NZQA Level 3 Calculus 2024 91578 Differentiation

NCEA Level 3 91578 2024. (Differentiation)

1) a) $f(x) = \sqrt{(4-9x^4)}$

$$= (4-9x^4)^{\frac{1}{2}}$$

$$\frac{df}{dx} = \frac{1}{2}(4-9x^4)^{-\frac{1}{2}}\left(\frac{d}{dx}(4-9x^4)\right)$$

$$= \frac{1}{2}\left(\frac{1}{\sqrt{(4-9x^4)}}\right)(-36x^3)$$

$$= \frac{-18x^3}{\sqrt{(4-9x^4)}}$$

b) $y = (x^2+3x+2)\sin x$

$$\frac{dy}{dx} = (x^2+3x+2)\cos x + (2x+3)\sin x.$$

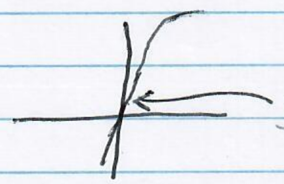
At $x=0$ this gives

$$\frac{dy}{dx} = (0+0+2)\cos 0 + (0+3)\sin 0$$

$$= 2\times 1 + 3\times 0$$

$$= \underline{\underline{2}}$$

(which is rather what you'd expect:



$y(\varepsilon) \approx 2\sin\varepsilon$.)

c) $y = 3(2x-7)^2 + 60\ln x + 12 \quad x > 0$

The function is decreasing when $\frac{dy}{dx} < 0$

And

$$\frac{dy}{dx} = 3 \times 2(2x-7) \times 2 + \frac{60}{x}$$

$$= 24x - 6 \times 7 \times 2 + \frac{60}{x}$$

$$= 24x - 84 + \frac{60}{x}$$

we can divide through by 4 (doesn't affect the sign)

and work with $6x - 21 + \frac{15}{x}$

and the condition we need is $6x + 21 + \frac{15}{x} < 0$

we can also multiply by x, since x is +ve:

$$6x^2 + 21x + 15 < 0$$

also, divide by 3:

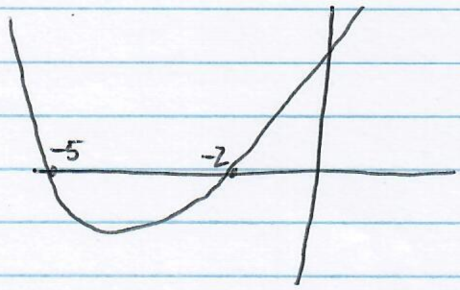
$$2x^2 + 7x + 5 < 0$$

Finally, set the expression to 0 and find its roots:

$$2x^2 + 7x + 5 = 0$$

$$(x+5)(x+2) = 0$$

So the roots are -2 and -5.



By inspection we can see this is a parabola of the form ∪ (as $x \to \pm\infty$, both sides are +ve)

So $\frac{dy}{dx} < 0$ in the interval $(-5, -2)$

So the original function y is decreasing in the range $\underline{\underline{-5 < x < -2}}$

(strict inequalities).

d) $y = (2x-1)e^{-2x}$

Stationary points occur when $\frac{dy}{dx} = 0$:

$$\frac{dy}{dx} = (2x-1)(-2e^{-2x}) + 2.e^{-2x}$$
$$= -4xe^{-2x} + 2e^{-2x} + 2e^{-2x}$$
$$= -4xe^{-2x} + 4e^{-2x}$$
$$= 4e^{-2x}(1-x)$$

This is 0 when $e^{-2x} = 0$ (never)
or $x = 1$.

So the only stationary point is when $x = 1$.

To determine the nature of the stationary point we consider $\frac{d^2y}{dx^2}$,

$$\frac{d^2y}{dx^2} = 4e^{-2x}(-1) + 4.-2.e^{-2x}(1-x)$$

$$= -4e^{-2x} - 8e^{-2x} + 8e^{-2x}x$$

$$= -12e^{-2x} + 8xe^{-2x}$$

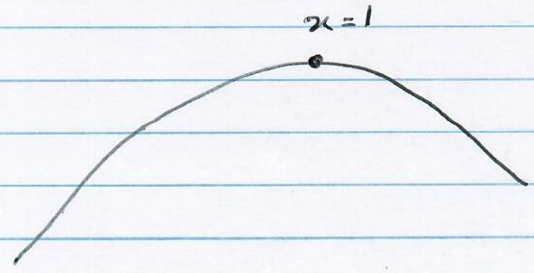
$$= 4e^{-2x}(2x-3)$$

When $x=1$, this equals

$$4e^{-2.1}(2-3)$$

The exp term is +ve but $(2-3)$ is −ve, so the second derivative here is −ve.

