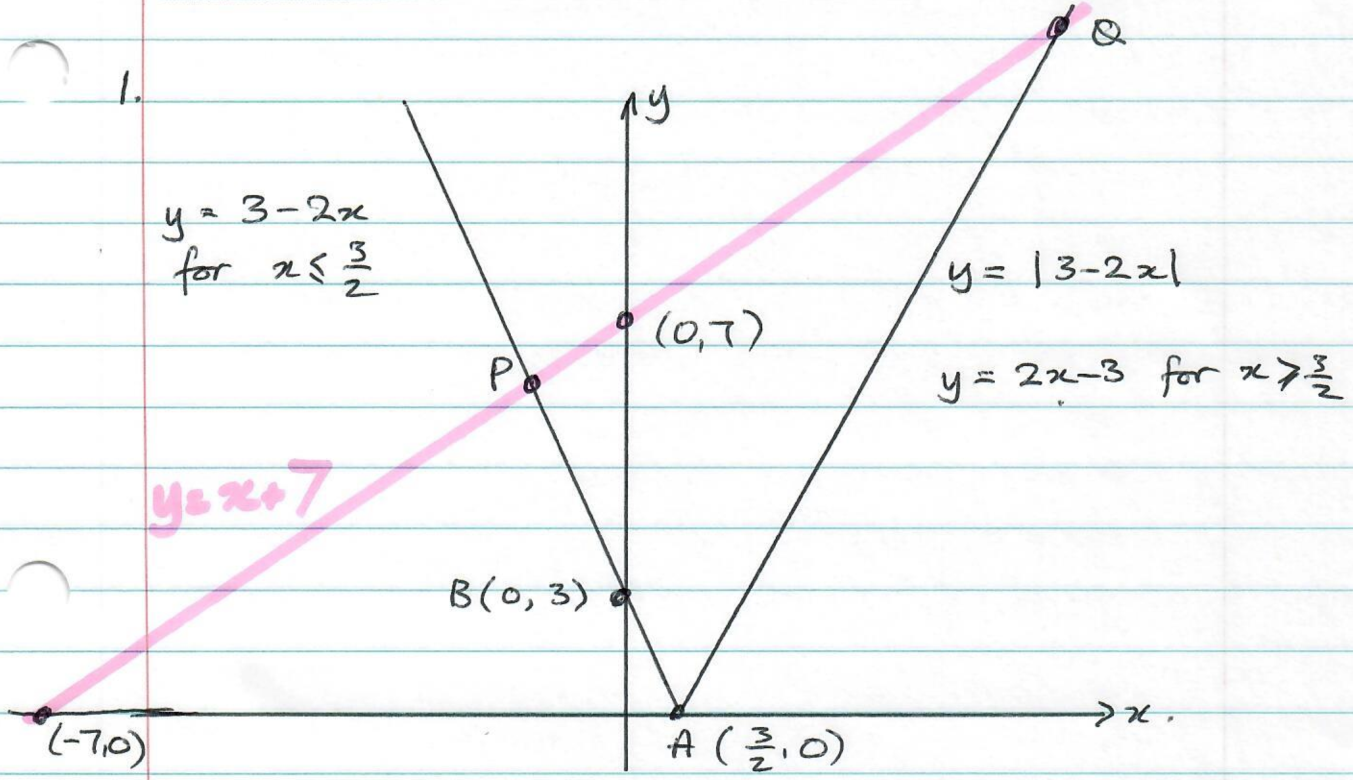
(Includes corrections advised by Vince Milner) ①
(and a couple I've found myself)

Edexcel A Level Pure Maths 2 : June 2022

1.



This one looks simple but it's worth doing a bit of curve investigation. ⓐ The expression $y = 3 - 2x$ is negative on the right hand of the plane – i.e. when $x \geq \frac{3}{2}$, so it's there that we need to reformulate it. To the left we can use it simply as $y = 3 - 2x$.

ⓑ The line $y = 3 - 2x$ has gradient -2

$y = 2x - 3$ 2

$y = x + 7$ 1

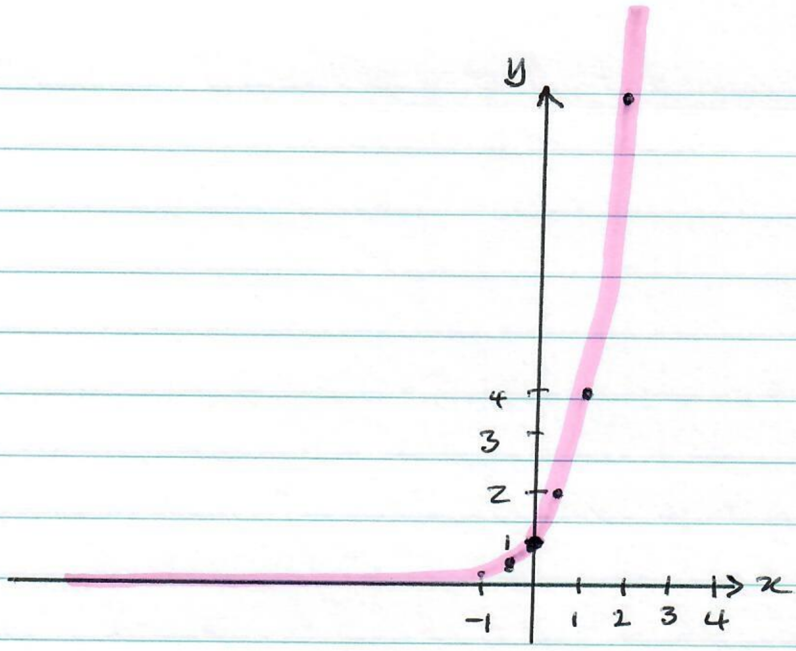
and they cross the axes as shown

--- so we can sketch the basic layout. Then:

Point P: $y = x + 7 = 3 - 2x$: $3x = -4$, $P = \left(\cancel{-\frac{3}{4}}\ -\frac{4}{3}, 6\frac{1}{4}\right)$

