NZQA Level 3 Calculus 2024 91577 Complex Numbers

NCEA Level 3 91577 Calculus 2024.

1) a) $(x+3)$ is a factor of the function, so $x = -3$ is a root.

So $(-3)^3 + p(-3)^2 + 5(-3) - 12 = 0$

$-27 + 9p - 15 - 12 = 0$

$9p - 54 = 0$

$9p = 54$

So $\underline{\underline{p = 6}}$

b) $z = m \operatorname{cis}\left(\frac{n\pi}{5}\right)$

$z^{15} = \left(m \operatorname{cis}\left(\frac{n\pi}{5}\right)\right)^{15}$

$= m^{15}\left(\operatorname{cis}\left(\frac{n\pi}{5}\right)\right)^{15}$

$= m^{15} . \operatorname{cis}\left(\frac{15n\pi}{5}\right)$

$= \underline{\underline{m^{15} \operatorname{cis}(3n\pi)}}$

which is in polar form.

c) $4 - \sqrt{kx} = \sqrt{(kx+4)}$

Squaring both sides:

$(4 - \sqrt{kx})^2 = kx + 4$

$16 - 8\sqrt{kx} + \cancel{kx} = \cancel{kx} + 4$

$$16 - 8\sqrt{kx} = 4$$

$$-8\sqrt{kx} = -12$$

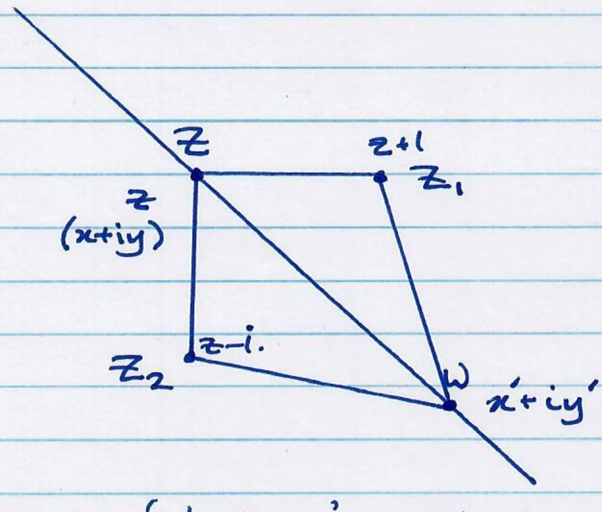
$$2\sqrt{kx} = 3$$

$$\sqrt{kx} = \frac{3}{2}$$

Squaring, $\quad kx = \frac{9}{4}$

So $\quad x = \frac{9}{4k}$

d) $|z-i| = |z+1|$ essentially says the distance of a point P from two fixed points is equal: and we know such a locus is a straight line perp and bisector of the line between them. Analysis here will not quite prove this but will use the known facts.



The answer is 'obvious': the gradient is −1.

But it's not clear whether the NZ curriculum wants us to work it out analytically.

Anyway, we get:

$$|z_1 w| = |z_2 w| \quad \text{so} \quad |z_1 w|^2 = |z_2 w|^2$$

So $(x' - (x+1))^2 + (y' - y)^2 = (x' - x)^2 + (y' - (y-1))^2$

$$\cancel{x'^2} - 2x'(x+1) + (x+1)^2 + \cancel{y'^2} - 2yy' + y^2$$

$$= \cancel{x'^2} - 2xx' + x^2 + \cancel{y'^2} - 2y'(y-1) + (y+1)^2$$

$$-\cancel{2x'x} - 2x' + \cancel{x^2} + 2x + \cancel{1} - \cancel{2yy'} + \cancel{y^2}$$

$$= -\cancel{2xx'} + \cancel{x^2} - \cancel{2yy'} + 2y' + \cancel{y^2} + 2y + \cancel{1}.$$

$$-2x' + 2x = 2y' + 2y$$

$$-x' + x = y' + y$$

$$y' = -x' + x - y$$

or $\quad y' = -x' + (x - y)$

which is the equation of a straight line

of gradient $\underline{-1}$.

e) $u = 2+3ki$

$v = 4+5ki$

$$w = \frac{u}{v} = \frac{2+3ki}{4+5ki}$$

To lie on the line $y=x$, the real and imag. parts of w would have to be equal.

So we express w in standard form:

$$w = \frac{2+3ki}{4+5ki}$$

$$= \frac{2+3ki\,(4-5ki)}{(4+5ki)(4-5ki)}$$

$$= \frac{(2+3ki)(4-5ki)}{16+25k^2}$$

$$= \frac{8+12ki-10ki+15k^2}{16+25k^2}$$

$$= \frac{8+15k^2}{16+25k^2} + \frac{2ki}{16+25k^2}$$

We can ignore the denominators and say the condition would require

$$8+15k^2 = 2k$$

$$15k^2-2k+8 = 0$$

We can solve this using the Quadratic formula but we see the discriminant (b^2-4ac) would be $(-2)^2-4.15.8$ $= 4-480 = -476$. This is negative so any value of k would be complex, which contradicts the condition $k \in \mathbb{R}$.