This indicates the first derivative is decreasing, so the point on the curve is a maximum.



e) $y = \frac{2x^2 - 1 - 2x\ln x}{x}$ $x > 0$.

(This seems a rather silly way to represent it)

Divide by x: $y = 2x - \frac{1}{x} - 2\ln x$ $x > 0$.

—— ①

A point of inflexion doesn't nec. have $\frac{dy}{dx} = 0$,

but it does have $\frac{d^2y}{dx^2} = 0$.

So: $\frac{dy}{dx} = 2 + \frac{1}{x^2} - \frac{2}{x}$ (2)

and $\frac{d^2y}{dx^2} = \frac{1}{-2x^3} + \frac{2}{x^2}$

$= \frac{1}{x^2}\left(\frac{-1}{2x} + 2\right)$

This is 0 when $\frac{-1}{2x} + 2 = 0$,

i.e. $\frac{1}{2x} = 2$

$4x = 1$

$\underline{\underline{x = \frac{1}{4}}}$.

The tangent line in general form will be

$y = mx + c$

where m is the gradient, given by $\frac{dy}{dx}$.

So in (2), $m = 2 + \frac{1}{(\frac{1}{4})^2} - \frac{2}{(\frac{1}{4})}$

$= 2 + 16 - 8 = \underline{\underline{10}}$

So the tangent is $y = 10x + C$. —— 3

To find c, substitute into 1:

$$y = 2\left(\frac{1}{4}\right) - \frac{1}{\left(\frac{1}{4}\right)} - 2\ln\left(\frac{1}{4}\right)$$

$$= \frac{1}{2} - 4 - 2\ln\left(\frac{1}{4}\right)$$

$$= \frac{10}{4} + C \quad \text{from } 3.$$

So $C = \frac{1}{2} - \frac{5}{2} - 4 - 2\ln\left(\frac{1}{4}\right)$

$= -6 - 2\ln\left(\frac{1}{4}\right)$

$= -3.2274$ (inelegant, but GC confirms)

So the tangent is $\underline{\underline{y = 10x - 3.2274}}$.

2) a) $x = 3t^2 + 1$

$y = \cos t$

$$\frac{dy}{dx} = \frac{dy}{dt} \Big/ \frac{dx}{dt} = \frac{-\sin t}{6t} = \frac{-\sin t}{6t}$$

b) distance $s = \ln(3t^2 + 5t + 2) \qquad t > 0$

$$\text{velocity} = \frac{ds}{dt} = \frac{1}{3t^2 + 5t + 2} \cdot (6t + 5)$$

$$= \frac{6t + 5}{3t^2 + 5t + 2}$$

So when $t = 1$,

velocity $= \dfrac{6.1+5}{3.1+5.1+2} = \dfrac{11}{10} = 1.1$ m/s

(GC confirms)

c) $\dfrac{d^2y}{dx^2} + 4x^2 y = 2\cos x^2 + (1-4x^2)\cos x$ ①

Test a solution $y = \sin x^2 - \cos x$:

$$\frac{dy}{dx} = (\cos x^2)(2x) + \sin x$$

$$= 2x\cos x^2 + \sin x$$

$$\frac{d^2y}{dx^2} = 2x \cdot x^2(-\sin x^2)(2x) + 2\cos x^2 + \cancel{\sin x} \cdot \cos x$$

$$= -4x^4 \sin x^2 + 2\cos x^2 + \cancel{\sin x} \; \cos x$$

Subs into ①,

$$\text{LHS} = \cancel{-4x^4 \sin x^2} + 2\cos x^2 + \overset{\cos}{\cancel{\sin}}\, x + \cancel{4x^2(\sin x^2)} - 4x^2\cos x$$

$$= 2\cos x^2 + \cos x - 4x^2\cos x$$

$$= 2\cos x^2 + (1-4x^2)\cos x$$

which = the RHS. So the proposed expression is indeed a solution.

d) $f(x) = \dfrac{\ln x}{x} \qquad x > 0$

For a point of inflexion, $\dfrac{d^2f}{dx^2} = 0$.

$$\frac{df}{dx} = \frac{x \cdot \frac{1}{x} - \ln x \cdot 1}{x^2}$$

$$= \frac{1 - \ln x}{x^2}$$

$$\frac{d^2f}{dx^2} = \frac{x^2\left(-\frac{1}{x}\right) - (1 - \ln x) \cdot 2x}{x^4}$$

$$= \frac{-x - 2x + 2x \ln x}{x^4}$$

$$= \frac{-3 + 2\ln x}{x^3}$$

This is 0 when $-3 + 2\ln x = 0$

i.e. $\ln x = \dfrac{3}{2}$

$x = 4.4817$.