Point Q: $y = x + 7 = 2x - 3$: $x = 10$, $Q = (10, 17)$

2a)



The curve rises rapidly for $x > 1$

It crosses the y axis at $y = 4^0$ ie $(0, 1)$

It approaches the x axis assymptotically as $x \to -\infty$

2b) $4^x = 100$

$x \log 4 = \log 100 = 2$

$x = \dfrac{2}{\log 4} = \underline{\underline{3.32}}$

(Check: $4^{3.32} = 99.73$ ✓)

3) a i) $a_1 = 3$

$a_{n+1} = 8 - a_n$

By inspection, $a_{n+2} = 8 - a_{n+1}$

$= 8 - (8 - a_n)$

$= 8 - 8 + a_n$

$= a_n$

This is true for all n so we can see

$a_1 = a_3 = a_5 = \cdots a_{2p+1}$

$a_2 = a_4 = a_6 = \cdots a_{2p}$

So the sequence is periodic...

a) ii) And the period is 2.

b) we observe:

$a_1 = a_3 = a_5 \cdots = a_{85} = 3$

$a_2 = a_4 = \cdots = a_{84} = 5$

Pairing these,

$$\left.\begin{aligned} a_1 + a_2 &= 8 \\ a_3 + a_4 &= 8 \\ &\vdots \\ &\vdots \\ a_{83} + a_{84} &= 8 \end{aligned}\right\} \text{42 rows}$$

$(a_{85} = 3)$

So $\sum_{n=1}^{85} a_n = 8 + 8 + 8 + 8 + 3$

$= 42 \times 8 + 3$

$= \underline{\underline{339}}$

4) $y = 2x^2$

$$\frac{dy}{dx} = \lim_{h \to 0} \frac{y(x+h) - y(h)}{h}$$

$$= \lim_{h \to 0} \frac{2(x+h)^2 - 2x^2}{h}$$

$$= \lim_{h \to 0} 2 \, \frac{\cancel{x^2} + 2xh + h^2 - \cancel{x^2}}{h}$$

$$= \lim_{h \to 0} 2\left(2x + h\right)$$

$$= \lim_{h \to 0} (4x + 2h)$$

As $h \to 0$ $2h \to 0$

So $\frac{dy}{dx} = 4x$ as required.

5a) Using the trapezium rule:

$$I = \int_3^9 \log_3 2x \, dx \approx \frac{1}{2} h \left[y(3) + 2\left(y(4.5) + y(6) + y(7.5)\right) + y(9) \right]$$

where $h = 1.5$

$$\text{So } I \approx \frac{1}{2} \times 1.5 \times \left[1.63 + 2\left(2 + 2.26 + 2.46\right) + 2.63 \right]$$

$$= \frac{1.5}{2} \left[1.63 + 2(6.72) + 2.63 \right]$$

$$= 0.75 \left(1.63 + 13.44 + 2.63 \right)$$

$$= 0.75 \times 17.7$$

$$= \underline{\underline{13.275}}$$

b)i) $\log_3 (2x)^{10} = 10 \log_3 (2x)$

$$\text{So } \int_3^9 \log_3 (2x)^{10} dx = 10 \int_3^9 \log_3 (2x) \, dx$$

$$\approx 10 \times 13.275 = \underline{\underline{132.75}}$$

ii) $\log_3 18x = \log_3 (9 \times 2)x$

$$= \log_3 (9) + \log_3 (2x)$$

$$= 2 + \log_3 (2x)$$

$$\text{So } \int_3^9 \log_3 18x \, dx = \int_3^9 \left[2 + \log_3 (2x) \right] dx$$

$$= \int_3^9 2\,dx + \int_3^9 \log_3(2x)\,dx$$

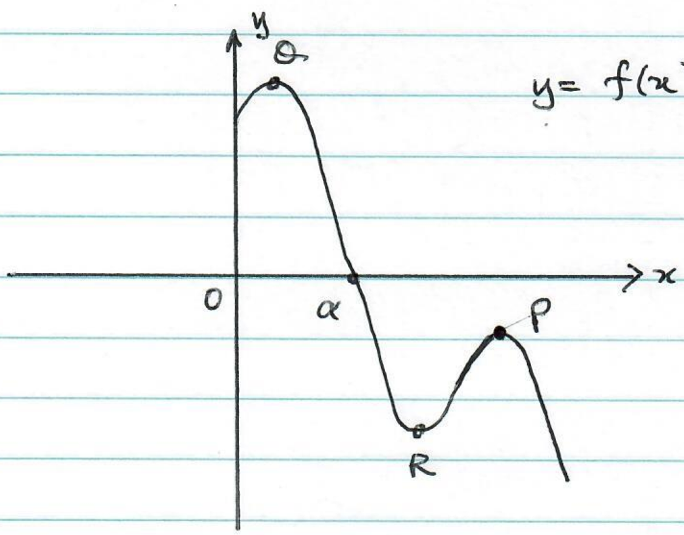
first term $= \left[2x\right]_3^9 = 2(9-3) = 2\cdot 6 = 12$

Second term ≈ 13.275

So the total integral $\int_3^9 \log_3 18x\,dx$

$$\approx 12 + 13.275 = \underline{\underline{25.275}}$$

6a.



