1) $f(x) = 3x^3 + 2ax^2 - 4x + 5a$

$(x+3)$ is a factor, so $f(-3) = 0$:

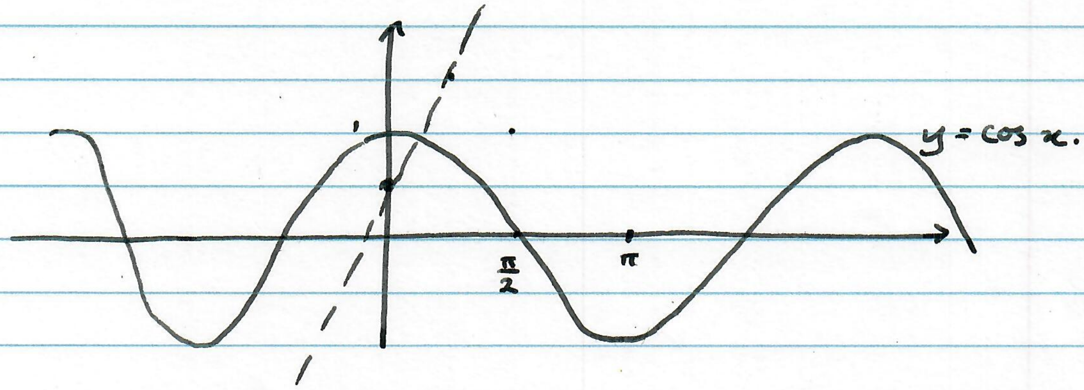
$0 = 3(-27) + 2a(9) - 4(-3) + 5a$

$= -81 + 18a + 12 + 5a$

$= -69 + 23a$

$23a = 69$ so $\underline{\underline{a = 3}}$.

2)



Also draw $y = 2x + \frac{1}{2}$ (dotted line).

This passes through the points $(0, \frac{1}{2})$ and $(\frac{\pi}{4}, \frac{\pi}{2} + \frac{1}{2})$ which are on opposite sides of $y = \cos x$: so there is an intersection...

but only 1, since the gradient of $y = 2x + \frac{1}{2}$ is 2 so the curves never meet again.

(This seems an adequate answer given that we're asked to use the diagram - though you could say more, eg that the gradient of $y = \cos x$ is never >1 so again the curves won't meet.)

Anyway, this means

$\cos x = 2x + \frac{1}{2}$ has only one root

and hence

$\cos x - 2x - \frac{1}{2}$ has only one root.

b) The small angle approximation says

for small x, $\cos x \approx 1 - \frac{x^2}{2}$

Substituting, this gives an equation for α:

$$1 - \frac{\alpha^2}{2} - 2\alpha - \frac{1}{2} = 0$$

$$-\frac{\alpha^2}{2} - 2\alpha + \frac{1}{2} = 0$$

$$\alpha^2 + 4\alpha - 1 = 0$$

$$(\alpha = \frac{-4 \pm \sqrt{16+4}}{2}$$

$$= -2 \pm \sqrt{5}$$

This gives 2 roots but the -ve one appears to be spurious (and is not small). So we can say

$$\alpha = -2 \pm \sqrt{5} = \underline{\underline{0.236}}$$

3) $$y = \frac{5x^2+10x}{(x+1)^2} \qquad x \neq -1.$$

Using the quotient rule:

$$\frac{dy}{dx} = \frac{(2\times5x+10)(x+1)^2 - (5x^2+10x).2.(x+1).1}{(x+1)^4}$$

$$\text{Numerator} = (x+1)^2(10x+10) + (x+1)(-10x^2-20x)$$

$$= (x+1)\left(10(x+1)(x+1) - 10x^2 - 20x\right)$$

$$= (x+1)\left(\cancel{10x^2} + \cancel{20x} + 10 - \cancel{10x^2} - \cancel{20x}\right)$$

$$= 10(x+1).$$

So $\frac{dy}{dx} = \frac{10}{(x+1)^3}$. So $\underline{\underline{A = 10}}$, $\underline{\underline{n = 3}}$.

