1) a) £ $(1+8x)^{\frac{1}{2}} = 1 + \frac{1}{2}(8x) + \frac{\frac{1}{2}(-\frac{1}{2})(8x)^2}{2} + \frac{\frac{1}{2}(-\frac{1}{2})(-\frac{3}{2})(8x)^3}{6}$

$+o(x^4)$

$= 1 + 4x - 8x^2 + \frac{\cancel{3}}{\cancel{8}\times\cancel{6}} \overset{32}{\cancel{64}} \times \cancel{8}x^3 + o(x^4)$

$= 1 + 4x - 8x^2 + 32x^3 + o(x^4)$

b) using $x = \frac{1}{32}$ would give $(1+\frac{8}{32})^{\frac{1}{2}} = (1+\frac{1}{4})^{\frac{1}{2}}$

$= \sqrt{\frac{5}{4}} = \frac{1}{2}\sqrt{5}.$

So the approximation would be found by using the expansion then multiplying by 2.

(The answer is 2.2363, which squares to 5.0012)

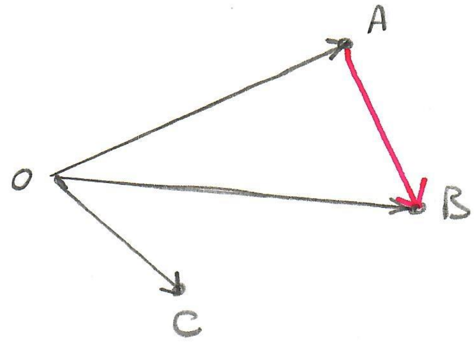
2) $4^{3p-1} = 5^{210}$

$(3p-1)\log 4 = 210 \log 5$

$3p-1 = 210 \frac{\log 5}{\log 4} = 210 \times 1.1610 = 243.80$

$3p = 244.80$: $\underline{\underline{p = 81.6}}$

3) 'Conceptually' ...



$\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA} = (3-2)\underline{i} + (-3-5)\underline{j} + (-4+6)\underline{k}$

$= \underline{i} - 8\underline{j} + 2\underline{k}$

we observe $\overrightarrow{OC} = 2\underline{i} - 16\underline{j} + 4\underline{k}$ which is $2 \times \overrightarrow{AB}$

This means $\overrightarrow{OC}$ and $\overrightarrow{AB}$ are parallel (the same 'gradient') so OABC is a trapezium which OC and AB the parallel sides.

4) a) $f(x) = \dfrac{3x-7}{x-2}$

Set $7 = \dfrac{3x-7}{x-2}$:

$7x - 14 = 3x - 7$

$4x = 7$

$x = \dfrac{7}{4}$

So $f^{-1}(7) = \underline{\underline{\dfrac{7}{4}}}$

b) $$ff(x) = \frac{3\left(\frac{3x-7}{x-2}\right) - 7}{\left(\frac{3x-7}{x-2}\right) - 2}$$

$$= \frac{\frac{3(3x-7) - 7(x-2)}{\cancel{x-2}}}{\frac{3x-7 - 2(x-2)}{\cancel{x-2}}}$$

$$= \frac{9x-21-7x+14}{3x-7-2x+4}$$

$$= \underline{\underline{\frac{2x-7}{x-3}}}$$

5) Label the relevant fastest speeds $V_1 \cdots V_6$.

a) Arithmetic sequence:

$V_1 = 28$

$V_n = V_1 + (n-1)d$

$V_6 = V_1 + 5d \quad = 28 + 5d \quad = 115$ (given)

So $5d = 115 - 28 = 87$

$d = \frac{87}{5} = 17.4$

So $V_3 = 28 + 2 \times 17.4 \quad = 28 + 34.8 = \underline{\underline{62.8\text{km/h}}}$

b) Geometric sequence:

$V_1 = 28$

$V_n = 28r^{n-1}$

$V_6 = 28r^5 = 115$ (given)

So $r^5 = \dfrac{115}{28} = 4.1071$

$r = 1.3265$

So $V_5 = 28 \times 1.3265^4$

$= 28 \times 3.0962$

$= \underline{\underline{86.6915}}$ km/h.

6) a) Treat $\sin x + 2\cos x$ as

$R(\sin x \cos \alpha + \cos x \sin \alpha)$

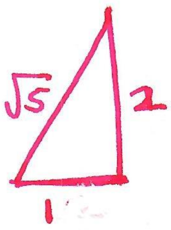
where $R\cos\alpha = 1$ ——— ①

and $R\sin\alpha = 2$ ——— ②

(we can then use the identity

$R\sin(x+\alpha) = R(\sin x \cos\alpha + \cos x \sin\alpha)$

② ÷ ①: $\tan\alpha = 2$



$\alpha = 63.435^\circ$
$= 1.107$ radians.

