3+1)\underline{i} + (-4+1)\underline{j} + (-5+4)\underline{k}$$

$$= -2\underline{i} - 3\underline{j} - \underline{k}$$

b) $|AC| = \sqrt{2^2+3^2+1^2} = \sqrt{4+9+1} = \sqrt{14}$

$$|BC| = \sqrt{1^2+1^2+4^2} = \sqrt{1+1+16} = \sqrt{18} = 3\sqrt{2}$$

$$|BA| = \sqrt{3^2+4^2+5^2} = \sqrt{9+16+25} = \sqrt{50} = 5\sqrt{2}$$

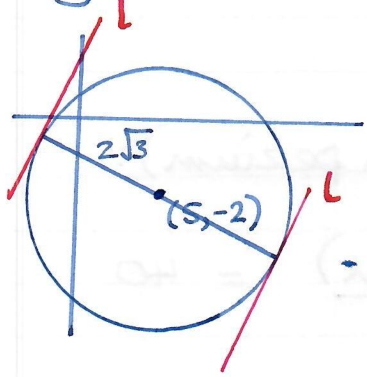
Using the cosine rule,

$$\cos ABC = \frac{|AB|^2 + |BC|^2 - |AC|^2}{2\,|AB||BC|}$$

$$= \frac{50+18-14}{2 \times 5\sqrt{2} \times 3\sqrt{2}} = \frac{\cancel{54}^{\cancel{18}\ 9}}{2\ \cancel{4} \times 5 \times \cancel{3}} = \frac{9}{10}$$

as required.

7) C: $x^2+y^2-10x+4y+11=0$



a) Completing both x and y squares:

$$(x-5)^2+(y+2)^2+11-25-4=0$$

$$(x-5)^2+(y+2)^2=18=(2\sqrt{3})^2$$

So i) centre is $\underline{\underline{(5,-2)}}$

ii) radius is $\underline{\underline{2\sqrt{3}}}$.

b) l: $y=3x+k$

$\exists$ 3 ways to do this:
1) geometry
2) Find $\frac{dy}{dx}$
3) Subs. into the eq^n for C.

(3) Leads to the most useful eq^n:

$$x^2+(3x+k)^2-10x+4(3x+k)+11=0$$

$$x^2+9x^2+6xk+k^2-10x+12x+4k+11=0$$

$$10x^2+(6k+2)x+(4k+11+k^2)=0$$

For l to be a tangent there must be only one solution for x, so "b^2-4ac" $=0$:

$$(6k+2)^2 - 4\times10\times(k^2+4k+11) = 0$$

$$36k^2 + 24k + 4 - 40k^2 - 160k - 440 = 0$$

$$-4k^2 - 136k - 436 = 0$$

$$4k^2 + 136k + 436 = 0$$

$$k^2 + 34k + 109 = 0$$

So $k = \dfrac{-34 \pm \sqrt{34^2 - 4\times109}}{2}$

$$= -17 \pm \sqrt{17^2 - 109}$$

$$= -17 \pm \sqrt{289 - 109}$$

$$= -17 \pm \sqrt{180}$$

$$= -17 \pm 6\sqrt{5}$$

8) a) $N = Ae^{kt}$

At $t=0$, $N = 1000 = Ae^{0} = A$ so $A = 1000$

At $t=5$, $N = 2000 = 1000e^{5k}$

$$e^{5k} = 2$$

$$5k = \ln 2$$

$$k = \frac{1}{5}\ln 2 = 0.139.$$ So $\underline{\underline{N = 1000e^{0.139t}}}$

b) $\frac{dN}{dt} = 1000 \times 0.139e^{0.139t}$

$= 139e^{0.139t}$

So when $t = 8$ the rate of increase is

$139 \times e^{0.139 \times 8} = 422.6$ bacteria/h

or $\underline{\underline{420}}$ to 2 sf.

c) We want to solve

$$500e^{1.4 \times 0.139T} = 1000e^{0.139T}$$

Divide B by $500e^{0.139T}$:

$$e^{0.4 \times 0.139T} = 2$$

$$0.4 \times 0.139T = \ln 2$$

$$T = 12.467$$

or $\underline{\underline{12.5 \text{ hours}}}$.

