1) There's a wrinkle here that we mustn't divide by 0 – but happily things work out.

Use the small angle approximations:

$$\sin x \approx x$$

$$\cos x \approx 1 - \frac{x^2}{2}$$

So $\dfrac{1 - \cos 4\theta}{2\theta \sin 3\theta}$ can be approximated

as $$\frac{\left(1 - \left(1 - \frac{4^2\theta^2}{2}\right)\right)}{2\theta \times 3\theta}$$

This appears to have a $\frac{1}{\theta^2}$ term, but it disappears:

$$\frac{1 - 1 + 4^2\theta^2/2}{6\theta^2} = \frac{4^2\theta^2}{12} = \frac{16\theta^2}{12\theta^2} = \underline{\underline{\frac{4}{3}}}$$

2) a) $y = x^2 - 2x - 24\sqrt{x}$

i) $\frac{dy}{dx} = 2x - 2 - 24 \times \frac{1}{2} \frac{1}{\sqrt{x}}$

$= 2x - 2 - \frac{12}{\sqrt{x}} = \underline{\underline{2x - 2 - 12x^{-\frac{1}{2}}}}$

ii) $\frac{d^2y}{dx^2} = 2 + \frac{12}{2} x^{-\frac{3}{2}}$

$= \underline{\underline{2 + \frac{6}{\sqrt{x^3}}}}$

b) when $x = 4$,

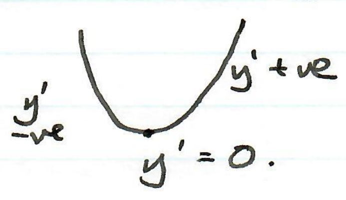
$\frac{dy}{dx} = 2 \times 4 - 2 - \frac{12}{\sqrt{4}} = 8 - 2 - 6 = 0$

So y has a $\underline{\underline{\text{stationary}}}$ point at $x = 4$.

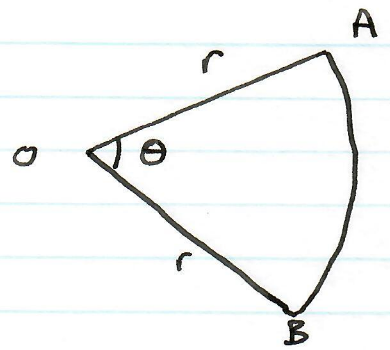
c) when $x = 4$,

$\frac{d^2y}{dx^2} = 2 + \frac{6}{\sqrt{4^3}} = 2 + \frac{6}{2^3} = 2\frac{3}{4}$

This is +ve, indicating $\frac{dy}{dx}$ is increasing, which means the curve is turning at $\underline{\underline{\text{a minimum}}}$:



3)



Area of AOB = $11cm^2$

Area of the sector is $\frac{\theta r^2}{2} = 11$ (given) — ①

Length of arc AB = $r\theta$

And we know $2r + \theta r = 4\theta r$

So $2r = 3\theta r$

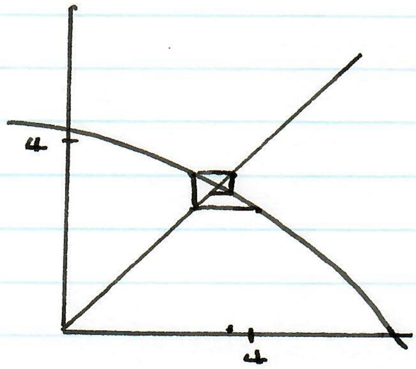
$\theta = \frac{2}{3}$

In ①: $\frac{\cancel{2}}{3}\frac{r^2}{\cancel{2}} = 11$

$r^2 = 33$

$r = 5.745$ cm.