$y = f(x) = 8\sin\frac{x}{2} - 3x + 9$

$x > 0$

$\frac{dy}{dx} = 8.\frac{1}{2}\cos\frac{x}{2} - 3 = 0$ at a local maximum

i.e $4\cos\frac{x}{2} - 3 = 0$

$$\cos\frac{x}{2} = \frac{3}{4} = 0.75$$

$\frac{x}{2} = \pm 0.7227$ rad (noting negative solutions are possible)
$+ 2\pi n$ (noting periodicity of cos)

and therefore

$$x = \pm 1.445 + 4\pi n$$

i.e. $x = 1.445 + 0\pi = 1.445$ Q
or $1.445 + 4\pi = 14.011$ P
etc

or $x = -1.445 + 4\pi = 11.121$ R

Inspecting the sketch (though I'm not sure this is rigorons) it appears these are the points P Q and R as labelled above,

i.e. P is the point where $x = 14.011$
(14.0 3 s.f)

(We could prove this more rigorously by finding the second derivative $\frac{d^2y}{dx^2}$ at these points,

but I think the hint 'using ... the sketch' points to doing it this way. And the question is already horrible enough.)

b) Assuming the graph is a fair and complete representation (ie there are no roots anywhere else) then α must lie in the interval $[4, 5]$ because

i) the curve is continuous
ii) the values $f(4)$ and $f(5)$ have opposite signs.

c) Newton Raphson method:

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$$

$x_0 = 5$
$f(x_0) = -1.212$
$$f'(x_0) = 4\cos\frac{5}{2} - 3 = 4 \times -0.801 - 3$$
$$= -3.20 - 3$$
$$= -6.209$$

So
$$x_1 = 5 - \frac{-1.212}{-6.209}$$
$$= 5 - \frac{1.212}{6.209}$$
$$= 5 - 0.195$$
$$= \underline{\underline{4.805}}$$

7a) $(4-9x)^{\frac{1}{2}}$

$= (4)^{\frac{1}{2}}\left(1-\frac{9}{4}x\right)^{\frac{1}{2}}$

$= 2\left(1-\frac{9}{4}x\right)^{\frac{1}{2}}$

Using the binomial expansion...

$$= 2\left\{1+\frac{1}{2}\left(\frac{-9}{4}x\right)+\frac{\frac{1}{2}\left(-\frac{1}{2}\right)\left(\frac{-9}{4}x\right)^2}{2!}+\frac{\frac{1}{2}\left(\frac{-1}{2}\right)\left(\frac{-3}{2}\right)\left(\frac{-9}{4}x\right)^3}{3!}+\ldots\right\}$$

To four terms this is:

$$2\times 1$$
$$+\frac{2}{1}\cdot\frac{1}{2}\cdot\frac{-9}{4}x$$
$$+\frac{2}{2}\cdot\frac{1}{2}\cdot\frac{-1}{2}\cdot\left(\frac{-9}{4}\right)^2 x^2$$
$$+\frac{2}{6}\cdot\frac{1}{2}\cdot\frac{-1}{2}\cdot\frac{-3}{2}\left(\frac{-9}{4}\right)^3 x^3$$

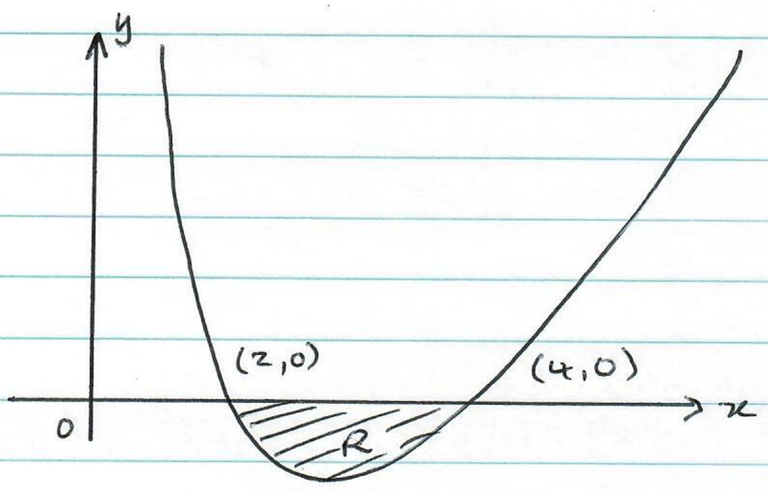
$$= 2-\frac{9}{4}x-\frac{81}{2.2.16}x^2-\frac{\cancel{3}}{\cancel{6}_2.4}\cdot\frac{729}{64}x^3$$

$$= 2-\frac{9}{4}x-\frac{81}{64}x^2-\frac{729}{512}x^3 \quad \text{(first 4 terms)}$$

b) The student will get an estimate for $\sqrt{3}$ but it will be an overestimate, since all subsequent terms are negative -

i.e x^n term is $\frac{1}{n!}\underbrace{\frac{1}{2}\cdot\frac{-1}{2}\cdot\frac{-3}{2}\cdot\frac{-5}{2}\ldots}_{n-1\ \text{-ves}}\underbrace{\left(\frac{-9}{4}x\right)^n}_{n\ \text{-ves}}$

8)



$$y = \frac{(x-2)(x-4)}{4\sqrt{x}} \qquad x > 0$$

y clearly has 2 roots $x=2$ and $x=4$, which lie at either end of R and define the integral, so

$$\text{Area } R = \int_2^4 y\,dx$$

$$y = \frac{(x-2)(x-4)}{4\sqrt{x}} = \frac{1}{4\sqrt{x}}(x^2-6x+8)$$

$$= \frac{1}{4}x^{3/2} - \frac{3}{2}x^{\frac{1}{2}} + 2x^{-\frac{1}{2}}$$

$$\text{So } R = \int_2^4 \frac{x^{3/2}}{4} - \frac{3x^{\frac{1}{2}}}{2} + 2x^{-\frac{1}{2}}\,dx$$

$$= \left[\frac{x^{5/2}}{4 \cdot \frac{5}{2}} - \frac{3x^{\frac{3}{2}}}{2 \cdot \frac{3}{2}} + \frac{2x^{\frac{1}{2}}}{\frac{1}{2}} \right]_2^4$$

$$= \left[\frac{x^{5/2}}{10} - \frac{3x^{3/2}}{3} + 4x^{\frac{1}{2}} \right]_2^4$$

$$= \left[\frac{x^{5/2}}{10} - x^{3/2} + 4x^{1/2} \right]_2^4$$

$$= \frac{4^{5/2}}{10} - 4^{3/2} + 4.4^{1/2}$$

$$- \left[\frac{2^{5/2}}{10} - 2^{3/2} + 4.2^{1/2} \right]$$

$$= \frac{4^2.4^{1/2}}{10} - \cancel{4.4^{1/2}} + \cancel{4.4^{\frac{1}{2}}}$$

$$- \frac{2^2.2^{1/2}}{10} + 2.2^{1/2} - 4.2^{1/2}$$

$$= \frac{16.2}{10} - \cancel{4.2} + \cancel{4.2} - \frac{4}{10}\sqrt{2} + 2\sqrt{2} - 4\sqrt{2}$$