2) a) $$\frac{i}{2k+i} = \frac{i}{2k+i} \times \frac{(2k-i)}{(2k-i)}$$

$$= \frac{2ki - i^2}{4k^2+1}$$

$$= \frac{1+2ki}{4k^2+1}$$

$$= \frac{1}{4k^2+1} + \frac{2k}{4k^2+1} i$$

which is in the required form.

b) $$2x^2 + (3+2r)x + 3 - 2r = 0$$

Using the Quadratic Formula, if the roots are equal then the Discriminant "b^2-4ac" is zero:

$$(3+2r)^2 - 4.2.(3-2r) = 0$$

$$9 + 12r + 4r^2 - 24 + 16r = 0$$

$$4r^2 + 28r - 15 = 0$$

$$(2r-1)(2r+15) = 0$$

So $r = \frac{1}{2}$ or $-\frac{15}{2}$ for the original eq^n to have equal roots.

roots are actually -1 and 3

c) $\dfrac{w}{w-i} = 2-i$

$w = (2-i)(w-i)$

$= 2w - iw - 2i + i^2$

$= 2w - iw \quad -2i - 1$

$0 = w - iw \quad -2i-1$

$1+2i = w(1-i)$

so $w = \dfrac{1+2i}{1-i}$ ⊛

$= \dfrac{1+2i}{1-i} \times \dfrac{(1+i)}{(1+i)}$

$= \dfrac{(1+2i)(1+i)}{1-i^2}$

$= \dfrac{1+2i+i+2i^2}{2}$

$= \dfrac{1-2}{2} + \dfrac{3i}{2}$

$= \dfrac{1}{2} + \dfrac{3}{2}i$

So $|w| = \sqrt{\left(\frac{1}{2}\right)^2 + \left(\frac{3}{2}\right)^2} = \sqrt{\frac{10}{4}} = \underline{\frac{1}{2}\sqrt{10}}$

⊛ Could say here: $|w| = \dfrac{|1+2i|}{|1-i|} = \underline{\dfrac{\sqrt{5}}{\sqrt{2}}}$ which is the same thing.

d) $2z^3 + dz^2 + 140z - 200 = 0$

One root is $z = 6 - 2i$

If d is real then the roots are in conjugate pairs, so another root is $\underline{6+2i}$ and the other root is real.

Also, $(z-(6-2i))(z-(6+2i))$ is a factor

i.e. $z^2 - (6-2i)z - (6+2i)z + (6-2i)(6+2i)$

$= z^2 - 12z + (36 - 4i^2)$

$= z^2 - 12z + 40$

(Check: roots of this are $\dfrac{12 \pm \sqrt{144 - 4.40}}{2}$

$= 6 \pm \dfrac{\sqrt{-16}}{2}$

$= 6 \pm 2i$ which is correct)

So factorising the polynomial, by inspection:

$2z^3 + dz^2 + 140z - 200$ —— ①

$= (z^2 - 12z + 40)(2z - 5)$ —— ②

So the other root is $\underline{z = \frac{5}{2}}$

Multiplying ② out again gives

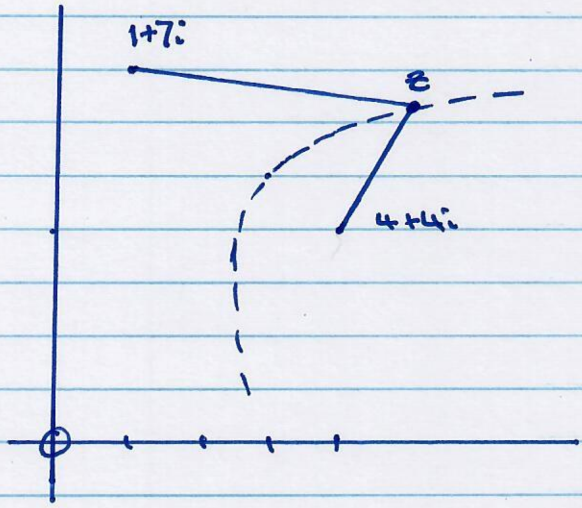
$$
\begin{array}{rl}
 & 2z^3 - 24z^2 + 80z \\
+ & \quad\;\; - 5z^2 + 60z - 200 \\
\hline
 & 2z^3 - 29z^2 + 140z - 200
\end{array}
$$

from which by comparison $\underline{d = -29}$

e) $|z - 1 - 7i| = 2|z - 4 - 4i|$ ①

Rewrite as: $|z - (1+7i)| = 2|z - (4+4i)|$

Then we can draw the Argand diagram as:



If we let $z = x + iy$ and substitute in ①:

$$|x + iy - 1 - 7i| = 2|x + iy - 4 - 4i|$$

or $|x - 1 + i(y-7)| = 2|x - 4 + i(y-4)|$

Squaring these moduli and using Pythagoras' theorem, we get

$$(x-1)^2 + (y-7)^2 = 4(x-4)^2 + 4(y-4)^2$$

or $\cancel{x^2} - \cancel{2x} + 1 + \cancel{y^2} - 14y + 49 = \cancel{4x^2} - \cancel{32x} + 64 + \cancel{4y^2} - \cancel{32y} + 64$

$$0 = 3x^2 - 30x - 3y^2 - 18y + 78$$

or $x^2 - 10x - y^2 - 6y + 26 = 0$

By inspection this can be written as

$$(x-5)^2 - 25 + (y-3)^2 - 9 + 26 = 0$$

or $\underline{(x-5)^2 + (y-3)^2 = 8}$ as required.