This is -ve when $(x+1)^3 < 0$

which means $x+1 < 0$

or $x < -1$.

So $\frac{dy}{dx} < 0$ when $\underline{\underline{x < -1}}$.

4) a) $\dfrac{1}{\sqrt{4-x}} = (4-x)^{-\frac{1}{2}} = (4)^{-\frac{1}{2}}\left(1-\frac{x}{4}\right)^{-\frac{1}{2}}$

$= \frac{1}{2}\left(1-\frac{x}{4}\right)^{-\frac{1}{2}}$

Binomial expansion:

$$= \frac{1}{2}\left[1^{-\frac{1}{2}} + \left(-\tfrac{1}{2}\right)1^{-\frac{3}{2}}\left(-\frac{x}{4}\right) + \frac{\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)}{2}1^{-\frac{5}{2}}\left(-\frac{x}{4}\right)^2 + \dots\right]$$

$$= \frac{1}{2} - \frac{1}{4}\left(\frac{-x}{4}\right) + \frac{3}{16}\left(\frac{-x}{4}\right)^2 + \dots$$

$$= \frac{1}{2} + \frac{x}{16} + \frac{3x^2}{256} + \dots \quad \text{as required.}$$

b) The expansion works when in the $(1+a)$ term, $|a|<1$: i.e. $\left|\frac{x}{4}\right|<1$ or $|x|<1$.

i) for $x=-14$: this breaches $|14|<1$. So $x=-14$ should not be used.

ii) for $x=2$; this will lead to $\left(1-\frac{1}{2}\right)^{-\frac{1}{2}}$ for which the expansion will converge slowly. (it gives 1.3438)

for $x=-\frac{1}{2}$; this will lead to $\left(1+\frac{1}{8}\right)^{-\frac{1}{2}}$ which will converge faster.

So $x=-\frac{1}{2}$ should be used. (it gives 1.4150)

(accurate answer is 1.4142)

5) a) $f(x) = 2x^2 + 4x + 9$

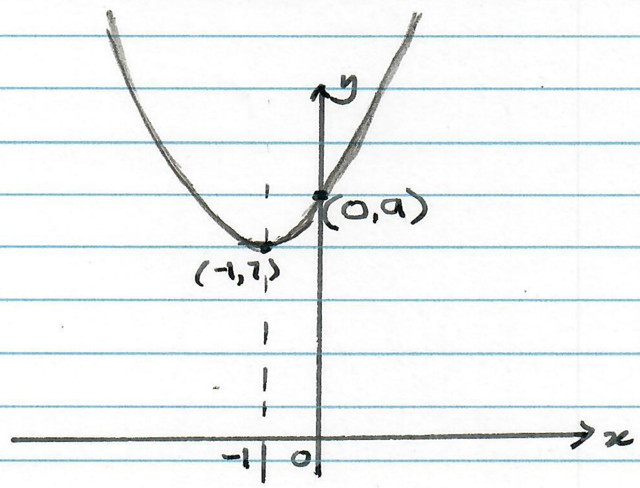
$= 2(x^2 + 2x) + 9$

$= \underbrace{2(x+1)^2 - 2}_{(= 2x^2 + 2x + 2 - 2)} + 9$

$= 2(x+1)^2 + 7$

So $a = 2$, $b = 1$, $c = 7$.

b) • The curve will be a parabola, (x^2 power)
• symmetric about $x = -1$, ($(x+1)^2$ term)
• ∪ shape $2x^2$ term
• zeroes will be when .... never, because it's always +ve
• turning point will be when $(x+1)^2$ is least, ie $x = -1$ and $y = 7$.
• value when $x = 0$ is $y = 9$.



c) i) This looks horrible, but we suspect...

$g(x) = 2(x-2)^2 + 4x - 3$
$= 2x^2 - 8x + 8 + 4x - 3$
$= 2x^2 - 4x + 5$

$$= 2(x^2 - 2x) + 5$$
$$= 2(x^2 - 2x + 1) - 2 + 5$$
$$= 2(x-1)^2 + 3.$$

So to map $f(x)$ onto $g(x)$,

a) shift $f(x)$ 2 units to the right
b) shift it down by 4 units.