At this point $f(x) = \dfrac{\ln x}{x} = \dfrac{\frac{3}{2}}{4.4817} = 0.3347$.

So the point of inflexion is $(4.4817, 0.3347)$

(messy, but GC confirms. I suspect the examiners couldn't be bothered to find a nicer solution.)

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d) $f(x) = \dfrac{\ln(x)}{x}$ $\quad x > 0.$

for a point of inflexion, $\dfrac{d^2f}{dx^2} = 0.$

$$\frac{df(x)}{dx} = \frac{x \cdot \frac{1}{x} - \ln(x) \cdot 1}{x^2}$$

$$= \frac{1 - \ln(x)}{x^2} = \frac{1}{x^2} - \frac{1}{x^2}\ln x.$$

$$\frac{d^2f(x)}{dx^2} = \frac{1}{-2x^3} - \frac{x^2 \frac{1}{x} - 2x\ln x}{x^4}$$

$$= \frac{1}{-2x^3} - \frac{1 - 2\ln x}{x^3}$$

$$= \frac{1}{x^3}\left(-\frac{1}{2} - 1 + 2\ln x\right)$$

$=$

e) "A single turning point at Q"
means there is one point and only one point where $\frac{dy}{dx} = 0$.

$$y = \frac{xe^{3x}}{2x+k}$$

$$\text{So } \frac{dy}{dx} = \frac{-(xe^{3x})(2) + (2x+k)\left[x3e^{3x} + e^{3x}\right]}{(2x+k)^2}$$

$$= \frac{-\cancel{2xe^{3x}} + 6x^2e^{3x} + 3xke^{3x} + \cancel{2xe^{3x}} + ke^{3x}}{(2x+k)^2}$$

Since we're only looking for this to be 0, we can ignore the denominator and focus on the numerator, which is:

$$6x^2e^{3x} + 3xke^{3x} + ke^{3x}$$

$$\text{or} \quad e^{3x}(6x^2 + 3xk + k).$$

for this to be 0, the quadratic expⁿ must be 0.

And if there is only one root, the discriminant ($b^2 - 4ac$) must be 0.

i.e. $(3k)^2 - 4.6.k = 0$

$$9k^2 - 24k = 0 \qquad 3k(3k - 8) = 0$$

$$k = \frac{8}{3}.$$

$$\text{So} \quad x = \frac{-3k \pm \sqrt{9k^2 - 4.6.k}}{2.6}$$

$k=0$ leads to a nonsense $\frac{0}{0}$

So $k = \frac{8}{3}$ and the required

$x =$

$$\therefore \; -\frac{3k}{12} = \frac{-k}{4} = -\frac{8}{12} = \underline{\frac{-2}{3}.}$$

(GC confirms)

try: $-5 < k < 5$ – it makes a rather pretty animation.

3) a) $y = \sqrt{x} \,.\, \sec(6x)$

using the product rule:

$$\frac{dy}{dx} = \sec(6x) \,.\, \frac{1}{2} \,.\, x^{-\frac{1}{2}} + x^{\frac{1}{2}} \frac{d}{dx}(\sec 6x)$$

$$= \frac{1}{2\sqrt{x}} \sec 6x + \sqrt{x} \,.\, 6 \,.\, \sec 6x \tan 6x$$

$$= \sqrt{x} \,.\, \sec 6x \left(\frac{1}{2x} + 6 \tan 6x \right)$$

b) i) $f(x)$ is continuous but not differentiable when ~~$x=-1$~~ and $x=5$.

Actually no, because it doesn't exist at $x=-1$.

ii) $f'(x) = 0$ when $x=1$ (turning pt)

and $3 < x \leq 5$ (constant)

iii) viewing $x \to -1$ from both sides the limit exists and appears to be 1, even though the value at $x = -1$ is not defined / specified.

c) $$f(x) = \frac{x^2 - 5x + 4}{x^2 + 5x + 4}.$$

$$\left(= \frac{(x-1)(x-4)}{(x+1)(x+4)}, \text{ though this doesn't help} \right)$$

using the quotient rule,

$$f'(x) = \frac{(x^2 + 5x + 4)(2x - 5) - (x^2 - 5x + 4)(2x + 5)}{(x^2 + 5x + 4)^2}$$

$$\begin{aligned}
\text{Numerator} &= \cancel{2x^3} + 10x^2 + \cancel{8x} - 5x^2 - \cancel{25x} - 20 \\
&\quad - \cancel{2x^3} + 10x^2 - \cancel{8x} - 5x^2 + \cancel{25x} - 20 \\
&= 20x^2 - 10x^2 - 40 \\
&= 10x^2 - 40
\end{aligned}$$