Given that $u = 3 + di$ lies on this locus, this means

$$(3-5)^2 + (d-3)^2 = 8$$

$$(-2)^2 + (d-3)^2 = 8$$

$$(d-3)^2 = 4$$

$$d-3 = \pm 2$$

So $d = 1$ or $d = 5$

So $\underline{\underline{u = 3+i}}$ or $\underline{\underline{3+5i}}$

6

3) a) $\text{exp} = \dfrac{\sqrt{2p}}{\sqrt{2p}-\sqrt{p}}$

Divide num. and den. by $\sqrt{p}$:

$$\text{exp} = \frac{\sqrt{2}}{\sqrt{2}-1} = \frac{\sqrt{2}}{\sqrt{2}-1} \cdot \frac{(\sqrt{2}+1)}{(\sqrt{2}+1)}$$

$$= \frac{\sqrt{2}(\sqrt{2}+1)}{2-1} = \frac{2+\sqrt{2}}{1}$$

$$= 2+\sqrt{2}$$ which is in the required form.

b) $z = -2+3i$

$w = z^2 = (-2+3i)(-2+3i)$

$= 4 - 12i + 3i^2$

$= 4-9 - 12i$

$= -5 - 12i$ which is $(-5, -12)$ in the Argand diagram.

(It would be nice to do this with polar coords, but there's no simple way to do that.
But as a check we can say the radius of z is $\sqrt{2^2+3^2} = \sqrt{13}$

whereas radius of $w = \sqrt{5^2+12^2} = \sqrt{169} = 13$

which equals $|z|^2$.)

c) $z = 3 + di$

$\bar{z} = 10dz^{-1} = \dfrac{10d}{z}$

So $z\bar{z} = 10d$

but we know if $z = r\,\text{cis}\,\theta$ where $r = |z|$

then $\bar{z} = r\,\text{cis}(-\theta)$

and $z\bar{z} = r^2$

and $|z| = \sqrt{3^2 + d^2} = |\bar{z}|$

So $z\bar{z} = \left(\sqrt{3^2 + d^2}\right)^2 = 10d$

i.e. $d^2 + 9 = 10d$

$d^2 - 10d + 9 = 0$

$(d-9)(d-1) = 0$

So $d = 1$ or $d = 9$.

(Check: $d = 1$:

$z\bar{z} = (3+i)(3-i) = 3^2 - i^2 = 10.1$ ✓

$d = 9$:

$z\bar{z} = (3+9i)(3-9i) = 3^2 - 81i^2$

$= 9 + 81$

$= 90$

$= 10.9$ ✓)

d) $z^4 + 81k^8 = 0$

so $z^4 = -81k^8$

k is real and this is really only a matter of taking repeated ± roots, working with $\sqrt{\pm i}$. We could do it ~~algebra~~ in Cartesian form but we need to finish in polar form, so perhaps do that from the start:

$$z^4 = -81k^8 = +81k^8 \operatorname{cis}(\pi)$$

$$\text{So} \quad z^2 = \sqrt{-81k^8} = \sqrt{81k^8} \operatorname{cis}\left(\pm\frac{\pi}{2}\right)$$

$$= 9k^4 \operatorname{cis}\left(\pm\frac{\pi}{2}\right)$$

Taking $+\frac{\pi}{2}$:

$$z^2 = 9k^4 \operatorname{cis}\left(\frac{\pi}{2}\right)$$

$$z = 3k^2 \operatorname{cis}\left(\frac{\pi}{4}\right) \text{ or } 3k^2 \operatorname{cis}\left(-\frac{3\pi}{4}\right)$$

Taking $-\frac{\pi}{2}$:

$$z^2 = 9k^4 \operatorname{cis}\left(\frac{-\pi}{2}\right)$$

$$z = 3k^2 \operatorname{cis}\left(\frac{-\pi}{4}\right) \text{ or } 3k^2 \operatorname{cis}\left(\frac{3\pi}{4}\right)$$

So $z = 3k^2 \operatorname{cis}(p)$

where $p = \frac{\pi}{4}, \frac{3\pi}{4}, \frac{-\pi}{4}$ or $-\frac{3\pi}{4}$.

e) $x + \frac{1}{x} = p$

find: $x^3 + \frac{1}{x^3} = q$

Try $\left(x + \frac{1}{x}\right)^3 = x^3 + 3x^2 \cdot \frac{1}{x} + 3x\frac{1}{x^2} + \frac{1}{x^3}$

$= x^3 + \frac{1}{x^3} + 3x + \frac{3}{x}$

$= q + 3\left(x + \frac{1}{x}\right)$

$= q + 3p.$

So $p^3 = q + 3p$

So q (the required expression)

$= x^3 + \frac{1}{x^3}$

$= \underline{\underline{p^3 - 3p}}.$