6) a) $5 \sin 2\theta = 9 \tan \theta$ $\quad -180^\circ \leq \theta \leq 180^\circ$

Use the formula $\sin 2\theta = 2 \sin\theta \cos\theta$

Then $5 \times 2 \cancel{\sin\theta} \cos\theta = 9 \dfrac{\sin\theta}{\cos\theta}$

$$\cos^2\theta = \frac{9}{10} = 0.9$$

$$\cos\theta = 0.9487$$

$$\underline{\theta = \pm 18.4^\circ}$$

b) $5 \sin(2x - 50^\circ) = 9 \tan(x - 25^\circ)$

The answer(s) is given by letting

$$2x - 50^\circ = \theta = \pm 18.4^\circ$$

$$2x = 31.6^\circ \text{ or } 68.4^\circ$$
$$x = 15.8^\circ \text{ or } 34.2^\circ$$

So the smallest +ve value of x is $\underline{\underline{15.8^\circ}}$.

7) $V(0) = 20000$

$V(1) = 16000$

a) Propose a model $V(t) = Ae^{\alpha t}$ (α will be -ve)

where $A = V(0) = 20000$.

We haven't been asked to find α, but we get

$$V(1) = 16000 = 20000e^{\alpha}$$

So $$e^{\alpha} = \frac{16000}{20000} = 0.8$$

$$\alpha = -\ln(0.8) = -0.223$$

and $$V(t) = \underline{\underline{e^{t\ln(0.8)}}} = \underline{\underline{e^{-0.223t}}}$$

b) The model would say

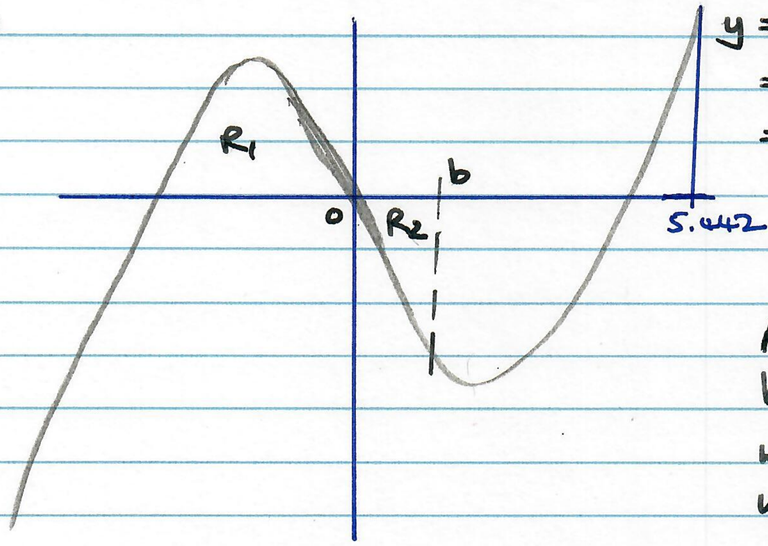
$$V(10) = 20k\,e^{-0.223\times 10} = 20k\,e^{-2.23} = £2150$$

This is not far off the actual value, but it suggests the model is a little generous in its valuation

c) Leave A the same but use a less severe depreciation constant α:

So $V(t) = Ae^{\beta t}$ where $|\beta| < |\alpha|$. and α, β are both -ve.