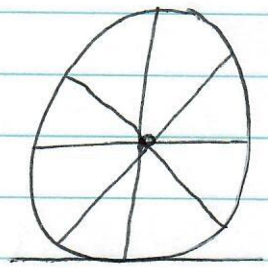
$$= \frac{16}{5} + \left(-2 - \frac{4}{10}\right)\sqrt{2}$$

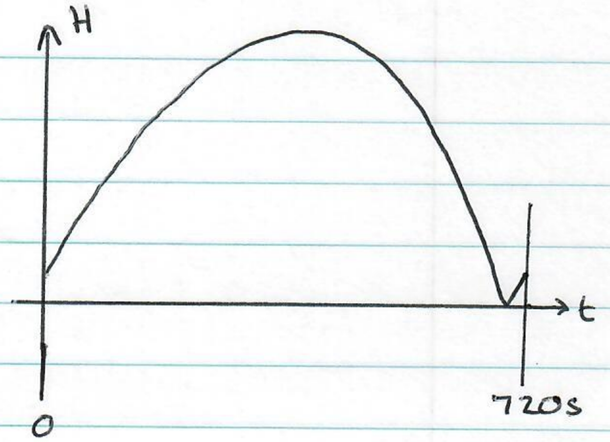
$$= \frac{16}{5} + \left(-2 - \frac{2}{5}\right)\sqrt{2} \qquad = \frac{16}{5} + \frac{1}{5}\left(-10-2\right)\sqrt{2}$$

$$= \frac{16}{5} + \frac{-12}{5}\sqrt{2}$$

$$= \frac{-12}{5}\sqrt{2} + \frac{16}{5} \quad \text{as required.}$$

a





This question is so badly set it's an embarrassment: it doesn't ask us to prove or derive anything so much as to observe or diagnose, particularly since its starting premise is obviously very wrong.

It ought to say ..... is badly modelled by the equation

$$H = |A \sin(bt+\alpha)^\circ|$$

a) Since the maximum value of $\sin(bt+\alpha)$ can be 1 and the maximum H is 50,

$$A = 50$$

We are clearly meant to understand the period 720s corresponds to one half-period (i.e $0° - 180°$) of the sine function

So (ignoring α for the moment) we map $b.0$ to $0°$ and $b.720$ to $180°$

so $b = \frac{1}{4}$.

So we know $H = \left| 50 \sin\left(\frac{t}{4} + \alpha\right)^{\circ} \right|$

and only α is left to find.

We know $H(0) = 1m$,

so $50 \sin \alpha = 1$

$\sin \alpha = \frac{1}{50}$

$\alpha = 1.146^{\circ}$

So $H = \left| 50 \sin\left(\frac{t}{4} + 1.146\right)^{\circ} \right|$

b) The equation in the model is hopeless because

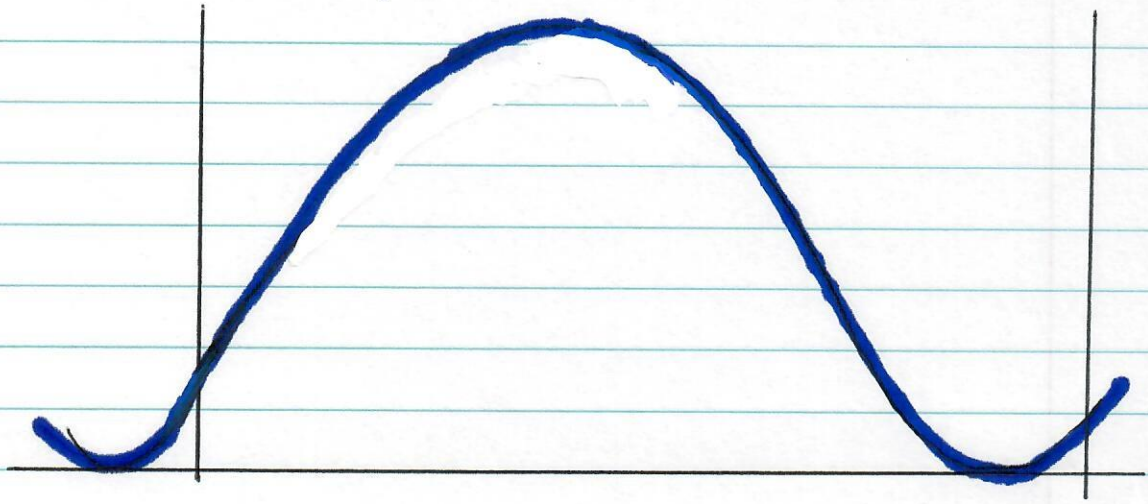
i) It doesn't model a smooth sinusoidal cyclic motion up and down
ii) It bounces the passenger up from the ground, probably killing them.
iii) It just doesn't model reality.

A much better model would be

$H = A \sin(bt + \alpha) + d$

i.e. $25 \sin\left(\frac{t}{2} + \alpha\right) + 25$

which would produce a curve like:



But even this doesn't have the | | form the question requires.

