1) a) It's worth knowing these first integration questions sometimes look terrifying, but if you use an intelligent guess it turns out they're really just reversing some fairly simple integration. So...

Use the fact

$$\frac{d}{dx}(\sec 2x) = 2 \sec 2x \tan 2x$$

So $\int 2 \sec 2x \tan 2x \, dx = \sec 2x \ (+c)$

So $\int 6 \sec 2x \tan 2x \, dx = 3 \sec 2x + c$
(different c).

b) Similarly, consider

$$\frac{d}{dx}(5x^2-3)^4 = 4.(10x)(5x^2-3)^3.$$

So $\int 40x(5x^2-3)^3 dx = (5x^2-3)^4 + c$

So the required integral

$$= \int_{-0.4}^{0} 40x(5x^2-3)^3 dx$$

$$= \Big[(5x^2-3)^4\Big]_{-0.4}^{0}$$

$$= (-3)^4 - (-2.2)^4$$
$$= 81 - 23.4256$$
$$= 57.5744 \text{ units}^2.$$

c) $v(t) = 26.4\sqrt[3]{t}$ ①

Let the object's distance from P be s

Then $s(t=0) = 360$ m.

In general $s = \int v\,dt$

$= 26.4\int t^{\frac{1}{3}}dt$

$= 26.4 \cdot \dfrac{t^{\frac{4}{3}}}{4/3} + c$

$= 26.4 \times \frac{3}{4} t^{\frac{4}{3}} + c$

$= 19.8\, t^{\frac{4}{3}} + c$ ——— ②

Also, from the equation ①, when $v = 264$,

$264 = 26.4\, t^{\frac{1}{3}}$

So $t^{\frac{1}{3}} = 10$

$t = 1000.$

From ②, $s(1000) = 19.8(1000)^{\frac{4}{3}} + c$

$= 19.8(10000) + c$

$= 198000 + c.$

So between the times $t=0$ and $t=1000$

the object has travelled $198000 + 360$

$= 198360$ m. from the point P
(note it's not the difference.)

d) $\dfrac{dy}{dx} = 24\cos 3x \sin x$

The expression has the form used in the identity:

$$\sin C - \sin D = 2\cos\frac{C+D}{2}\sin\frac{C-D}{2}$$

So $\sin\ x - \sin 2x = 2\cos 3x \sin x$

So $\dfrac{dy}{dx} = 12(\sin\ x - \sin 2x)$

$$y = 12\int(\sin 4x - \sin 2x)\,dx$$

$$= 12\left(-\frac{1}{4}\cos 4x + \frac{1}{2}\cos 2x\right) + c$$

$$= -3\cos 4x + 6\cos 2x + c.$$

when $x = \dfrac{\pi}{3}$, $y = 6 = -3\cos\dfrac{4\pi}{3} + 6\cos\dfrac{2\pi}{3} + c$

$$= -3 \times -\frac{1}{2} + 6 \times -\frac{1}{2} + c$$

$$= \frac{3}{2} - 3 + c$$

$$= -1.5 + c \qquad \text{So } c = 7.5$$

So $y = -3\cos 4x + 6\cos 2x + 7.5$

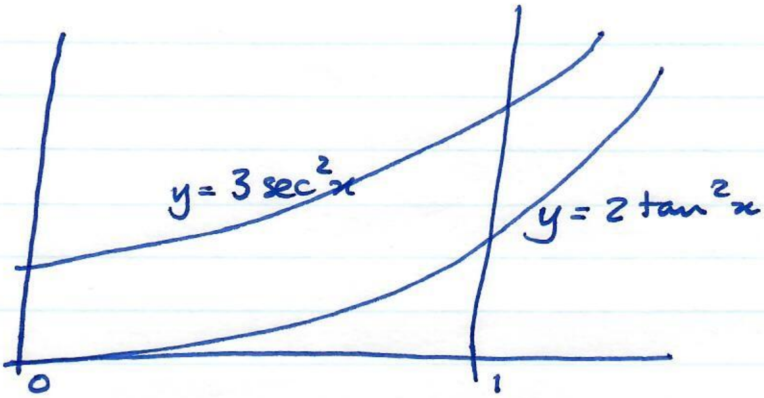
when $x = \dfrac{\pi}{2}$,

$$y = -3\cos 2\pi + 6\cos\pi + 7.5$$

$$= -3.1 + 6.-1 + 7.5$$

$$= -3 - 6 + 7.5 = \underline{\underline{-1.5}}$$

(No reason at all why it says 'value(s)') e)



we want $I = \int_0^1 3\sec^2 x - 2\tan^2 x \, dx.$

we know $\frac{d}{dx} \tan x = \sec^2 x$ so we'll be able to use that

but also $\tan^2 x + 1 = \sec^2 x.$

$2\tan^2 x = 2\sec^2 x - 2$

So $I = \int_0^1 3\sec^2 x - 2\sec^2 x + 2 \, dx$

$= \int_0^1 \sec^2 x + 2 \, dx$

$= \left[\tan x + 2x\right]_0^1$

$= \tan 1 + 2 - \tan 0 - 0$

$= \underline{\underline{3.557}}$ units^2

2) a) $\int (3x^4+4)^2\,dx$

Doesn't look like a chain rule question, so just multiply it out.