8)



$y = x(x+2)(x-4)$
$= x(x^2-2x-8)$
$= x^3-2x^2-8x$

Always treat areas below the x-axis with caution, and usually as <u>negative</u>

From the factors in the given expression, the curve crosses the x-axis at $x = -2, 0$ and 4.

$$\text{So } R_1 = \int_{-2}^{0} y \, dx$$

$$= \int_{-2}^{0} x^3 - 2x^2 - 8x \, dx$$

$$= \left[\frac{x^4}{4} - \frac{2}{3}x^3 - \frac{8x^2}{2} \right]_{-2}^{0} \quad \text{—— (1)}$$

$$= 0 - 0 - 0 - \frac{16}{4} - \frac{16}{3} + 16$$

$$= -4 + 16 - \frac{16}{3}$$

$$= 12 - \frac{16}{3}$$

$$= \frac{36-16}{3}$$

$$= \underline{\underline{\frac{20}{3}}} \text{ as required.}$$

b) Saying $R_1 = R_2$ actually means $R_2 = -\frac{20}{3}$ as an integral, i.e. $\int_0^b y\,dx = -\frac{20}{3}.$

Substituting the expression in ① this gives

$$\frac{b^4}{4} - \frac{2}{3}b^3 - 4b^2 = -\frac{20}{3}$$

$$3b^4 - 8b^3 - 48b^2 = -80 \quad ②$$

Working backwards from the given expression:

$$(b+2)^2(3b^2 - 20b + 20)$$

$$= (b^2 + 4b + 4)(3b^2 - 20b + 20)$$

$$= 3b^4 + 12b^3 + 12b^2$$
$$-20b^3 - 80b^2 - 80b$$
$$+20b^2 + 80b + 80$$

$$= 3b^2 - 8b^3 - 48b^2 + 80 \quad ③$$

Taking -80 to the LHS in ②, the two expressions are the same so the result is proved.

Alternately, consider ① as

$$3b^4 - 8b^3 - 48b^2 + 80 = 0$$

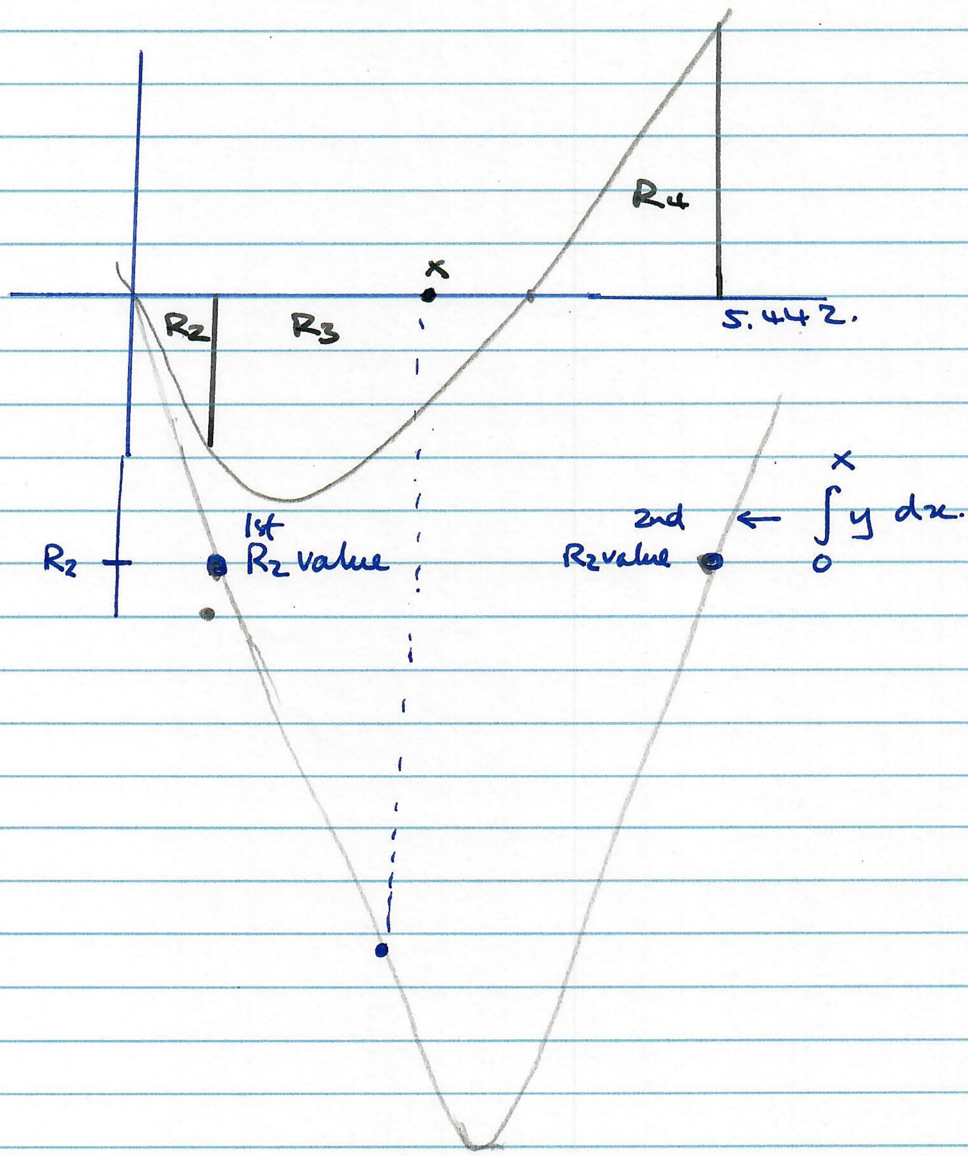
factorise with $(b+2)$:

$$(b+2)(3b^3 - 14b^2 - 20b + 40) = 0$$

$$(b+2)(b+2)(3b^2 - 20b + 20) = 0$$

which is as required.

c) The significance of the root 5.442 relates to the +ve and -ve values of area under the curve. R_2 is actually a negative area; so is R_3 below, but R_4 is +ve. As x increases above 4, the value of $R_2 + R_3 + R_4$ will rise until it reaches the value R_2 – and this occurs at $x = 5.442$.



9) a) $\log a - \log b = \log(a-b)$

but $\log a - \log b = \log \frac{a}{b}$

So $\frac{a}{b} = a-b$

$a = ab - b^2$

$a - ab = -b^2$

$a = \frac{-b^2}{1-b} = \frac{b^2}{b-1}$ as required.

b) For the original equation to be viable,

- b must be +ve
- a must be greater than b

Also for $a = \frac{b^2}{b-1}$ to be valid, $b \neq 1$

and since a must be +ve,
b must be >1.

So the full restriction is

$\underline{\underline{1 < b < a}}.$

10) i) A) by inspection:

— if n is even then $n = 2m$ for some m

So $n^2+2 = 4m^2+2$

So $(n^2+2) \bmod 4$ has a remainder 2.

— if n is odd then $n = 2m+1$ for some m

So $n^2+2 = (2m+1)^2+2$
$= 4m^2+4m+1+2$
$= 4m^2+4m+3$

So $(n^2+2) \bmod 4$ has a remainder 3.

n even or odd exhausts all cases and none has 0 remainder mod 4 — so the result is proved.

B) by induction:

Suppose n^2+2 is not divisible by 4,

~~Consider~~ $\cancel{(n+1)^2+2}$

$\cancel{= n^2+2n+1+2}$
$\cancel{=(n^2+2) + 2n + 1.}$

~~If this were divisible~~ by 4

This doesn't really work, but consider

$(n+2)^2+2 = n^2+4n+4+2$
$=(n^2+2) + 4n+4.$

If this were divisible by 4, then we could remove the $4n+4$ term and the remainder - n^2+2 - would be divisible by 4. But this is a contradiction.

Now consider starting cases:

$n=1$: $n^2+2=3$, which is not div. by 4.