10 $$f(x) = \frac{8x+5}{2x+3} \qquad x > -\frac{3}{2}$$

a) I've a feeling we ought to find a general expression for $f^{-1}(x)$ but... If $f(x) = \frac{3}{2}$ then

$$\frac{8x+5}{2x+3} = \frac{3}{2}$$

$$2(8x+5) = 3(2x+3)$$
$$16x + 10 = 6x + 9$$
$$10x = -1$$
$$x = \frac{-1}{10}$$

so $f^{-1}\left(\frac{3}{2}\right) = \frac{-1}{10}$ $\left(\text{Check: } f\left(\frac{-1}{10}\right) = \frac{-0.8+5}{-0.2+3} = \frac{4.2}{2.8} = \frac{3}{2} \checkmark\right)$

b) By inspection,

$$f(x) = \frac{4(2x+3) - 12 + 5}{2x+3}$$

$$= 4 - \frac{7}{2x+3}$$ as required.

c) $g = 16 - x^2 \qquad 0 \leq x \leq 4$

Trick question: Range of g^{-1} $\equiv$ domain of g

<u>unless</u> we allow $\sqrt{x^2}$ to be negative

– in which case the range of g^{-1} is

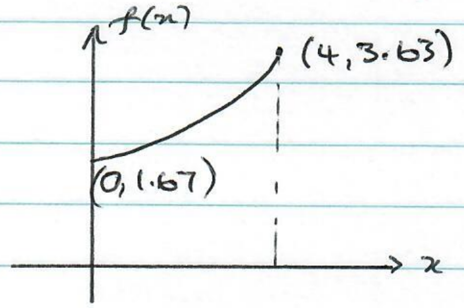
$[-4, 4]$ (but this would contradict general Edexcel view that $\sqrt{x^2}$ is the +ve root.)

d) Assuming range of g^{-1} is $[0, 4]$

Then f applied to $[0, 4]$ is found by:

$$f(0) = 4 - \frac{7}{3} = \frac{5}{3} = 1.67$$

$$f(4) = 4 - \frac{7}{11} = \frac{44-7}{11} = \frac{37}{11} = 3.36$$



So the range of fg^{-1} is $[1.67, 3.36]$

11) If n is even, $n(n^2+5)$ is even since one factor is even

If n is odd, n^2 is odd so n^2+5 is even:
so $n(n^2+5)$ is even since one factor is even.

(Perhaps they wanted more ...)

12) $$f(x) = \frac{e^{3x}}{4x^2+k}$$

using differentiation of a quotient,

$$f'(x) = \frac{(4x^2+k)\frac{d}{dx}(e^{3x}) - e^{3x}\frac{d}{dx}(4x^2+k)}{(4x^2+k)^2}$$

$$= \frac{(4x^2+k)(3e^{3x}) - e^{3x}(8x)}{(4x^2+k)^2}$$

$$= (12x^2+3k-8x)\frac{e^{3x}}{(4x^2+k)^2}$$

$$= (12x^2-8x+3k)\frac{e^{3x}}{(4x^2+k)^2}$$

which is the form required:

$$g(x) = \frac{e^{3x}}{(4x^2+k)^2}$$

b) For a stationary point one factor of $f'(x)$ must be 0. $g(x)$ cannot ever be 0 ($e^{3x} > 0 \ \forall x$)

so this requires $12x^2 - 8x + 3k = 0$

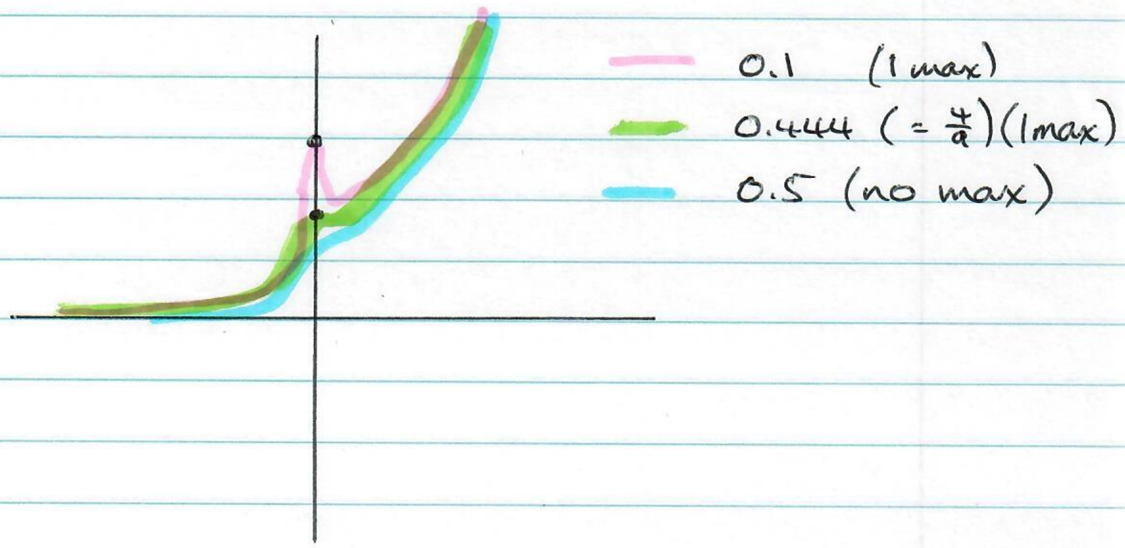
Roots of this equation are

$$x = \frac{8 \pm \sqrt{64 - 4.12.3k}}{24}$$

This requires the discriminant $\geqslant 0$, i.e.

$$64 - 4.12.3k \geqslant 0$$
$$64 - 144k \geqslant 0$$
$$8 - 18k \geqslant 0$$
$$4 - 9k \geqslant 0$$
$$k \leqslant \frac{4}{9}$$

The curve is worth looking at in Geogebra for different values of k:



If you let $K < 0$ then all kinds of wierd things happen – but happily we're told k is positive.