$\sin\alpha = \dfrac{2}{\sqrt{5}}$

$\dfrac{R \times 2}{\sqrt{5}} = 2$ $\quad$ $R = \sqrt{5}$

So $\sin x + 2\cos x = \sqrt{5}\sin(x + 1.107)$

b) using (a),

$\theta = 5 + \sqrt{5}\sin\left(\dfrac{\pi t}{12} - 3 + 1.107\right)$

$= 5 + \sqrt{5}\sin\left(\dfrac{\pi t}{12} - 1.893\right)$

Since the sin term has a max. of 1
(and this does occur during the day - specifically when $t = \dfrac{1.893 \times 12}{\pi} + 6$

$= 13.231$ h)

then the max. value of θ is $5 + \sqrt{5}$

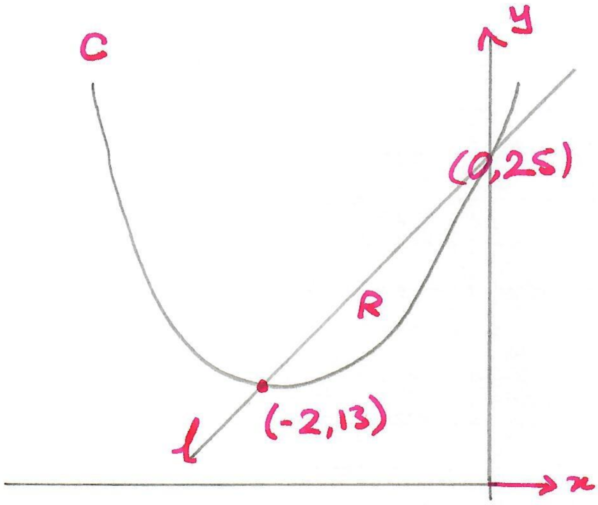
$= 7.236\,^\circ C$

c) We already found this as 13.231 h

or 13h 14m

i.e. $\underline{\underline{13:14}}$

7)



Since we know $f(x)$ is a quadratic with a minimum at $(-2, 13)$ we know it is a parabola of the form

$$y = f(x) = a(x+2)^2 + 13$$

and since we know it passes through $(0, 25)$:

$$a(2)^2 + 13 = 25$$

$$4a = 12$$

$$\underline{\underline{a = 3.}}$$

So $f(x) = 3(x+2)^2 + 13$

And for the line l we know

$$\text{gradient} = \frac{25-13}{0-(-2)} = \frac{12}{2} = 6$$

So its equation is $y = 6x + c$ for some c

and at the point $(0,25)$:

$$25 = 0 + c \quad \text{so } c = 25.$$

So the bounding curves for R are:

for $-2 \leqslant x \leqslant 0$,

l: $y = 6x + 25$

C: $y = 3(x+2)^2 + 13$

And so a point (x, y) is in R when

$$3(x+2)^2 + 13 \leqslant 6x + 25$$

for $-2 \leqslant x \leqslant 0$

or $3(x+2)^2 + 13 \leqslant y \leqslant 6x + 25$.

8) The observation indicates

$$\frac{dn}{dt} = kn \quad \text{for some } n.$$

Separating variables,

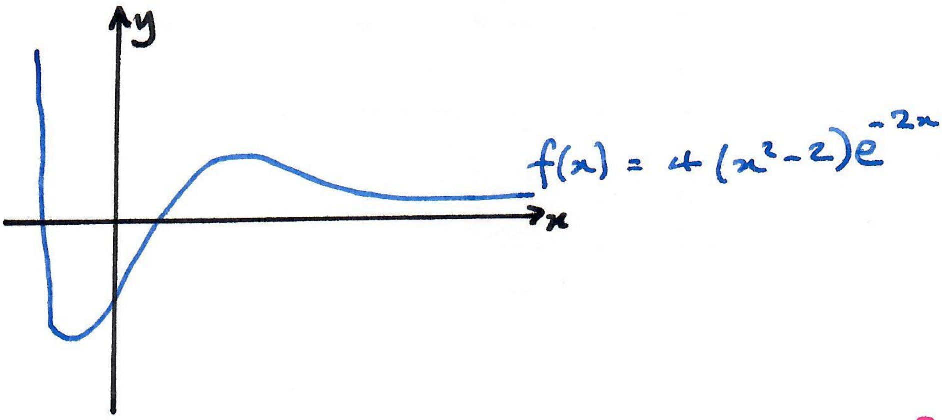
$$\frac{dn}{n} = k\,dt$$

So (integrating):