9.) $$f(x) = \frac{50x^2 + 38x + 9}{(5x+2)^2(1-2x)}$$

$$= \frac{A}{5x+2} + \frac{B}{(5x+2)^2} + \frac{C}{1-2x}$$

Multiplying this out...

$$= \frac{A(5x+2)(1-2x) + B(1-2x) + C(5x+2)^2}{(\quad)^2(\quad)}$$

$$= \frac{A(-10x^2 - 8x + 2) + B(1-2x) + C(25x^2 + 20x + 4)}{(\quad)^2(\quad)}$$

$$= \frac{x^2(-10A + 25C) + x(-8A - 2B + 20C) + 1(2A + B + 4C)}{(\quad)^2(\quad)}$$

Equating coefficients:

x^2: $-10A + 25C = 50$ —— ①

x: $-8A - 2B + 20C = 38$ —— ②

1: $2A + B + 4C = 9$ —— ③

③ × 2: $4A + 2B + 8C = 18$ —— ④

② + ④: $-4A + 28C = 56$ —— ⑤

~~③~~ + ⑤ × $\frac{5}{2}$: $-10A + 70C = 140$ —— ⑥

$45C = 90 \Rightarrow C = 2.$

In ①: $-10A + 50 = 50 \Rightarrow A = 0$

In ③: $-0 + B + 8 = C \Rightarrow B = 1.$

So $A = 0$ (answering ii)

$B = 1$ (answering i)

$C = 2$

b) we could presumably bash on with the original $f(x)$, but using the expanded form looks simpler:

$$f(x) = \frac{1}{(5x+2)^2} + \frac{2}{1-2x}.$$

Using the Binomial Expansion:

$$(5x+2)^{-2} = 2^{-2}\left(1 + \frac{5}{2}x\right)^{-2}$$

$$= \frac{1}{4}\left\{1 + (-2)\left(\frac{5}{2}x\right) + \frac{(-2)(-3)}{2}\left(\frac{5}{2}x\right)^2 + o(x^3)\right\}$$

$$= \frac{1}{4}\left\{1 - 5x + \frac{75}{4}x^2 + \dots\right\}$$

$$= \frac{1}{4} - \frac{5}{4}x + \frac{75}{16}x^2 \dots \quad ①$$

And $\frac{2}{1-2x} = 2\left\{1 + (-1)(-2x) + \frac{(-1)(-2)(-2x)^2}{2} + o(x^3)\dots\right\}$

$= 2(1-2x)^{-1}$

$$= 2\{1 + 2x + 4x^2 + \dots\}$$

$$= 2 + 4x + 8x^2 + \dots \quad ②$$

①+② gives $f(x) = \frac{9}{4} + \frac{11}{4}x + \frac{203}{16}x^2 + \dots$

and this is valid when

a) $\left|\frac{5x}{2}\right| < 1$ i.e. $|x| < \frac{2}{5}$

b) $|2x| < 1$ i.e. $|x| < \frac{1}{2}$

Take the lower bound - so $|x| < \frac{2}{5}$.

10) a) use the identities

$$\cos 2\theta = \cos^2\theta - \sin^2\theta$$

$$\sin 2\theta = 2\cos\theta \sin\theta$$

$$\cos^2\theta + \sin^2\theta = 1.$$

Then $$\frac{1 - \cos 2\theta + \sin 2\theta}{1 + \cos 2\theta + \sin 2\theta}$$

$$= \frac{1 - \cos^2\theta + \sin^2\theta + 2\sin\theta\cos\theta}{1 + \cos^2\theta - \sin^2\theta + 2\sin\theta\cos\theta}$$

$$= \frac{1 - (1 - \sin^2\theta) + \sin^2\theta + 2\sin\theta\cos\theta}{1 + \cos^2\theta - (1 - \cos^2\theta) + 2\sin\theta\cos\theta}$$

$$= \frac{2\sin^2\theta + 2\sin\theta\cos\theta}{2\cos^2\theta + 2\sin\theta\cos\theta}$$

$$= \frac{\cancel{(2\sin\theta + 2\cos\theta)}}{\cancel{(2\cos\theta + 2\sin\theta)}}\frac{\sin\theta}{\cos\theta} = \tan\theta$$

as required.

b) using (a) – and noting this is about $4x$, not $2x$ –

$$\frac{1-\cos 4x+\sin 4x}{1+\cos 4x+\sin 4x} = \tan 2x = 3\sin 2x$$

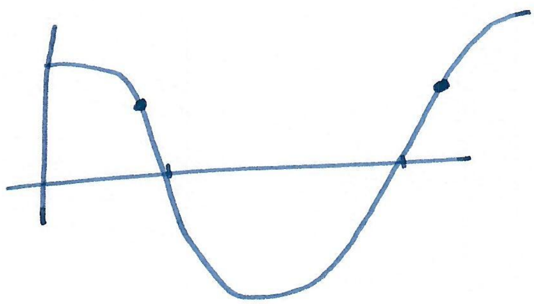
So $\dfrac{\sin 2x}{\cos 2x} = 3\sin 2x$