So the result is proved by induction for $n = 3, 5, 7, 9 \ldots$

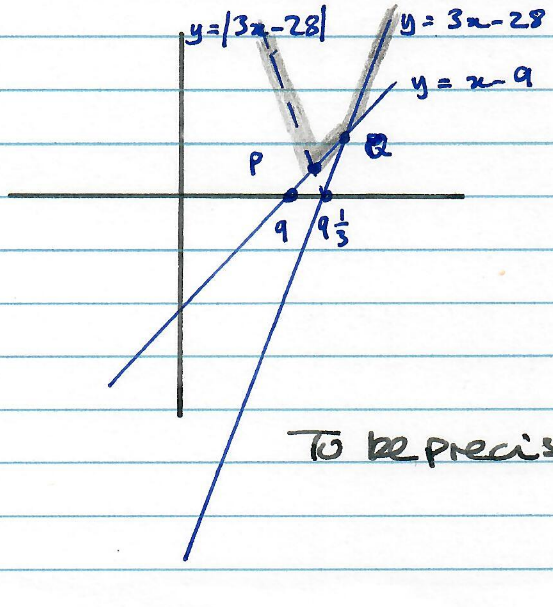
$n=2: n^2+2=6$ . . - — —
$n=0$ $\qquad\qquad 2$

So again proved for $n = 4, 6, 8 \ldots$

So in conclusion the result is proved $\forall n \in \mathbb{N}$.

ii) $x \in \mathbb{R}$: $|3x-28|$ and $(x-9)$.

This is an odd little question that seems best answered with a sketch graph:



This shows that for most x, $|3x-28|$ is indeed $\geqslant (x-9)$ - but for a small interval between points P and Q it is not true. So the answer is 'sometimes'.

To be precise: for P, $28-3x = x-9$: $x = \frac{37}{4}$

for Q, $3x-28 = x-9$: $x = \frac{19}{2}$

c) Using the formula for sum of a geometric series, and counting the first term as t_4, we get a sum over 17 terms:

$$S_{20} = 3 \times 6 + \frac{6(1-1.05^{17})}{1-1.05}$$

$$= 18 + 155.042$$

$$= 173.042 \text{ m}$$

$$= \underline{\underline{2h\ 53m\ 3s}}$$

11) This looks like a continuous-variable s/v/t question, but in fact it's about geometric series... with 3 extra terms on the front.

Let t_r be the time to run the r^{th} km.

Then $t_1 = t_2 = t_3 = t_4 = 6m$.

~~Let $V_4 = \frac{1 km}{t_4} = \frac{1}{6}$~~

~~Then $V_5 = \frac{1}{6}$~~

a) t_5 is 5% greater, so $t_5 = 6m \times 1.05$

$= 6m + .05 \times 360s = 6m\ 18s.$

again $t_6 = t_5 \times 1.05 = 6m\ 18s + \cdot 05 \times 378s$

$= 6m\ 18s + 18.9s$ (say 19s)

$= 6m\ 37s$

So adding, total time to run the first 6 km is:

$6+6+6+6 + 6m\ 18s + 6m\ 37s$

$= \underline{\underline{36m\ 55s.}}$

b) In similar fashion, $t_r = t_{r-1} \times 1.05$

for $r > 4$:
$$t_5 = t_4 \times 1.05$$
$$t_6 = t_5 \times 1.05 = t_4 \times 1.05^2$$
$$\vdots \qquad \vdots$$
$$t_r = \qquad = t_4 \times 1.05^{r-4}$$

(not sure what there is to show like a proof here: it seems to be a case of show you understand)

12) a) $f(x) = 10e^{-0.25x}\sin x$

$$\frac{df}{dx} = 10\left(e^{-0.25x}\cos x + (-0.25)e^{-0.25x}\sin x\right)$$

this is 0 when (ignoring $10e^{-0.25x}$ factor)