$$\ln n = kt + c \quad \text{for some } c$$

$$\text{or } \underline{\underline{n = Ae^{kt}}} \quad (A = e^c)$$

9)



a) $f'(x) = 4(x^2-2)(-2)e^{-2x} + 4(2x)e^{-2x}$ (product rule)

$$= -8x^2e^{-2x} + 16e^{-2x} + 8xe^{-2x}$$

$$= \underline{\underline{8(2+x-x^2)e^{-2x}}} \text{ as required.}$$

b) Stationary points are given by

$f'(x) = 0 = 8(2+x-x^2)e^{-2x}$

Since e^{-2x} is never 0, we look for

$2+x-x^2 = 0$

$(2-x)(1+x) = 0$

So $x = -1$ or 2.

At $x=-1$, $y = 4(1-2)e^{+2} = -4e^2$

So the point is $(-1, -4e^2)$

At $x=2$, $y = 4(4-2)e^{-2\times 2} = 8e^{-4}$

So the point is $(2, 8e^{-4})$

c) 'Range' is the extent of possible values of a function, so

i) range of $2f(x) = [2(-4e^2), \infty)$

$= [-8e^2, \infty)$

ii) Here $x \geqslant 0$, so the minimum point when $x=-1$ isn't in play, and the min of $f(x)$ is $f(0) = 4(0-2)e^0$

$= -8$.

Also the max. of $f(x)$ when $x \geq 0$ is (what we've already found) $8e^{-4}$

So the range of $h(x) = 2f(x) - 3$ is:

$$-8 \times 2 - 3 = -19$$

to $16e^{-4} - 3$

i.e. $[-19, 16e^{-4} - 3]$ (allowing [ ] at the ends).

10) $$I = \int_5^{10} \frac{3\,dx}{(x-1)(3+2\sqrt{x-1})}$$

Let $x = u^2 + 1$ then $x - 1 = u^2$

$\sqrt{x-1} = u$

and $dx = 2u\,du$

So $$I = \int_p^q \frac{3 \times 2u\,du}{u^2(3+2u)} = \int_p^q \frac{6u\,du}{u^2(3+2u)}$$

where $p = \sqrt{5-1} = 2$

$q = \sqrt{10-1} = 3.$

$$I = \int_2^3 \frac{6}{u(3+2u)}\,du$$

as required.

b) $$I = \int_2^3 \frac{6\,du}{u(3+2u)} = 6\int_2^3 \frac{1}{u(3+2u)}\,du$$

Separate the fraction:

$$\frac{1}{u(3+2u)} = \frac{A}{u} + \frac{B}{3+2u} = \frac{A(3+2u)+Bu}{u(3+2u)}$$

Equating coeffs:

$1{:}\quad 1 = 3A \qquad A = \frac{1}{3}$

$u{:}\quad 0 = 2A + B \qquad B = \frac{-2}{3}$

$$\text{So } I = \frac{6}{3}\int_2^3 \left(\frac{1}{u} - \frac{2}{3+2u}\right) du$$

$$= 2\int_2^3 \frac{1}{u}\,du = 2\ln u\Big|_2^3 \quad \text{——— (1)}$$

$$-4\int_2^3 \frac{1}{3+2u}\,du$$

Let $3+2u = t$: $2du = dt$, $du = \frac{dt}{2}$.

$$-4\int_7^9 \frac{1}{t}\frac{dt}{2} = -2\int_7^9 \frac{1}{t}\,dt$$

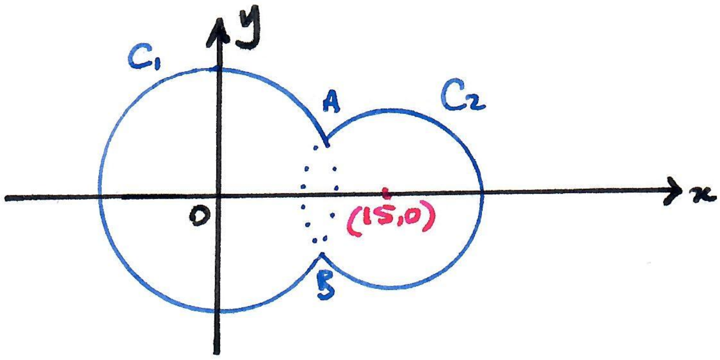
$$= -2\ln t\Big|_7^9 \quad \text{——— (2)}$$