Discounting $\sin 2x = 0$ ~~since $0<x<180°$,~~ for a moment,

this gives $\dfrac{1}{\cos 2x} = 3$

$$\cos 2x = \frac{1}{3}$$

So $2x = 70.529°$ or $289.471°$

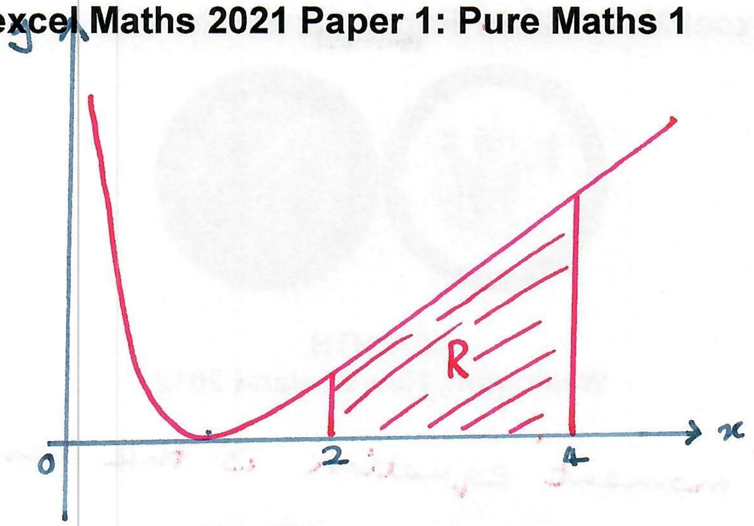


So $x = \underline{\underline{35.3°}}$ or $\underline{\underline{144.7°}}$ (1 dp)

And $\sin 2x = 0$ also gives a solution

$\underline{\underline{x = 90°.}}$

11)



$y = (\ln x)^2$

a) Using the trapezium rule:

$$R = \frac{0.5}{2}\left\{(0.4805 + 1.9218) + 2(0.8396 + 1.2069 + 1.5694)\right\}$$

$$= 2.4085 \text{ or } \underline{\underline{2.41 \text{ to } 3\text{sf}}}.$$

b) $$R = \int_2^4 (\ln x)^2 \, dx$$

(see next sheet)

Try integration by parts:

$$\frac{d}{dx}\left[x(\ln x)^2\right] = 2x\ln x\frac{1}{x} + (\ln x)^2$$

$$= 2\ln x + (\ln x)^2$$

So $\int(\ln x)^2 dx = x(\ln x)^2 - \int 2\ln x\, dx$

And $\frac{d}{dx}\int(x\ln x) = \frac{x}{x} + \ln x$

$$= 1 + \ln x$$

So $\int \ln x = x\ln x - \int 1\, dx$

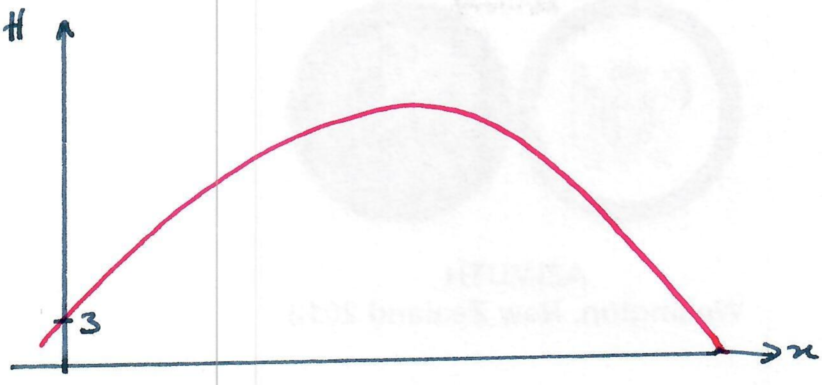
So $R = \left[x(\ln x)^2 - 2(x\ln x - x)\right]_2^4$

$$= 4(\ln 4)^2 - 2\times 4\ln 4 + 8$$
$$- 2(\ln 2)^2 + 4\ln 2 - 4$$

$$= 16(\ln 2)^2 - 16\ln 2 + 4$$
$$-2(\ln 2)^2 - 4\ln 2$$

$$= 14(\ln 2)^2 - 12\ln 2 + 4.$$

12)



(Interestingly everything here is done in terms of x, not t.)

$H(0) = 3$ (m)

max H is when $x = 90$ $\left(\frac{dH}{dx} = 0\right)$

$H(120) = 27$ (m)

a) It's 'obvious' to use a model based on $(x-90)$, but that didn't go well, so ...