$$(3x^4+4)^2 = 9x^8+24x^4+16$$

$$\int 9x^8+24x^4+16\,dx$$

$$= \frac{9}{9}x^9 + \frac{24}{5}x^5 + 16x + c$$

$$= x^9 + \frac{24}{5}x^5 + 16x + c$$

b) $\int_k^{16} 3\sqrt{x}\,dx = 112$

$$\text{LHS} = \int_k^{16} 3x^{\frac{1}{2}}\,dx$$

$$= \left.\frac{3x^{\frac{3}{2}}}{\left(\frac{3}{2}\right)}\right|_k^{16}$$

$$= \left.\frac{2.3}{3}x^{\frac{3}{2}}\right|_k^{16}$$

$$= 2\left(16^{\frac{3}{2}} - k^{\frac{3}{2}}\right)$$

$$= 2\left(64 - k^{\frac{3}{2}}\right) = \text{RHS} = 112$$

So $2\left(64-k^{\frac{3}{2}}\right) = 112$

$64 - k^{3/2} = 56$

$8 = k^{3/2}$

$k = 4.$

c) $\dfrac{dy}{dx} = 12y^2e^{3x}$

Integrate by separation:

$$\frac{dy}{y^2} = 12e^{3x}\,dx.$$

So $\dfrac{1}{-1.y} = \dfrac{1}{3}.12e^{3x} + c$

$$\frac{-1}{y} = 4e^{3x} + c \qquad \text{—— ①}$$

We know $y = 0.5$ when $x = 0$:

$$-2 = \frac{-1}{0.5} = 4e^{0} + c = 4 + c$$

$$-2 = 4 + c \quad \text{so} \quad \underline{c = -6}$$

So from ①,

$$\frac{-1}{y} = 4e^{3x} - 6$$

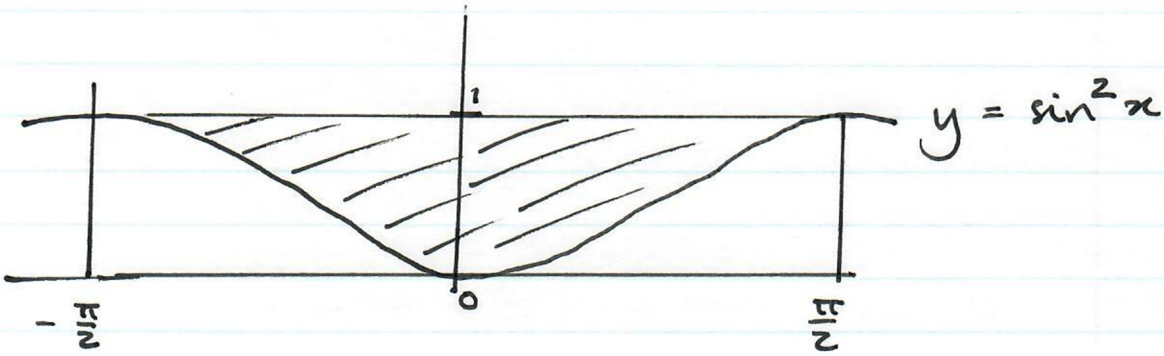
$$y = \frac{1}{6 - 4e^{3x}}$$

When $x = \frac{1}{3}$,

$$y = \frac{1}{6 - 4e^{3.\frac{1}{3}}} = \frac{1}{6 - 4e}$$

$$= \underline{\underline{-0.2052}}$$

d)