So $I = (1) + (2) = 2\ln 3 - 2\ln 2 - 2\ln 9 + 2\ln 7$

$$= 2\ln\left(\frac{3\times 7}{2\times 9}\right) = 2\ln\left(\frac{7}{6}\right) = \ln\frac{49}{36}$$

$(a = 49)$

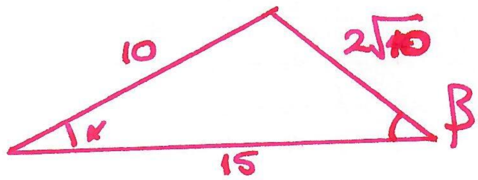
11)



C_1: $x^2+y^2=100$ (centre (0,0), radius 10)

C_2: $(x-15)^2+y^2=40$ (centre (15,0), radius $\sqrt{40} = 2\sqrt{10}$)

a) From the observations already made we can draw:



So α is given by the cosine rule:

$$\cos\alpha = \frac{10^2+15^2-(2\sqrt{10})^2}{2\times10\times15} = \frac{325-40}{300}$$

$$= \frac{285}{300} = \frac{95}{100} = 0.95$$

So $\alpha = 18.195^\circ = 0.3176$ rad

So $\angle AOB = 2\alpha = \underline{\underline{0.635}}$ to 3 sf. as required.

b) For this we'll also need angle β:

$$\cos\beta = \frac{15^2 + (2\sqrt{10})^2 - 10^2}{2 \times 15 \times 2\sqrt{10}}$$

$$= \frac{225 + 40 - 100}{60\sqrt{10}}$$

$$= \frac{165}{60\sqrt{10}} = 0.8696$$

So $\beta = 29.5848^\circ = 0.5164$ rad

$2\beta = 1.0327$ rad.

The complements of 2α and 2β are $(2\pi - \dots)$

i.e. 5.648 and 5.250

So the perimeter of the shape is:

$$10 \times 5.648 + 2\sqrt{10} \times 5.250$$

$$= 56.48 + 33.21$$

$$= 89.69 \text{ or } 89.7 \text{ to } 1 \text{ dp.}$$

12) a) Use the identity

$$\cos^2\theta + \sin^2\theta \equiv 1$$

$$\cos^2\theta \equiv 1 - \sin^2\theta$$

so $$\frac{\cos^2\theta}{\sin\theta} \equiv \operatorname{cosec}\theta - \sin\theta$$

so $\cos\theta \cot\theta \equiv \operatorname{cosec}\theta - \sin\theta$ as required.

b) $\operatorname{cosec} x - \sin x = \cos x \cot(3x - 50°)$

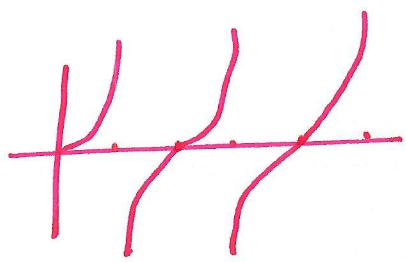
using (a) this gives

$$\cos x \cot x = \cos x \cot(3x - 50°)$$

one solution is clearly $\cos x = 0$: $\underline{\underline{x = 90°}}$

otherwise,

$$\cot x = \cot(3x - 50°)$$



so $x = 3x - 50°$:

$2x = 50°$ $\underline{\underline{x = 25°}}$

or $x = 3x + 50° + n\,180°$

$2x = 50 + 180n°$

$x = 25 + 90n$

in the range given, this allows also $\underline{\underline{x = 115°}}$

So $\underline{x = 25°, 90° \text{ or } 115°.}$

13) a) $a_{n+1} = \dfrac{k(a_n+2)}{a_n}$

we know $a_1 = 2$ and $a_4 = a_1 = 2$

So $a_2 = \dfrac{k(2+2)}{2} = 2k$

$a_3 = \dfrac{k(2k+2)}{2k} = k+1$

$a_4 = \dfrac{k(k+1+2)}{k+1} = \dfrac{k(k+3)}{k+1} = a_1 = 2.$

So $k^2 + 3k = 2(k+1) = 2k+2$

or $\underline{\underline{k^2+k-2=0}}$ as required.