Model H as $H(x) = ax^2 + bx + c$

Then $\frac{dH}{dx} = 2ax + b.$

So from the given conditions:

At $x = 0$: $3 = c$

At $x = 90$: $0 = 180a + b$ —— ①

At $x = 120$: $27 = 120^2 a + 120b + c$

$= 14400a + 120b + 3$

$24 = 14400a + 120b$

$1 = 600a + 5b$ —— ②

① × 5: $0 = 900a + 5b$ ——— ③

③ − ②: $300a = -1$

$$a = \frac{-1}{300}$$

So $b - \frac{180}{300} = 0$: $b = \frac{180}{300} = \frac{3}{5}$

So the model is $H(x) = \frac{-x^2}{300} + \frac{3}{5}x + 3$

b) i) when $x = 90$,

$$H = \frac{-90^2}{300} + \frac{3}{5}90 + 3$$

$$= -27 + 54 + 3 = \underline{\underline{30\text{m}}}.$$

ii) when $H = 0$,

$$\frac{-x^2}{300} + \frac{3}{5}x + 3 = 0$$

$$x^2 - 180x - 900 = 0$$

$$x = \frac{180 \pm \sqrt{32400 + 3600}}{2}$$

$$= 90 \pm 94.87$$

$= -4.87\text{m}$ (discount since <0)

or $\underline{\underline{184.87\text{m}}}$

c) Spin, air-resistance, gravitational pull of the moon, etc.

13) $x = \dfrac{t^2+5}{t^2+1}$ $\qquad x - 3 = \dfrac{t^2+5}{t^2+1} - \dfrac{3(t^2+1)}{t^2+1}$

$$= \frac{t^2 - 3t^2 + 5 - 3}{t^2+1}$$

$$= \frac{-2t^2+2}{t^2+1}$$

$$= \frac{2-2t^2}{t^2+1}$$

$y = \dfrac{4t}{t^2+1}$

So $(x-3)^2 + y^2 = \dfrac{(2-2t^2) + (4t)^2}{(t^2+1)^2}$

$$= \frac{4 - 8t^2 + 4t^4 + 16t^2}{(t^2+1)^2}$$

$$= \frac{4t^4 + 8t^2 + 4}{(t^2+1)^2}$$

$$= \cancel{\frac{4(t^4+2t+1)}{(t^2+1)^2}}$$

$= \underline{4}$ as required.

So all points on C satisfy $\underline{\underline{(x-3)^2 + y^2 = 4}}$.

14) $y = \dfrac{x-4}{2+\sqrt{x}}$

Using the quotient rule,

$$\frac{dy}{dx} = \frac{(2+\sqrt{x})(1) - (x-4)\left(\frac{1}{2\sqrt{x}}\right)}{(2+\sqrt{x})^2}$$

$$= \frac{2+\sqrt{x} - \frac{x}{2\sqrt{x}} + \frac{4}{2\sqrt{x}}}{(2+\sqrt{x})^2}$$

$$= \frac{\;\frac{4\sqrt{x} + 2x - x + 4}{2\sqrt{x}}\;}{4 + 4\sqrt{x} + x}$$

$$= \frac{\;\frac{\cancel{4\sqrt{x} + x + 4}}{2\sqrt{x}}\;}{\cancel{4 + 4\sqrt{x} + x}}$$

$$= \frac{1}{2\sqrt{x}}$$

As required, with $A = 2$.

15) i) I'm exhausted...

There are (only) 4 cases to consider:

n	$(n+1)^3$	3^n	is $(n+1)^3 > 3^n$?
~~0~~	~~1~~	~~1~~	~~no~~
1	8	3	yes
2	27	9	yes
3	64	27	yes
4	125	81	yes.

Assuming $0 \notin \mathbb{N}$ (debatable point) then the required result is demonstrated by exhaustion for all n.

ii) $m^3 + 5$ is odd: $m^3+5 = 2q+1$ for some integer q.

Suppose m is ~~even~~ odd: $m = 2p+1$ for some integer p.
pay attention!

$$\cancel{\text{So } (2p)^3 + 5 = 2q+1}$$

$$\cancel{8p^3 + 8 = 2q+1}$$

$$\cancel{8p^3 + 4 = 2q}$$

$$\cancel{4p^3 = q - 2}$$

Then $(2p+1)^3 + 5 = 2q+1$

$$8p^3 + 6p^2 + 6p + 1 + 5 = 2q+1$$

$$2(4p^3 + 3p^2 + 3p + 3) = 2q+1$$

LHS has a factor of 2: RHS doesn't – this is a contradiction.

So m cannot be odd: m can only be even

(and indeed we could prove m^3+5 is odd.)