4)



$y = 2\ln(8-x)$: $y = x$. single point α

a) Given the statement in the question we need to show

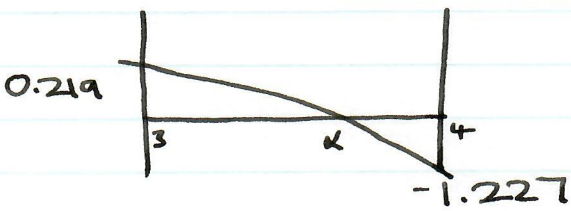
$y(x=3) > x=3$ and $y(x=4) < x=4$

or vice versa

So $y(x=3) = 2\ln(8-3) = 2\ln 5 = 3.219$

$y(x=4) = 2\ln(8-4) = 2\ln 4 = 2.773$

By Rolle's theorem or equivalent this means the equation $2\ln(8-x) - x$ has a 0 between 3 and 4, which is the root α:



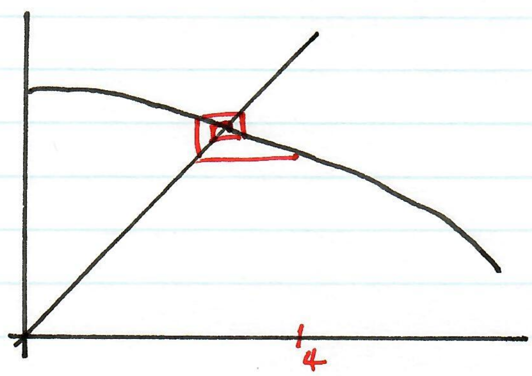
b) ~~Taking~~ $\cancel{x_1 = 4}$ $x_2 = \cancel{2\ln(8-x_1) = 2\ln 4} = 2.773$

$\cancel{x_3} = 2\ln(\cancel{8}-2.773) = 2\ln(\cancel{5.227}) = 3.308$

~~After this~~ it ~~converges rapidly~~ to ~~3.156~~.

(~~But I suspect this wants more of a proof~~)

Apparently what's wanted is not a 'proof' as such (i.e. in analytic terms) but a 'cobweb diagram' drawn on the given diagram:



... and an assertion that 'it spirals inwards'.

How this <u>proves</u> anything I don't know.

(I've tried proving it analytically but I got stuck.)

The question also doesn't want us to show it <u>works</u> – even though it goes rather nicely:

$x_1 = 4$
$x_2 = 2.773$
$x_3 = 3.308$
$\vdots$
3.156

5) $$y = \frac{3\sin\theta}{2\sin\theta + 2\cos\theta}$$

using the quotient rule,

$$\frac{dy}{dx} = \frac{3(2\sin\theta + 2\cos\theta)\cos\theta - 3\sin\theta(2\cos\theta - 2\sin\theta)}{(2\sin\theta + 2\cos\theta)^2}$$

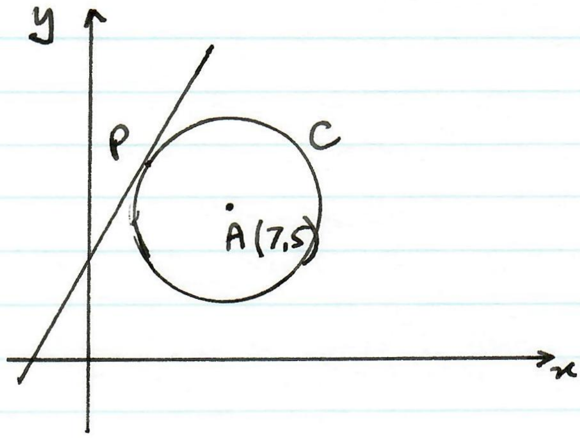
$$= \frac{\cancel{6\sin\theta\cos\theta} + 6\cos^2\theta - \cancel{6\sin\theta\cos\theta} + 6\sin^2\theta}{2^2(\sin^2\theta + 2\sin\theta\cos\theta + \cos^2\theta)}$$

$$= \frac{6 \times 1}{4(1 + \sin 2\theta)}$$

$$= \frac{3}{2}\frac{1}{1+\sin 2\theta}$$ as required, with $A = \frac{3}{2}$.

(and the interval condition $-\frac{\pi}{4} < \theta < \frac{3\pi}{4}$ is satisfied).

6)



tangent at P: $y = 2x + 1$.

gradient of the tangent is 2, so the gradient of the perpendicular (which is PA) is $-\frac{1}{2}$

(the negative reciprocal).