The shaded area is $\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (1 - \sin^2 x)\, dx$

$$= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos^2 x \; dx. \qquad \text{——— } ①$$

use the identity $2\cos^2 x = \cos 2x + \cos 0$
$= \cos 2x + 1$

(this is probably a long way round)

$$① = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{1}{2}(\cos 2x + 1)\, dx$$

$$= \left[\frac{1}{4}\sin 2x + \frac{x}{2} \right]_{-\frac{\pi}{2}}^{\frac{\pi}{2}}$$

$$= \frac{1}{4}\sin \pi + \frac{\pi}{4} - \frac{1}{4}\sin(-\pi) + \frac{\pi}{4}$$

$$= 0 + \frac{\pi}{4} - 0 + \frac{\pi}{4}$$

$$= \frac{\pi}{2}$$

e) $$M = \int_0^p 4\pi r^2 \left(\frac{a}{1+br^3}\right) dr$$

Let $1 + br^3 = u^3$

then $3br^2 dr = 3u^2 du$

Assume this is OK It feels dodgy but we assume everything is continuous, + ve, ok.

$$r^2 = \frac{u^2 du}{b\, dr} = \frac{u^2}{b}\frac{du}{dr}.$$

$$\text{So } M = \int_{r=0}^{r=p} 4\pi \frac{\cancel{u^2}}{b} \frac{a}{\cancel{u^3}\ u} \frac{du}{\cancel{dr}} \cancel{dr}$$

$$= \int_{r=0}^{r=p} \frac{4\pi a}{b} \cdot \frac{1}{u}\, du$$

$$= \left[\frac{4\pi a}{b} \ln u\right]_{r=0}^{r=p}$$

when $r=0$, $u^3 = 1$ so $u = 1$

$r = p$ $u^3 = 1+bp^3$ so $u = \sqrt[3]{1+bp^3}$.

$$\text{So } M = \frac{4\pi a}{b}\left(\ln \sqrt[3]{1+bp^3} - \ln 1\right)$$

$$= \frac{4\pi a}{b} \ln\left(\sqrt[3]{1+bp^3}\right)$$

$$= \underline{\underline{\frac{4\pi a}{3b} \ln(1+bp^3)}}$$

NZQA Level 3 Calculus 2024 91579 Integration

3) a) $\int e^{2x} + \frac{3}{e^{4x}} \, dx$

$= \int e^{2x} + 3e^{-4x} dx$

$= \frac{1}{2} e^{2x} - \frac{1}{4} 3e^{-4x} + c$

$= \frac{e^{2x}}{2} - \frac{3}{4} e^{-4x} + c$

b) $\frac{dy}{dx} = \frac{5}{4x-3}$ $\qquad y = \int \frac{5}{4x-3} dx$

Let $4x - 3 = u$. Then $4dx = du$

$dx = \frac{1}{4} du.$

So $y = \int \frac{5}{u} \cdot \frac{1}{4} du = \frac{5}{4} \ln u + c$

$= \frac{5}{4} \ln(4x-3) + c$

we know when $y = 10$, $x = 1$

So $10 = \frac{5}{4} \ln(4-3) + c$

$= \frac{5}{4} \ln(1) + c$

$= \frac{5}{4} \cdot 0 + c$

So $c = 10$

and $y = \frac{5}{4} \ln(4x-3) + 10$

should be | | throughout.

c) $$\int_{-1}^{m} \left(\frac{4x+5}{2x+3} \right) dx = 2m.$$

Let $2x+3 = u$ then $2dx = du$
$dx = \frac{1}{2} du.$

and $4x+5 = 2(2x+3)-1 = 2u-1.$

So the integral is

$$\int_{x=1}^{x=m} \frac{2u-1}{u} \cdot \frac{1}{2} du$$

$$= \int_{x=1}^{x=m} 1 - \frac{1}{2u} du$$

$$= \left[u - \frac{1}{2} \ln|u| \right]_{x=1}^{x=m.}$$

when $x=1$, $u=5$
$x=m$, $u=3+2m$ (assume $m \geq 1$).

So this is $\left[2x+3 - \frac{1}{2} \ln|2x+3| \right]_{-1}^{m}$

$= 2m+3 - \frac{1}{2} \ln(2m+3)$
$+ 2 - 3 + \frac{1}{2} \ln(-1.2+3)$
$= 2m + 2 - \frac{1}{2} \ln(2m+3) + \ln(1)$
$= 2m + 2 - \ln\sqrt{(2m+3)}$

we know this $= 2m$, so $2 - \ln\sqrt{(2m+3)} = 0$

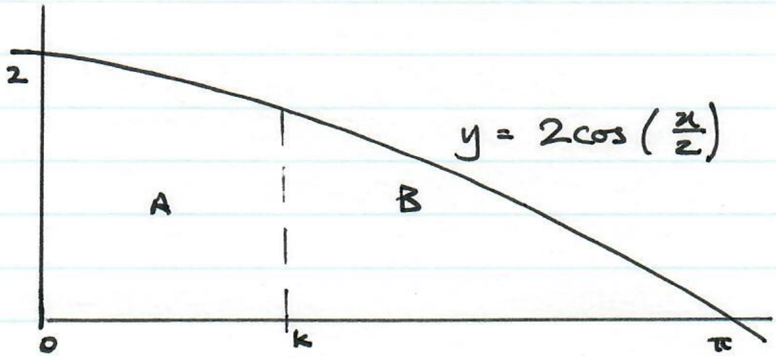
So $\qquad 2 = \ln\sqrt{2m+3}$

$$e^2 = \sqrt{2m+3}$$

$$e^4 = 2m+3$$

$$m = \frac{e^4-3}{2} = 25.7991$$

d)