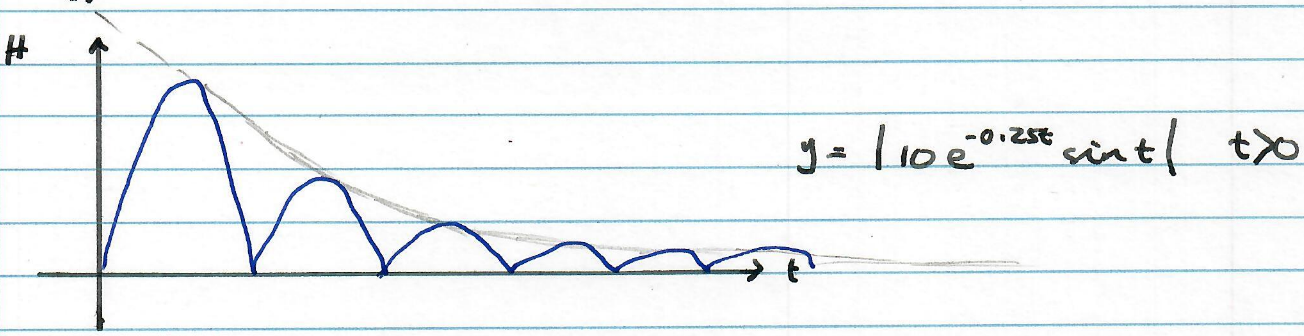
$$0 = \cos x - 0.25\sin x$$

i.e. $$0.25\sin x = \cos x$$

$$\tan x = \frac{\sin x}{\cos x} = 4$$ as required.

(this is surprising since I'd have expected the TPs to be the ordinary $\frac{\pi}{2} + n\pi$ for a sine curve. Evidently the exp. term has a significant effect.)

b)



The curve is basically the same as in (a) but with negative parts reflected in the x axis.

Its long term behaviour is to decline and tend towards 0 since the $e^{-0.25t}$ term will act as a limit on its value.

c) We want to find the second turning point,

i.e. when $\tan t = 4$ and $\pi < t < 2\pi$

$180° < t < 360°$

$\tan^{-1}(4) = 75.96° = 1.326$ rad

So the value we need is $180° + 75.96° = 255.96°$

$= 4.467$ rad.

At this point $H(4.467) = \left| 10 e^{-0.25 \times 4.467} \sin(4.467) \right|$

$= 10 \times e^{-1.1167} \times 0.9701$

$= 10 \times 0.3273 \times 0.9701$

$= 3.1755$ m.

d) (Not very well worded).

1) the curve has zeros equalled spaced - at $t = n\pi$. But a ball would actually take less time for later bounces.

2) the curve never becomes a constant zero, but the ball would eventually stop bouncing.

13) $$y = \frac{p-3x}{(2x-q)(x+3)}$$

a) i) Vertical assymptotes occur when the denominator is 0, ie when $2x-q=0$ or $x+3=0$.

Since $x+3=0$ defines the assymptote $x=-3$,

we know $2x-q=0$ defines $x=2$

So $4-q=0$

$$\underline{\underline{q=4.}}$$

ii) Knowing $y = \dfrac{p-3x}{(2x-4)(x+3)}$ we can substitute for the point $\left(3, \frac{1}{2}\right)$:

$$\frac{1}{2} = \frac{p-3.3}{(2.3-4)(3+3)} = \frac{p-9}{\cancel{2}.6}$$

$$6 = p-9 \qquad \underline{\underline{p=15}}$$

So $$y = \frac{15-3x}{(2x-4)(x+3)}$$

b) First we'll find the upper x-limit of R: i.e. where $y=0$. This occurs when $15-3x=0$, i.e. when $x=5$. So we require

$$\int_3^5 y\,dx.$$

As for $\dfrac{15-3x}{(2x-4)(x+3)}$, we 'know' that this can be expressed as $\dfrac{A}{2x-4}+\dfrac{B}{x+3}$.

Putting these over a common denominator:

$$\frac{A(x+3)+B(2x-4)}{(2x-4)(x+3)} = \frac{(A+2B)x+(3A-4B)}{(\;)(\;)}$$

So equating coefficients:

$$-3 = A+2B \quad ①$$
$$15 = 3A-4B \quad ②$$

$2\times①$: $-6 = 2A+4B$ ③

$②+③$: $9 = 5A$ So $A = \frac{9}{5}$

$3\times①$: $-9 = 3A+6B$ ④

$②-④$: $24 = -10B$ So $B = -\frac{24}{10} = \frac{-12}{5}$

So $y = \frac{9}{5}\frac{1}{2x-4} - \frac{12}{5}\frac{1}{x+3}$

Let $2x-4 = s$ then $2dx = ds$ $dx = \frac{ds}{2}$

$x+3 = t$ then $dx = dt$.