So this line is $y = -\frac{1}{2}x + c$ for some c.

Subs. for (7,5):

$$5 = -\frac{7}{2} + c$$

$$10 = -7 + 2c$$

$$c = \frac{17}{2}.$$

So the line is $y = -\frac{x}{2} + \frac{17}{2}$

$$2y = -x + 17$$

i.e. $\underline{\underline{2y + x = 17}}$. as required.

At the point P both line equations hold:

$$y = 2x + 1 \quad \text{——} ①$$

$$2y + x = 17 \quad \text{——} ②$$

① in ②: $2(2x+1) + x = 17$

$$4x + 2 + x = 17$$

$$5x = 15$$

$$x = 3.$$

$$y = 2 \times 3 + 1 = 7.$$

So P is $(3,7)$.

and the radius of C is $|AP|$

which is $\sqrt{(3-7)^2 + (7-5)^2}$

$$= \sqrt{4^2 + 2^2}$$

$$= \sqrt{20}$$

So since the centre of the circle is $(7,5)$,

its equation is

$$\underbrace{(x-7)^2 + (y-5)^2}_{\text{pythagoras}} = \underbrace{20}_{r^2}$$

Expanding:

$$x^2+y^2-2\times7x-2\times5y+49+25=20$$

$$\underline{\underline{x^2+y^2-14x-10y+54=0}}$$

7) $k \in \mathbb{Z}$ (not $\mathbb{R}$)

a) Consider $\displaystyle\int_k^{3k} \frac{2}{3x-k}\,dx$.

Let $3x-k=t$ then $3dx = dt$

$dx = \frac{1}{3}dt$.

and the limits are:

$x=k$: $t = 3k-k = 2k$

$x=3k$: $t = 9k-k = 8k$

The integral is $\displaystyle\frac{1}{3}\int_{2k}^{8k} \frac{2}{t}\,dt$

$$= \frac{2}{3}\ln t \Big|_{2k}^{8k}$$

$$= \frac{2}{3}(\ln 8k - \ln 2k)$$

$$= \frac{2}{3}\ln\left(\frac{8k}{2k}\right) = \frac{2}{3}\ln 4$$

which is ind^t of k.

b) $\displaystyle\int_k^{2k} \frac{2}{(2x-k)^2}\, dx.$

Let $2x - k = t$ $\qquad 2dx = dt$

$dx = \frac{1}{2} dt$

and limits are: $x = k$: $t = 2k - k = k$

$x = 2k$: $t = 4k - k = 3k$

So the integral is $\displaystyle\int_k^{3k} \frac{2}{t^2} \frac{dt}{2}$

Here it's clear that since we're integrating t^{-2} the result will be a multiple of t^{-1}: and the limits won't change this ... but here goes:

$$\int_k^{3k} \frac{1}{t^2} dt = \left[-\frac{1}{t} \right]_k^{3k}$$

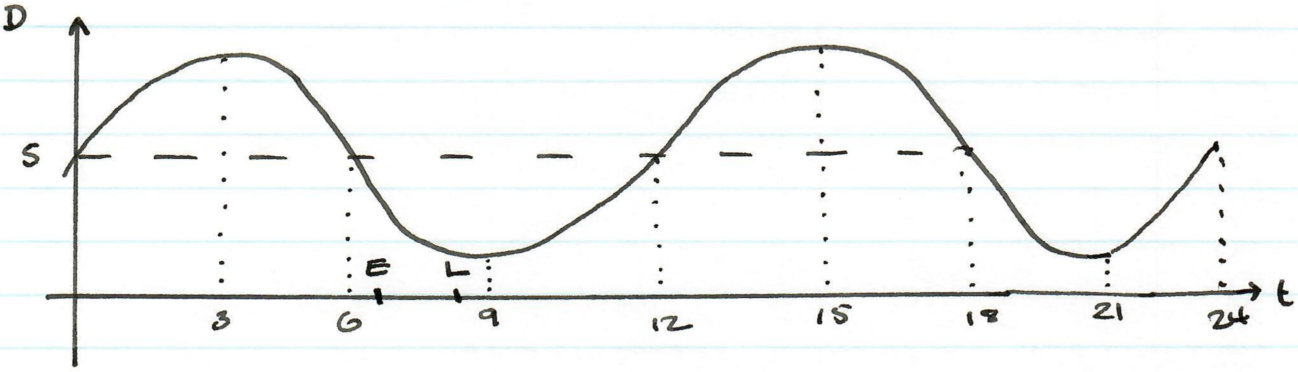
$$= -\frac{1}{3k} + \frac{1}{k}$$

$$= \frac{-1+3}{3k} = \frac{2}{3k}$$

which is inversely proportional to k, as required.

8) $D = 5 + 2\sin(30t)^\circ$

Boat E enters at 6.30 am
Finishes loading at L 8.30am.



a) Depth of water when $t = 6.30$ is

$$5 + 2\sin(30 \times 6.5)^\circ$$
$$= 5 + 2\sin(195)^\circ$$
$$= \underline{\underline{4.482\text{ m}}}$$

b) To leave the harbour after L (8:30) we require a solution between 180° and 360° to

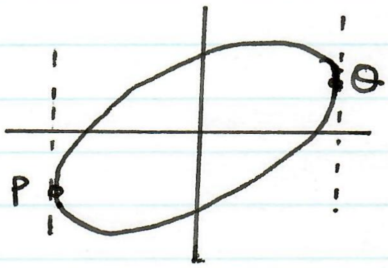
$$3.8 = 5 + 2\sin(30t)$$

so $2\sin(30t) = -1.2$
$\sin(30t) = -0.6$

$30t = 180^\circ + 36.87^\circ = 216.87^\circ$
$t = 7.23$ (which is too early)

or $30t = 360^\circ - 36.87^\circ = 323.13^\circ$
$t = 10.771$
$= \underline{\underline{10:46}}$ which is acceptable

9)



$$x^2 - 2xy + 3y^2 = 50 \quad \text{—} ①$$

a) Differentiate ①:

$$2x - 2\left(x\frac{dy}{dx} + y\right) + 6y\frac{dy}{dx} = 0$$

$$\frac{dy}{dx}(6y - 2x) + 2x - 2y = 0$$

$$\frac{dy}{dx} = \frac{2y - 2x}{6y - 2x} = \frac{y - x}{3y - x}$$

as required.

b) At P the gradient $\frac{dy}{dx}$ becomes infinite,

i.e. the denominator is 0: so $3y - x = 0$

and hence $x = 3y$. $y = \frac{x}{3}$

Substitute this into ①:

$$x^2 - \frac{2x^2}{3} + \frac{3x^2}{9} = 50$$

$$3x^2 - 2x^2 + x^2 = 150$$

$$2x^2 = 150$$
$$x^2 = 75$$
$$x = \pm 5\sqrt{3} \quad (-5\sqrt{3} \text{ at } P)$$
$$y = \frac{x}{3} = \pm\frac{5}{3}\sqrt{3} \quad \left(-\frac{5}{3}\sqrt{3} \text{ at } P\right).$$

So P is $\left(-5\sqrt{3},\ -\frac{5}{3}\sqrt{3}\right)$

c) To find the point furthest north we would set $\frac{dy}{dx} = 0$ and follow a similar procedure.

10) a) $\frac{dH}{dt} = \frac{H\cos(0.25t)}{40}$

Solve this by separating variables.

$$\frac{dH}{H} = \frac{1}{40}\cos(0.25t)\,dt$$

Integrating:

$$\ln H = \frac{1}{40}\,\frac{\sin(0.25t)}{0.25} + c$$

$$= \frac{1}{10}\sin(0.25t) + c.$$

At the start of the ride $t = 0$, $H = 5$:

$$\ln 5 = \frac{1}{10}\sin(0.25\times 0) + c$$

$$= c$$

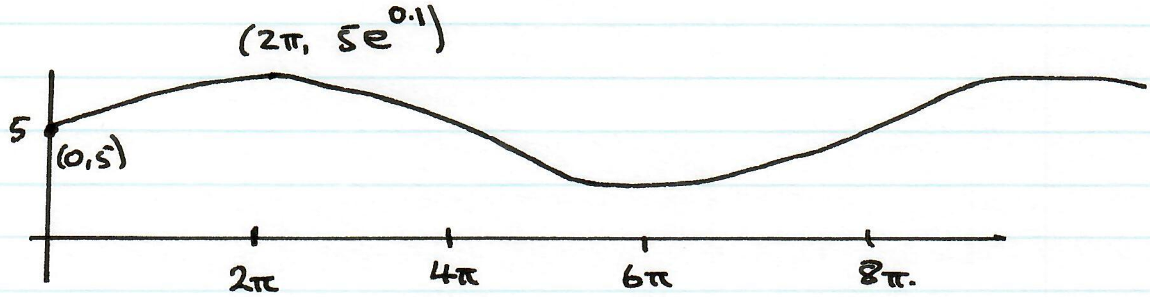
So $c = \ln 5$

So $\ln H = \frac{1}{10}\sin(0.25t) + \ln 5$

$$\ln\left(\frac{H}{5}\right) = \frac{1}{10}\sin(0.25t)$$

Taking exp'ts: $H = 5e^{(0.1\ \sin(0.25)t)}$ as required.

10) b)



Maximum height is when the $\sin((0.25)t)$ term is maximum, i.e. when

$$\sin(0.25t) = 1$$

$$0.25t = \frac{\pi}{2}$$

$$\underline{\underline{t = 2\pi.}}$$

(or do this the hard way with $\frac{dH}{dt} = 0$)

c) T is given by 2π + the period of $\sin(0.25t)$, which is 8π.

So $T = 2\pi + 8\pi = \underline{\underline{10\pi}}$

11) a) $\sqrt{\dfrac{1+4x}{1-x}} = (1+4x)^{\frac{1}{2}}(1-x)^{-\frac{1}{2}}$

Expanding both of these binomially (and checking both expansion are valid for small x):

$$\left(1 + \frac{1}{2}4x + \frac{1}{2}\left(-\frac{1}{2}\right)\frac{(4x)^2}{2} + o(x^3)\ldots\right)$$

$$\times \left(1 - \frac{1}{2}(-x) + \left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)\frac{(-x)^2}{2} + o(x^3)\right)$$

$$= \left(1 + 2x - 2x^2 + o(x^3)\right)\left(1 + \frac{x}{2} + \frac{3}{8}x^2 + o(x^3)\right)$$

$$= 1 + \frac{x}{2} + \frac{3}{8}x^2 + o(x^3)$$

$$+ 2x + x^2 + o(x^3)$$

$$- 2x^2 + o(x^3)$$

$$= \underline{\underline{1 + \frac{5}{2}x - \frac{5}{8}x^2 + o(x^3)}} \text{ as required.}$$

b) The student should not use $x = \frac{1}{2}$ because the first expansion (for $(1+4x)^{\frac{1}{2}}$) would not converge - since $|4x| \nless 1$

c) $x = \frac{1}{11}$ gives $\sqrt{\dfrac{1+\frac{4}{11}}{1-\frac{1}{11}}} \approx 1 + \frac{5}{2}\cdot\frac{1}{11} - \frac{5}{8}\frac{1}{11^2}$

$$\sqrt{\frac{15}{11} \times \frac{11}{10}} \approx 1 + \frac{5}{22} - \frac{5}{8.121}$$

$$LHS = \sqrt{\frac{3}{2}}$$

$$RHS = \frac{968 + 220 - 5}{968} = \frac{1183}{968}$$

So $\sqrt{\frac{3}{2}} \approx \frac{1183}{968}$

$$\sqrt{2}\sqrt{\frac{3}{2}} = \frac{1183}{968}\sqrt{2}$$

$$\sqrt{6} = \frac{1183}{968}\sqrt{2}\times\sqrt{2} = \frac{1183}{484} \quad (= 2.444)$$

$484 = 2^2 \times 11^2$ and neither 2 or 11 divides 1183, so this fraction is in its simplest form.