If Area A = Area B then $\int_0^k y\,dx = \int_k^\pi y\,dx.$ ——— ①

$$\int y\,dx = \int 2\cos\frac{x}{2}\,dx = 2.\frac{\sin\frac{x}{2}}{\frac{1}{2}} \quad (+c)$$

$$= 4\sin\frac{x}{2}.$$

So in ①,

$$4\sin\frac{x}{2}\Big|_0^k = 4\sin\frac{x}{2}\Big|_k^\pi$$

$\therefore$ (discarding factor 4):

$$\sin\frac{k}{2} - \sin 0 = \sin\frac{\pi}{2} - \sin\frac{k}{2}.$$

So $2\sin\frac{k}{2} = \sin\frac{\pi}{2} = 1.$

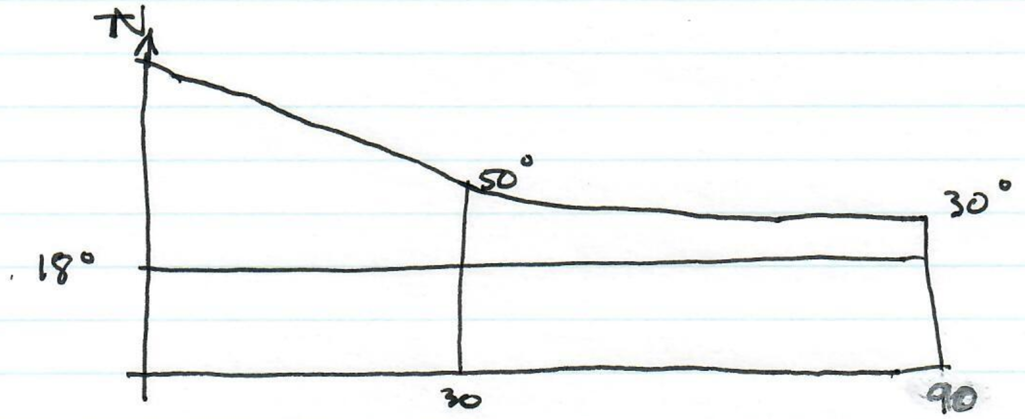
$$\therefore \quad \sin\frac{k}{2} = \frac{1}{2}$$

$$\frac{k}{2} = \frac{\pi}{6}$$

$$k = \frac{\pi}{3} \quad (60^\circ)$$

$$= \underline{\underline{1.0472}}$$

e)



$$\frac{dN}{dt} \propto N - 18$$

$$\frac{dN}{dt} = m(N-18)$$ where we expect: m is $-$ve

N is 50 at $t=30$

N is 30 at $t=90$

If $\frac{dN}{dt} = m(N-18)$

then $\qquad \frac{dN}{N-18} = m\,dt$

$=$

or $\displaystyle\int \frac{dN}{N-18} = \int m\,dt$

So $\quad \ln(N-18) = mt + c.$

This leads to 2 simultaneous eq^s:

$$\ln(50-18) = 30m + c$$

$$\ln(30-18) = 90m + c$$

$$\ln 32 = 30m + c \quad \text{——— (1)}$$

$$\ln 12 = 90m + c. \quad \text{——— (2)}$$

(1) - (2) : $\ln 32 - \ln 12 = -60m$

$$= \ln \frac{32}{12}$$

$$= \ln \frac{8}{3}.$$

So $m = \frac{-1}{60} \ln \frac{8}{3} = -0.0163$

(1) + (2) : $\ln 32 + \ln 12 = 120m + 2c$

$$c = \frac{\ln(32 \times 12) - 120m}{2}$$

$$= (5.9506 + 1.956)/2$$

$$= 3.9533.$$ (seems some way from the approved answer) of 3.95615

So $\ln(N-18) = -0.0163t + 3.9533$

When $t=0$ this gives $N-18 = e^{3.9533} = 52.1070$

So $\underline{\underline{N = 70.1070}}$ I