13. A : $4\underline{i} - 3\underline{j} + 5\underline{k}$

B : $4\underline{j} + 6\underline{k}$

C : $-16\underline{i} + p\underline{j} + 10\underline{k}$

A, B, C are on a straight line
So $\overrightarrow{AB}$ is a multiple of $\overrightarrow{AC}$

$$\overrightarrow{AB} = 4\underline{j} + 6\underline{k} - (4\underline{i} - 3\underline{j} + 5\underline{k})$$

$$= -4\underline{i} + 7\underline{j} + \underline{k}$$

$$\overrightarrow{AC} = -16\underline{i} + p\underline{j} + 10\underline{k} - (4\underline{i} - 3\underline{j} + 5\underline{k})$$

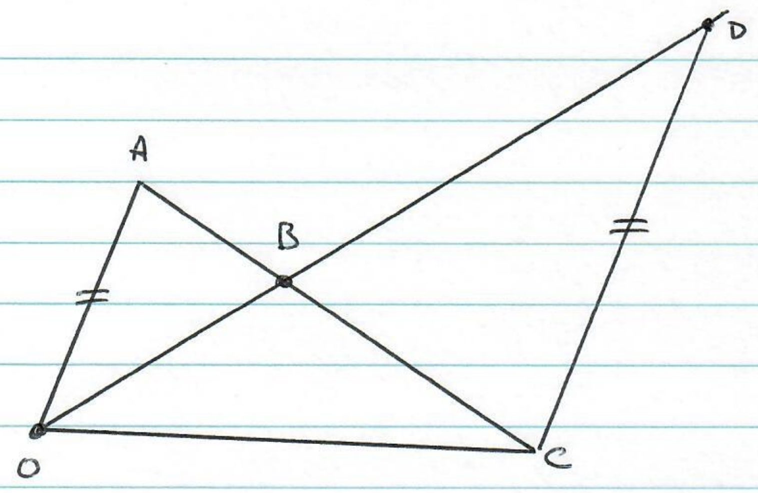
$$= -20\underline{i} + (p+3)\underline{j} + 5\underline{k}$$

Since the coefficients of $\underline{i}$ and $\underline{k}$ differ by a factor of 5, we must have:

$$(p+3) = 5 \times 7 = 35$$

$$\therefore \underline{\underline{p = 32}}$$

13b)



$$\vec{OD} = \lambda\vec{OB} \quad \text{(multiple of } \vec{OB}\text{)}$$
$$= \lambda.0\underline{i} + \lambda.4\underline{j} + \lambda.6\underline{k}$$

$$\vec{CD} = \vec{OD} - \vec{OC}$$
$$= \lambda.0\underline{i} + \lambda.4\underline{j} + \lambda.6\underline{k}$$
$$- (-16\underline{i} + 32\underline{j} + 10\underline{k})$$

$$= 16\underline{i} + (4\lambda - 32)\underline{j} + (6\lambda - 10)\underline{k}$$

But $\vec{CD}$ is parallel to $\vec{OA}$ and hence is a multiple of $\vec{OA}$

$$\vec{OA} = 4\underline{i} - 3\underline{j} + 5\underline{k}$$

By inspection of the $\underline{i}$ coefficient the multiple is 4, so

$$4\lambda - 32 = 4 \times (-3) \quad (\underline{j} \text{ coefficient})$$
$$\therefore \lambda = 5$$

and to check: $6\lambda - 10 = 4 \times (5)$ ($\underline{k}$ coefficient)

$$\therefore \lambda = 5$$

So $\vec{OD} = 5.4\underline{j} + 5.6\underline{k} = 20\underline{j} + 30\underline{k}$

So $|\vec{OD}| = \sqrt{20^2 + 30^2} = 10\sqrt{2^2 + 3^2} = \underline{\underline{10\sqrt{13}}}$

14a) Let $\dfrac{3}{(2x-1)(x+1)} = \dfrac{A(2x-1)+B(x+1)}{(2x-1)(x+1)}$

Then by inspection on the numerators:

Coeffs of x: $\quad 0 = 2A+B \quad$ ①

Coeffs of 1: $\quad 3 = -A+B \quad$ ②

① − ② : $\quad -3 = 3A \quad A = -1$

in ② : $\quad 3 = 1+B \quad B = 2.$

So the expression $= \dfrac{-1(2x-1)+2(x+1)}{(2x-1)\ (x+1)}$

$$= \frac{2}{2x-1} - \frac{1}{x+1}$$

b) $\dfrac{dV}{dt} = \dfrac{3V}{(2t-1)(t+1)}$

$\therefore \dfrac{1}{V}dV = \dfrac{1}{(2t-1)(t+1)}\,dt$

Integrating,

$$\int \frac{1}{V}\,dV = \int \frac{1}{(2t-1)(t+1)}\,dt$$

$$= \int \frac{2}{2t-1}\,dt - \int \frac{1}{t+1}\,dt \quad \text{using part(a)}$$

$$LHS = \int \frac{1}{V}\,dV = \ln V$$

$$RHS = \int \frac{2}{2t-1}\,dt - \int \frac{1}{t+1}\,dt$$

Let $u = 2t-1$: $du = 2dt$ $\quad v = t+1$: $dv = dt$

$$= \int \frac{1}{u}\,du - \int \frac{1}{v}\,dv$$

$$= \ln u - \ln v \quad + \ln C$$

$$= \ln\left(\frac{u}{v}\right) \quad + \ln C$$

$$= \ln \frac{(2t-1)}{t+1} \quad + \ln C$$

$$\therefore \quad \ln V = \ln \frac{(2t-1)}{(t+1)} + \ln C$$

$$\therefore \quad V = C\,\frac{(2t-1)}{(t+1)}$$

Using the conditions given,

$$3 = \frac{C(2.2-1)}{2+1} = C.\frac{3}{3}$$

So $C = 3$

and $V = \dfrac{3(2t-1)}{(t+1)}$ as required.

14c) i) For any $t \geq 0$ the denominator will be +ve, but for $t < \frac{1}{2}$ the numerator will be -ve, which would mean $V < 0$.