Check: $2.444^2 = 5.974\ldots$

12) a) i) $V = Ap^t$

In 2005, $V_4 = Ap^4 = 32000$ ①

In 2012, $V_{11} = Ap^{11} = 50000$ ②

② ÷ ①: $\frac{50000}{32000} = p^7 = ~~0.~~ 1.5625$

So $\underline{\underline{p = 1.0658}}$

ii) In ①, $A(1.0658)^4 = 32000$

$$A = \frac{32000}{(1.0658)^4}$$

$$= \underline{\underline{24797}}$$

which is approximately $\underline{\underline{24800}}$ as req'd.

b) i) A is the value of the car when $t = 0$,

i.e. $V_0 = Ap^0 = A$.

So it is the price originally paid.

ii) p is the ratio between the values in successive years: i.e

$p = 1 + \frac{r}{100}$ where r is the % increase each year.

($r = 6.58\%$ is pretty good.)

c) To find this, we set

$$100000 = V = Ap^t = 24797 \times (1.0658)^t$$

and solve for t:

$$4.0327 = 1.0658^t$$
$$\log 4.0327 = t \log 1.0658$$
$$t = 21.8819$$

so $\underline{\underline{\text{in year 2022}}}$

13) $I = \int_0^2 2x\sqrt{x+2}\,dx$

Let $x+2=u$ then $dx = du$

and limits become $x=0: \quad u=2$
$x=2: \quad u=4$

So $I = \int_2^4 2(u-2)\,u^{\frac{1}{2}}\,du$

$= \int_2^4 2u^{\frac{3}{2}} - 4u^{\frac{1}{2}}\,du$

$= \left[2\times\frac{2}{5}u^{\frac{5}{2}} - 4\times\frac{2}{3}u^{\frac{3}{2}} \right]_2^4$

$= \left[\frac{4}{5}u^{\frac{5}{2}} - \frac{8}{3}u^{\frac{3}{2}} \right]_2^4$

$= \frac{4}{5}\left(4^{\frac{5}{2}} - 2^{\frac{5}{2}}\right) - \frac{8}{3}\left(4^{\frac{3}{2}} - 2^{\frac{3}{2}}\right)$

$= \cancel{\left[\frac{4}{5}\left(4^2 - 2^3\right) - \frac{8}{3}\left(4-2\right)\right]\sqrt{2}}$

$= \cancel{\left(\frac{12\times4}{5} - \frac{8\times2}{3}\right)\sqrt{2}}$

$= \cancel{\left(\frac{48}{5} - \frac{16}{3}\right)\sqrt{2}}$

$$= \frac{4}{5}\left(32 - 4\sqrt{2}\right) - \frac{8}{3}\left(8 - 2\sqrt{2}\right)$$

$$= \frac{4 \times 32}{5} - \frac{64}{3} - \left(\frac{16}{5} - \frac{16}{3}\right)\sqrt{2}$$

$$= \frac{128}{5} - \frac{64}{3} - \left(\frac{48 - 80}{15}\right)\sqrt{2}$$

$$= \frac{384 - 320}{15} - \frac{32}{15}\sqrt{2}$$

$$= \frac{64}{15} - \frac{32}{15}\sqrt{2}$$

$$= \frac{32}{15}\left(2 - \sqrt{2}\right) \quad \text{as required.}$$

14) $x = 3 + 2\sin t$ $\quad 0 \leq t \leq 2\pi.$

$y = 4 + 2\cos 2t$

a) We are required to show:

$$\underset{LHS}{\overset{y}{4 + 2\cos 2t}} = \overset{6-(x-3)^2}{6 - (3 + 2\sin t - 3)^2}$$

$$= \underset{RHS}{6 - 4\sin^2 t}$$

Now $\cos 2t = \cos^2 t - \sin^2 t$

So LHS $= 4 + 2(\cos^2 t - \sin^2 t)$

$= 4 + 2(1 - \sin^2 t - \sin^2 t)$

$= 4 + 2 - 2 \cdot 2\sin^2 t$

$= \underline{\underline{6 - 4\sin^2 t}}$

$=$ RHS as required.