Adjusting the limits, we now require

$$\int_3^5 y\,dx = \frac{9}{5}\int_2^6 \frac{1}{s}\frac{1}{2}ds - \frac{12}{5}\int_6^8 \frac{1}{t}dt$$

$$= \frac{9}{5}\cdot\frac{1}{2}\ln s\Big|_2^6 - \frac{12}{5}\ln t\Big|_6^8$$

$$= \frac{9}{10}\ln 6 - \frac{9}{10}\ln 2 - \frac{12}{5}\ln 8 + \frac{12}{5}\ln 6$$

$$= \cancel{\frac{9}{10}}\ln 2 + \frac{9}{10}\ln 3 - \cancel{\frac{9}{10}}\ln 2 - \frac{36}{5}\ln 2 + \frac{12}{5}\ln 2 + \frac{12}{5}\ln 3$$

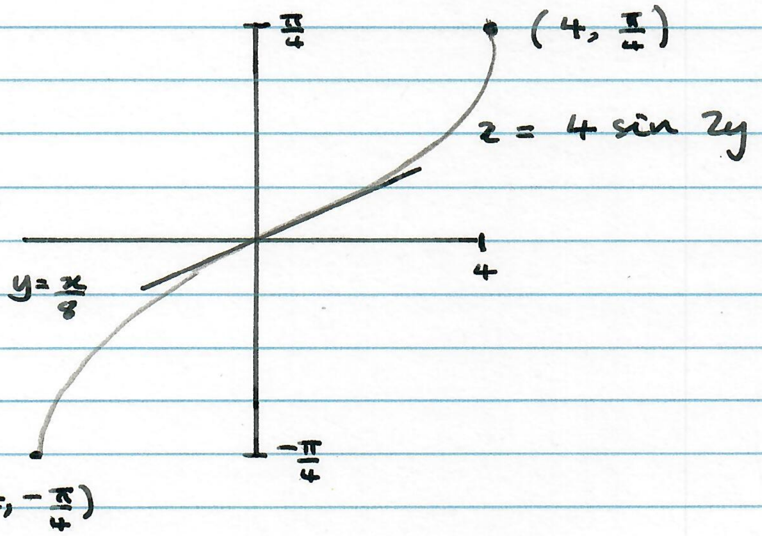
$$= \left(\frac{-36}{5} + \frac{12}{5}\right)\ln 2 + \left(\frac{9}{10} + \frac{24}{10}\right)\ln 3$$

$$= -\frac{24}{5}\ln 2 + \frac{33}{10}\ln 3.$$

So as required in the question,

$$a = -\frac{24}{5} \qquad b = \frac{33}{10}$$

14)



a) $$x = 4\sin 2y$$

$$\frac{dx}{dy} = 4.2.\cos 2y$$

So $$\frac{dy}{dx} = \frac{1}{8}\sec 2y \text{ or } \frac{1}{8}\frac{1}{\cos 2y}.$$

At the origin this is $\frac{1}{8}\frac{1}{\cos 0} = \underline{\frac{1}{8}}$

b) i) The small angle approximation says

for small θ, $\sin\theta \approx \theta$.

Substituting for $2y$ this gives

for small y, $x \approx 4.2y = 8y$

or $y = \frac{x}{8}$ ——— ①

ii) ① is the equation of a line through the origin with gradient $\frac{1}{8}$, which matches the local behaviour of the original curve - i.e. it's a tangent to it.

c) we recall $\frac{dy}{dx} = \frac{1}{8} \cdot \frac{1}{\cos 2y}$

and $\cos 2y = \sqrt{1 - \sin^2 2y}$

$= \sqrt{1 - \left(\frac{x}{4}\right)^2}$

$= \frac{1}{4}\sqrt{16 - x^2}$

So $\frac{dy}{dx} = \frac{1}{8} \cdot \frac{4}{1} \frac{1}{\sqrt{16 - x^2}}$

$= \frac{1}{2\sqrt{16 - x^2}}$

This in the form required, with

$a = 2$

$b = 16$