This does not make sense, so we conclude that the model does not work (is not intended to apply) when $t < \frac{1}{2}$ (hour)

This maps perfectly to saying there is a time delay of $\frac{1}{2}$ ~~minute~~ hour = 30 minutes

ii) The limit is given by

$$\lim_{t \to \infty} \frac{3(2t-1)}{t+1}$$

$$= \lim_{t \to \infty} \left(\frac{6(t+1)}{t+1} - \frac{9}{t+1} \right) \text{ (by inspection)}$$

$$= \lim_{t \to \infty} 6 - \frac{9}{t+1}$$

$$= \underline{\underline{6 \text{ m}^3}}$$ since the $\frac{1}{t+1}$ term $\to \infty$

15a) $u_1 = 12\cos\theta$

$u_2 = 5 + 2\sin\theta$

$u_3 = 6\tan\theta$

These have a common ratio, so:

$$\frac{u_2}{u_1} = \frac{u_3}{u_2} \quad \therefore \quad \frac{5+2\sin\theta}{12\cancel{\cos\theta}} = \frac{6\tan\theta}{5+2\sin\theta}$$

$$= \frac{6\sin\theta}{(5+2\sin\theta)\cancel{\cos\theta}}$$

$$\therefore (5+2\sin\theta)^2 = 72\sin\theta$$

$$25 + 20\sin\theta + 4\sin^2\theta = 72\sin\theta$$

$$25 - 52\sin\theta + 4\sin^2\theta = 0$$

or $4\sin^2\theta - 52\sin\theta + 25 = 0$

as required.

b) from (a),

$$(2\sin\theta - 25)(2\sin\theta - 1) = 0$$

so $\sin\theta = \frac{25}{2}$ (nonsense - at least before university)

or $\sin\theta = \frac{1}{2}$

$\theta = \cancel{\frac{\pi}{6} (30^\circ)}$ or $\frac{5\pi}{6}$

must be obtuse

So $\theta = \dfrac{5\pi}{6}$

The sum to ∞ of an infinite geometric series is: $S_\infty = \dfrac{a}{1-r}$ where $a = u_1$
$r =$ common ratio

<u>if</u> $|r| < 1$.

Check $|r| < 1$:

$$r = \frac{5 + 2\sin\theta}{12\cos\theta} = \frac{5 + 2 \cdot \frac{1}{2}}{12 \times \frac{-\sqrt{3}}{2}}$$

$\left(\theta = \dfrac{5\pi}{6}\right)$

$$= \frac{5+1}{12} \times \frac{-2}{\sqrt{3}}$$

$$= \frac{1}{2} \times \frac{-2}{\sqrt{3}} = -\frac{1}{\sqrt{3}}.$$

This certainly has modulus < 1, so the series converges.

$$\text{So } S_\infty = \frac{12\cos\theta}{1 - \left(\frac{-1}{\sqrt{3}}\right)}$$

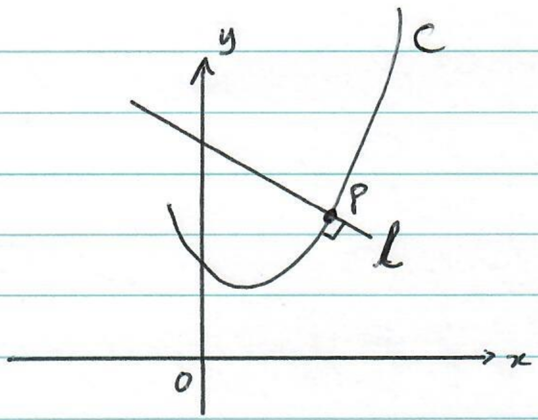
$$= 12 \times \frac{-\sqrt{3}}{2} \times \frac{1}{1 + \frac{1}{\sqrt{3}}}$$

$$= -6\sqrt{3} \times \frac{\sqrt{3}}{\sqrt{3}+1} = \frac{-18}{1+\sqrt{3}}$$

$$= \frac{-18(1-\sqrt{3})}{(1+\sqrt{3})(1-\sqrt{3})} = \frac{-18}{1-3}(1-\sqrt{3}) = \frac{-18}{-2}(1 \overset{-}{\cancel{+}} \sqrt{3})$$

$= 9(1 \overset{-}{\cancel{+}} \sqrt{3})$ so $k = 9$ as required.

16.



$$x = 2\tan t + 1 \qquad -\frac{\pi}{4} \leqslant t \leqslant \frac{\pi}{3}$$
$$y = 2\sec^2 t + 3$$

P is where $t = \frac{\pi}{4}$

To find $\frac{dy}{dx}$:

$$y = 2\sec^2 t + 3 = \frac{2}{\cos^2 t} + 3 = 2\cos^{-2} t + 3$$

$$\frac{dy}{dt} = -4\cos^{-3} t \times -\sin t = \frac{4\sin t}{\cos^3 t}$$

$$x = 2\tan t + 1 = \frac{2\sin t}{\cos t} + 1$$

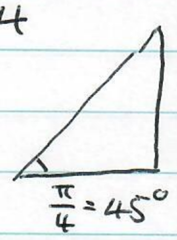
$$\frac{dx}{dt} = 2\,\frac{\cos t\cos t + \sin t\sin t}{\cos^2 t}$$

$$= \frac{2}{\cos^2 t}$$

So $\frac{dy}{dx} = \frac{dy}{dt} \times \frac{dt}{dx} = \frac{4\sin t}{\cos^3 t} \times \frac{\cos^2 t}{2}$

$$= 2\frac{\sin t}{\cos t} = 2\tan t.$$

So at P, $\frac{dy}{dx} = 2 \times \tan \frac{\pi}{4} = 2 \times 1 = 2$



So the normal at P, line l, has

gradient $\frac{-1}{2} = -\frac{1}{2}$.