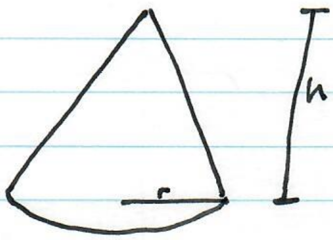
So $$f'(x) = \frac{10x^2 - 40}{(x^2 + 5x + 4)^2}$$

Stationary points are given by $f'(x) = 0$

which occurs when $10x^2 - 40 = 0$

i.e. $x^2 - 4 = 0$ So $\underline{\underline{x = \pm 2.}}$

d)



given $h = 2r$

Volume of cone $= \quad V = \frac{1}{3}\pi r^2 h$

$$= \frac{2}{3}\pi r^3$$

we require $\frac{dh}{dt}$ with knowledge of $\frac{dV}{dt}$,

and some calculus expressions to assist.

we can write: $\quad \frac{dh}{dt} = \frac{dh}{dr} \cdot \frac{dr}{dV} \cdot \frac{dV}{dt}$

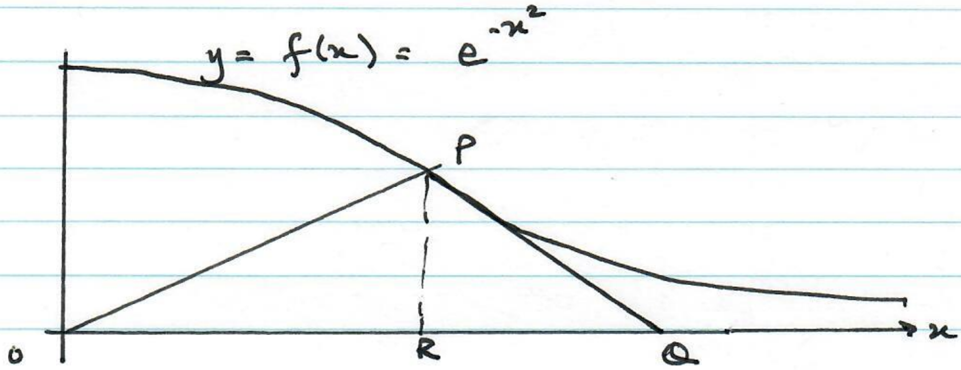
$\frac{dh}{dr} = 2$; $\quad \frac{dV}{dr} = \frac{3 \cdot 2\pi r^2}{3} = 2\pi r^2$; $\quad \frac{dV}{dt} = 3$.

So $\frac{dh}{dt} = 2 \cdot \frac{1}{2\pi r^2} \cdot 3 = \frac{3}{\pi r^2}$

So when $h = 4$, $r = 2$ and

$$\frac{dh}{dt} = \frac{3}{4 \cdot \pi} \simeq 0.2387 \quad (\text{cm s}^{-1})$$

e) $f(x) = e^{-x^2}$ $x \geqslant 0$



$OP = PQ$. Δs OPR, QPR are similar.

if P is the point $(x, f(x))$ then

area Δ OPR $= \dfrac{x \cdot f(x)}{2}$

area Δ OPQ $= 2 \Delta$ OPR $= x \cdot f(x)$.

So the question is one of finding the maximum of $x \cdot f(x)$. $= xe^{-x^2}$

Let this be $g(x) = xe^{-x^2}$

Then by the product rule,

$$\frac{dg}{dx} = x \frac{d}{dx}(e^{-x^2}) + 1 \cdot e^{-x^2}$$

$$= x(-2x)e^{-x^2} + e^{-x^2}$$

$$= (-2x^2 + 1)e^{x^2}$$

This is 0 when $1 - 2x^2 = 0$

So $x^2 = \frac{1}{2}$ $x = \pm \frac{1}{\sqrt{2}}$.

we discard the negative value to give:

$$x = \frac{1}{\sqrt{2}}$$

And the area of OPQ is

$$x \cdot f(x) = \frac{1}{\sqrt{2}} e^{-\left(\frac{1}{\sqrt{2}}\right)^2}$$

$$= \frac{1}{\sqrt{2}} e^{-\frac{1}{2}}$$

$$= \frac{1}{\sqrt{2e}}$$