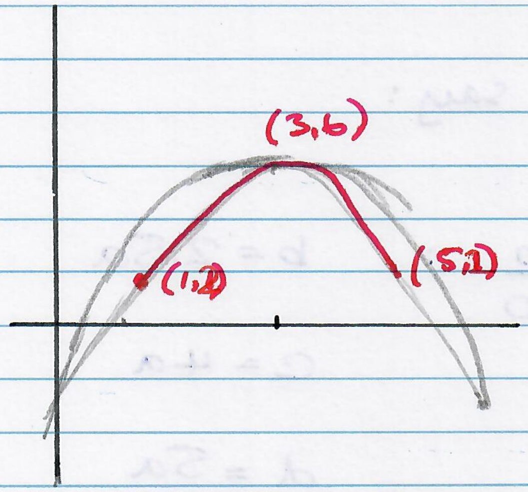
As well as proving the identity, this also shows that

$$y = 6 - (x-3)^2 \quad ①$$ and it's a parabola...

... or at least for all points defined by the values of t.

The parabola, at least, has an axis at $x = 3$, a max. value at $y = 6$ (the point $(3, 6)$) and is upturned.

And when $x = 0$ or 6, $y = 6 - (-3)^2 = -3$.



However, the values of t limit how much of the $\bigcap$ is used.

When $t = 0$ $\quad x = 3+0 = 3$
$\quad y = 4+2 = 6.$

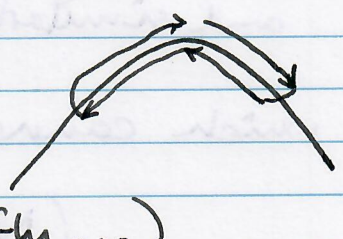
$t = 2\pi$ $\quad x = 3+0$
$\quad y = 4+2 = 6.$

$t = \frac{\pi}{2}$ $\quad x = 3+2 = 5$ and this is the highest x can get
$\quad y = 4+(-2) = 2$

$t = \frac{3\pi}{2}$ $\quad x = 3-2 = 1$
$\quad y = 4+(-2) = 2$ ---- lowest

So essentially x runs:

and y takes values appropriately:



(this isn't very briefly ...)

b) Line l: $x + y = k$

$y = -x + k.$ —— (2)

The line has gradient -1 so it's 'obvious' what the configuration is for it to have 2 points of intersection. But do it properly - firstly with the parabola:

(2) in (1): $6-(x-3)^2 = k-x$

$$6 - x^2 + 6x - 9 + x - k = 0$$

$$-x^2 + 7x - (3+k) = 0$$

$$x^2 - 7x + (3+k) = 0$$

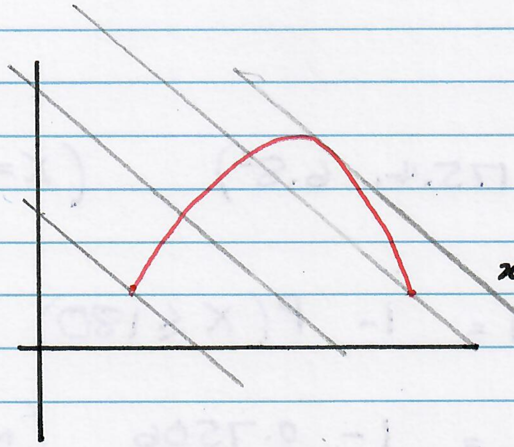
For 2 roots '$b^2 - 4ac$' >0:

$$49 - 4(3+k) > 0$$
$$37 - 4k > 0$$

$$k < \frac{37}{4}.$$

~~This gives (at the ends)~~ $\cancel{x = \frac{7 \pm \sqrt{49 - 4(3+\frac{37}{4})}}{2}}$

$\cancel{= \frac{7 \pm \sqrt{49-49}}{2}}$



This gives a maximum value of k, and works for the entire parabola - but on the parametric curve there's another lower limit at $(5, 2)$ which gives

$$5 + 2 = k = 7.$$

So the region for k is $k \in \left[7, \frac{37}{4}\right)$

or $\left\{k : 7 \leq k < \frac{37}{4}\right\}$