Solutions of this equation are

$(k+2)(k-1)=0$ so $k = -2$ or 1

b) But we discount $k=1$ because this would mean

$a_{n+1} = \dfrac{a_n+2}{a_n} = 1 + \dfrac{2}{a_n}$

which would not have periodicity 3

(alternatively: $a_1 = 2$

$a_2 = \dfrac{4}{2} = 2$

$a_3 = 2$

etc for the particular case $a_1 = 2$)

c) So with $k = -2$ we know

$$a_1 = 2$$

$$a_2 = \frac{-2(4)}{2} = -4$$

$$a_3 = \frac{-2(-4+2)}{-4} = -1$$

$$a_4 = \frac{-2(-1+2)}{-1} = 2 \checkmark$$

So $a_1 + a_2 + a_3 = 2 - 4 - 1 = -3$

We observe $\sum_{r=1}^{80} a_r = \sum_{r=1}^{78} a_r + a_{79} + a_{80}$

and since the terms are 3-periodic,

$$\sum_{r=1}^{78} a_r = \frac{78}{3} \times \sum_{r=1}^{3} a_r + a_{(79-78)} + a_{(80-78)}$$

$$= 26 \times (-3) + a_1 + a_2$$

$$= -78 + 2 - 4$$

$$= \underline{\underline{-80}}$$

14) we know $\frac{dV}{dt} = c$ for some constant

and $V = \frac{4}{3}\pi r^3$

So $\frac{dV}{dt} = \frac{4}{\cancel{3}}\pi . \cancel{3}r^2 \frac{dr}{dt} = c$

So $\frac{dr}{dt} = \frac{c}{4\pi r^2} = \underline{\underline{\frac{-k}{r^2}}}$ as required

where $k = -\frac{c}{4\pi}$.

Separating variables:

$r^2 dr = -k dt$

Integrating:

$\frac{1}{3}r^3 = -kt + c$ (different c)

Using the conditions given:

$\frac{1}{3}40^3 = 0 + c$: $c = \frac{64000}{3}$

$\frac{8000}{3} = \frac{1}{3}20^3 = -5k + \frac{64000}{3}$

$5k = \frac{56000}{3}$ $k = \frac{11200}{3}$

So $\frac{1}{3}r^3 = -\frac{11200}{3}t + \frac{64000}{3}$

or $r^3 = 64000 - 11200t$

Since $r \geqslant 0$ the model is only valid (or needed)

while $64000 - 11200t \geqslant 0$

i.e. $t \leqslant \frac{64000}{11200} = \frac{640}{112} = \underline{\underline{\frac{40}{7}\text{ s}}}$

15) $x^2 \tan y = 9$

Using product rule and chain rule:

$x^2 \frac{d}{dx}(\tan y) + 2x \tan y = 0$

$x^2 \sec^2 y \frac{dy}{dx} = -2x \tan y = -2x \frac{9}{x^2}$

$= \frac{-18}{x}.$

$\frac{dy}{dx} = \frac{-18}{x^3 \sec^2 y}$

$= \frac{-18}{x^3(1+\tan^2 y)}$

$= \frac{-18}{x^3\left(1+\frac{81}{x^4}\right)} = \frac{-18x^4}{x^3(x^4+81)} = \underline{\underline{\frac{-18x}{x^4+81}}}$

as required.

$$\frac{dy}{dx} = \frac{-18x}{x^4+81}$$

So using the quotient rule

$$\frac{d^2y}{dx^2} = \frac{(x^4+81)(-18) - (-18x)(4x^3)}{(\quad X \quad)}$$

$$= \frac{(-18x^4 - 1458 + 72x^4)}{(\quad X \quad)} = \frac{54x^4 - 1458}{(\;X\;)}$$

$= 0$ at a point of inflection.

Since the denominator is always +ve,

this means

$$54x^4 - 1458 = 0$$

$$x^4 = 27$$

So $\underline{\underline{x = \sqrt[4]{27}}}$ as required.

16) Suppose $\exists$ positive (*) integers p and q such that $4p^2 - q^2 = 25$

25 can be factorised as 1×25 or 5×5,

and $(4p^2 - q^2) = (2p-q)(2p+q)$

So either $\left.\begin{array}{l} 2p - q = 1 \\ 2p + q = 25 \end{array}\right\}$ $4p = 26 : p = \frac{13}{2}$ which isn't an integer

or $\left.\begin{array}{l} 2p - q = 5 \\ 2p + q = 5 \end{array}\right\}$ $q = 0$, which contradicts (*) so the result is proved by contradiction.