So line l is of the form $y = -\frac{1}{2}x + c$

To find c we subsitute for $t = \frac{\pi}{4}$:

$$x = 2\tan t + 1 = 2.1 + 1 = 3$$

$$y = 2\sec^2 t + 3 = \frac{2}{\cos^2 t} + 3$$

$$= \frac{2}{(1/\sqrt{2})^2} + 3$$

$$= \frac{2}{1/2} + 3 = 4 + 3 = 7.$$

So $7 = -\frac{1}{2} . 3 + c$

$$c = 7 + \frac{3}{2} = \frac{14+3}{2} = \frac{17}{2}$$

So the line l is $y = -\frac{1}{2}x + \frac{17}{2}$

as required.

(b) RHS $= \frac{1}{2}(x-1)^2 + 5$

$= \frac{1}{2}(2\tan t)^2 + 5$

$= \frac{2^2}{2}\tan^2 t + 5$

$= 2\tan^2 t + 5$

$= 2\frac{\sin^2 t}{\cos^2 t} + 5$

But $\sin^2 t = 1 - \cos^2 t$ so RHS

$= 2\frac{(1-\cos^2 t)}{\cos^2 t} + 5$

$= \frac{2}{\cos^2 t} - 2 + 5$

$= \frac{2}{\cos^2 t} + 3$

$= 2\sec^2 t + 3$

$= y$ $=$ LHS as required.

So all points on C satisfy

$y = \frac{1}{2}(x-1)^2 + 5.$

c) Substitute the line $y = -\frac{x}{2} + k$

into the equation in part b:

$$-\frac{x}{2} + k = \frac{1}{2}(x-1)^2 + 5$$

$$-x + 2k = (x-1)^2 + 10$$

$$-x + 2k = x^2 - 2x + 1 + 10$$

$$0 = x^2 - x + (11 - 2k)$$

$$\therefore \quad x = \frac{+1 \pm \sqrt{(-1)^2 - 4.1.(11-2k)}}{2.1}$$

This is real if the discriminant > 0
(and gives 2 solutions)

i.e. $$(-1)^2 - 4.1.(11-2k) > 0$$

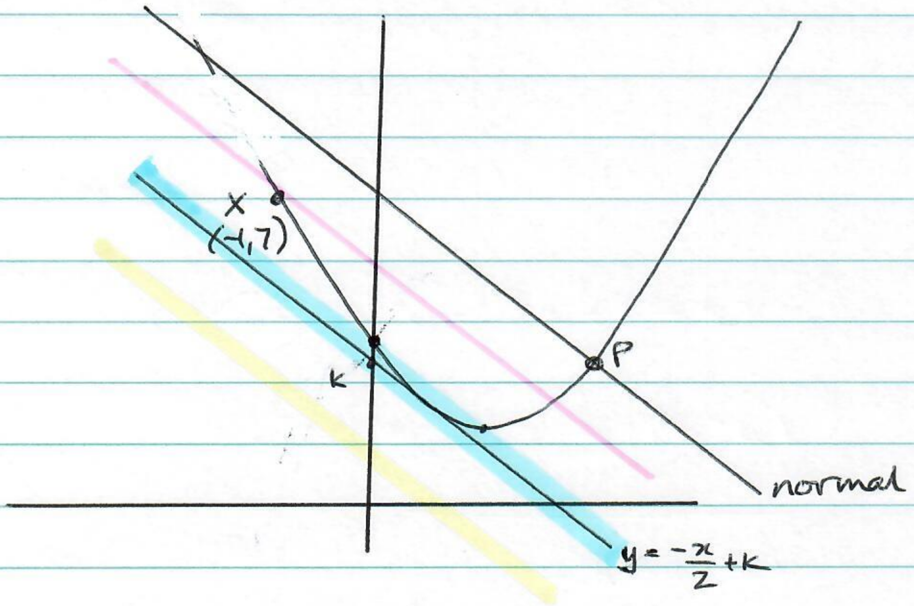
$$1 - 44 + 8k > 0$$

$$8k > 43$$

$$k > \frac{43}{8}$$

(I'm not sure I believe this,
but in fact it's correct – see next sheet)

On inspection (with geogebra) it turns out this problem is a parametrised version of something simpler - namely a parabola. It actually looks like:



The line $y = \frac{-x}{2} + k$ clearly intersects the parabola once when blue or twice when red - and none at all when yellow.

The determining factor for one and above is $k = \frac{43}{8}$, as we've found.

k is in fact the intercept on the y axis and could obviously take any value up to ∞ to produce two intercepts with the parabola. <u>However</u> the curve in this question stops when $t = \frac{-\pi}{4}$, i.e. the point X:

$$x = 2\tan\left(\frac{-\pi}{4}\right) + 1 \quad = -2 + 1 = -1$$

$$y = 2\sec^2\left(\frac{-\pi}{4}\right) + 3 \quad = \ldots$$

(better, simpler) = from part b $\quad \frac{1}{2}(x-1)^2 + 5 = \frac{1}{2}(-2)^2 + 5$

$$= \frac{4}{2} + 5 = 7.$$

If the line $y = -\frac{x}{2} + k$ is not to lie above the point X, the limit is

$$7 = \frac{-(-1)}{2} + k$$

$$7 = \frac{1}{2} + k$$

$$k = \cancel{\frac{15}{2}} \quad \frac{13}{2}$$

So we have a lower bound for k of $\frac{43}{8}$

and an upper bound of $\cancel{\frac{15}{2} = \frac{60}{8}}$ $\frac{13}{2}$

Inspecting the inequalities this gives:

$$\frac{43}{8} < k \leqslant \cancel{\frac{60}{8}} \ \frac{13}{2} \quad \left(\frac{52}{8}\right)$$

As a final check we should make sure the <u>upper</u> bound on t ($\frac{\pi}{3}$) doesn't further limit k – but since the line I already found is within the curve and its y-intercept is $\frac{13}{2} = \cancel{\frac{68}{8}}$, this shows the new line has "run out of 'opportunity' to make 2 intercepts" some time earlier – i.e. at $k = \cancel{\frac{60}{8}}$ $\frac{13}{2}$

(This doesn't sound very rigorous but I could sketch it – very like on page 30.)